Equivalent Definitions of Luce's Semiorders

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Abstract

The equivalence of some definitions of semiorders which are binary relations is shown by using Boolean matrices over the two element Boolean algebra. Semiorders are represented in various forms by using well-known properties of Boolean matrices, and the properties of semiorders are given as properties of Boolean matrices.

1 Introduction

A semiorder is usually represented by a relation or a pair of relations, and it is a fundamental concept in preference, utility, measurement and so on. Many studies of semiorders have been performed in various fields (Fishburn, 1970; Roberts, 1979; Roubens and Vincke, 1985), and some definitions of semiorders exist. In this paper we represent semiorders by Boolean matrices over the two element Boolean algebra (Kim, 1982), and show the equivalence of these definitions. Although the logical equivalence of these definitions is well known among researchers, the proof of equivalence by Boolean matrices is simple and clear. Semiorders are represented in various forms by using well-known properties of Boolean matrices, and many properties of semiorders are shown as properties of Boolean matrices.

2 Operations and notation

For $x, y \in \{0, 1\}$, we define $x \vee y = \max(x, y)$, $x \wedge y = \min(x, y)$, and $\bar{x} = 1 - x$. We treat Boolean matrices over $\{0, 1\}$ and generally denote the (i, j)-element of a Boolean matrix A by $(A)_{ij}$ or a_{ij}. For $m \times n$ Boolean matrices A,

B, and $n \times p$ Boolean matrix C, we define matrix operations as follows:

$(A \vee B)_{ij} = (A)_{ij} \vee (B)_{ij}$,

$(A \wedge B)_{ij} = (A)_{ij} \wedge (B)_{ij}$,

$(\bar{A})_{ij} = \overline{(A)_{ij}}$,

$(A')_{ij} = (A)_{ji}$,

$(A \times C)_{ij} = \vee_k ((A)_{ik} \wedge (C)_{kj})$,

$(A \diamond C)_{ij} = \wedge_k ((A)_{ik} \vee (C)_{kj})$,

$A \leqq B \Leftrightarrow (A)_{ij} \leqq (B)_{ij}$.

As special matrices, we denote the identity matrix by $I = [\delta_{ij}]$ (δ_{ij} is the Kronecker delta), the zero matrix by O, and the universal matrix all of whose elements are 1 by J.

3 Results

We represent semiorders given by Luce (1956, 1959) and Scott and Suppes (1958) by Boolean matrices. We then show the equivalence of the definitions of semiorders represented by Boolean matrices. First we consider the definitions of semiorders given by Luce. Luce's semiorder consists of a pair of relations. In the following, $S = [s_{ij}]$ and $T = [t_{ij}]$ are $n \times n$ Boolean matrices, the elements of which are zero or one.

Definition 1 (Luce, 1956). A Boolean matrix system (S, T) represents a semiorder if for every i, j, k, and l

(1) exactly one of $(S)_{ij} = 1$, $(S)_{ji} = 1$, or $(T)_{ij} = 1$ holds,

(2) $(T)_{ii} = 1$,

(3) $(S)_{ij} = 1$, $(T)_{jk} = 1$, $(S)_{kl} = 1 \Rightarrow (S)_{il} = 1$,

(4) $(S)_{ij} = 1$, $(S)_{jk} = 1$, $(T)_{jl} = 1 \Rightarrow$ not both $(T)_{il} = 1$ and $(T)_{kl} = 1$.

Definition 2. A Boolean matrix system (S, T) represents a semiorder if for every i, j, k, and l

(1) exactly one of $(S)_{ij} = 1$, $(S')_{ij} = 1$, or $(T)_{ij} = 1$ holds,

(2) $(T)_{ii} = 1$,

(3) $(S)_{ij} = 1,\ (T)_{jk} = 1,\ (S)_{kl} = 1 \Rightarrow (S)_{il} = 1$,

(4) $(S)_{jk} = 1,\ (T)_{kl} = 1,\ (S')_{ji} = 1,\ (T)_{il} = 1 \Rightarrow (\overline{T})_{jl} = 1$.

Proposition 1. Definition 1 ⇔ Definition 2.

Proof. As for condition (1) of Definition 1, clearly $(S)_{ji} = 1 \Leftrightarrow (S')_{ij} = 1$. Condition (4) of Definition 1 can be rewritten as follows:

$(S)_{ij} = 1,\ (S)_{jk} = 1,\ (T)_{jl} = 1 \Rightarrow$ not both $(T)_{il} = 1$ and $(T)_{kl} = 1$

if and only if

$(S)_{ij} = 1,\ (S)_{jk} = 1,\ (T)_{jl} = 1 \Rightarrow (T)_{il} \neq 1$ or $(T)_{kl} \neq 1$

if and only if

$(S)_{ij} = 1,\ (T)_{il} = 1,\ (S)_{jk} = 1,\ (T)_{jl} = 1 \Rightarrow (T)_{kl} \neq 1$

if and only if

$(S)_{ij} = 1,\ (T)_{il} = 1,\ (S)_{jk} = 1 \Rightarrow (T)_{jl} \neq 1$ or $(T)_{kl} \neq 1$

if and only if

$(S)_{ij} = 1,\ (T)_{il} = 1,\ (S)_{jk} = 1,\ (T)_{kl} = 1 \Rightarrow (T)_{jl} \neq 1$

if and only if

$(S)_{ij} = 1,\ (T)_{il} = 1,\ (S)_{jk} = 1,\ (T)_{kl} = 1 \Rightarrow (T)_{jl} = 0$

if and only if

$(S')_{ji} = 1,\ (T)_{il} = 1,\ (S)_{jk} = 1,\ (T)_{kl} = 1 \Rightarrow (\overline{T})_{jl} = 1$

if and only if

$(S)_{jk} = 1,\ (T)_{kl} = 1,\ (S')_{ji} = 1,\ (T)_{il} = 1 \Rightarrow (\overline{T})_{jl} = 1$.□

Definition 3. A Boolean matrix system (S, T) represents a semiorder if

(1) $(S \wedge \overline{S'} \wedge \overline{T}) \vee (\overline{S} \wedge S' \wedge \overline{T}) \vee (\overline{S} \wedge \overline{S'} \wedge T) = J$,

(2) $I \leqq T$,

(3) $S \times T \times S \leqq S$,

(4) $(S \times T) \wedge (S' \times T) \leqq \overline{T}$.

Proposition 2. Definition 2 ⇔ Definition 3.

Proof. We consider the corresponding conditions.

(1)－(3) Obvious.

(4) $(S)_{jk}=1,\ (T)_{kl}=1,\ (S')_{ji}=1,\ (T)_{il}=1 \Rightarrow (\overline{T})_{jl}=1$

if and only if

$(S)_{jk}\wedge(T)_{kl}\wedge(S')_{ji}\wedge(T)_{il}=1 \Rightarrow (\overline{T})_{jl}=1$

if and only if

$(S)_{jk}\wedge(T)_{kl}\wedge(S')_{ji}\wedge(T)_{il}\leqq(\overline{T})_{jl}$

if and only if

$\vee_k\vee_i\,((S)_{jk}\wedge(T)_{kl}\wedge(S')_{ji}\wedge(T)_{il})\leqq(\overline{T})_{jl}$

if and only if

$\vee_k\,((S)_{jk}\wedge(T)_{kl})\wedge\vee_i\,((S')_{ji}\wedge(T)_{il})\leqq(\overline{T})_{jl}$

if and only if

$(S\times T)_{jl}\wedge(S'\times T)_{jl}\leqq(\overline{T})_{jl}$

if and only if

$((S\times T)\wedge(S'\times T))_{jl}\leqq(\overline{T})_{jl}$

if and only if

$(S\times T)\wedge(S'\times T)\leqq\overline{T}$.□

Definition 4 (Luce, 1959; Luce and Galanter, 1963). A Boolean matrix system (S, T) represents a semiorder if for every $i, j, k,$ and l

(1) exactly one of $(S)_{ij}=1,\ (S)_{ji}=1,$ or $(T)_{ij}=1$ holds,

(2) $(T)_{ii}=1,$

(3) $(S)_{ij}=1,\ (T)_{jk}=1,\ (S)_{kl}=1 \Rightarrow (S)_{il}=1,$

(4) $(S)_{ij}=1,\ (S)_{jk}=1 \Rightarrow$ not both $(T)_{il}=1$ and $T_{lk}=1.$

Definition 5. A Boolean matrix system (S, T) represents a semiorder if

(1) $(S\wedge\overline{S'}\wedge\overline{T})\vee(\overline{S}\wedge S'\wedge\overline{T})\vee(\overline{S}\wedge\overline{S'}\wedge T)=J,$

(2) $I\leqq T,$

(3) $S\times T\times S\leqq S,$

(4) $S\times S\leqq\overline{T\times T}.$

Proposition 3. Definition 4 ⇔ Definition 5.

Proof. We consider the corresponding conditions.

(1) − (3) Obvious.

(4) $(S)_{ij}=1,\ (S)_{jk}=1 \Rightarrow$ not both $(T)_{il}=1$ and $(T)_{lk}=1$

if and only if

$(S)_{ij}=1,\ (S)_{jk}=1 \Rightarrow (T)_{il} \neq 1$ or $(T)_{lk} \neq 1$

if and only if

$(S)_{ij}=1,\ (S)_{jk}=1 \Rightarrow (T)_{il}=0$ or $(T)_{lk}=0$

if and only if

$(S)_{ij}=1,\ (S)_{jk}=1 \Rightarrow (\overline{T})_{il}=1$ or $(\overline{T})_{lk}=1$

if and only if

$(S)_{ij} \wedge (S)_{jk}=1 \Rightarrow (\overline{T})_{il} \vee (\overline{T})_{lk}=1$

if and only if

$(S)_{ij} \wedge (S)_{jk} \leqq (\overline{T})_{il} \vee (\overline{T})_{lk}$

if and only if

$\vee_j ((S)_{ij} \wedge (S)_{jk}) \leqq \wedge_l ((\overline{T})_{il} \vee (\overline{T})_{lk})$

if and only if

$(S \times S)_{ik} \leqq (\overline{T} \diamond \overline{T})_{ik}$

if and only if

$S \times S \leqq \overline{T} \diamond \overline{T} = \overline{T \times T}$. □

Proposition 4 (Luce and Galanter, 1963). If $I \leqq T$ and $S \times T \times S \leqq S$, then $S \times S \leqq S$.

Proof. $S \times S = S \times I \times S \leqq S \times T \times S \leqq S$. □

Lemma 1. For $m \times n$ Boolean matrices A, B, and C,

$(A \wedge B) \vee (B \wedge C) \vee (C \wedge A) = O, A \vee B \vee C = J$

$\Leftrightarrow (A \wedge \overline{B} \wedge \overline{C}) \vee (\overline{A} \wedge B \wedge \overline{C}) \vee (\overline{A} \wedge \overline{B} \wedge C) = J.$

Proof. $(A \wedge B) \vee (B \wedge C) \vee (C \wedge A) = O, A \vee B \vee C = J$

$\Leftrightarrow (\overline{A} \vee \overline{B}) \wedge (\overline{B} \vee \overline{C}) \wedge (\overline{C} \vee \overline{A}) = J, A \vee B \vee C = J$

$\Leftrightarrow (\overline{A} \vee \overline{B}) \wedge (\overline{B} \vee \overline{C}) \wedge (\overline{C} \vee \overline{A}) \wedge (A \vee B \vee C) = J$

$\Leftrightarrow((\bar{A}\wedge\bar{C})\vee\bar{B})\wedge(\bar{C}\vee\bar{A})\wedge(A\vee B\vee C)=J$

$\Leftrightarrow((\bar{A}\wedge\bar{C})\vee(\bar{B}\wedge\bar{C})\vee(\bar{A}\wedge\bar{B}))\wedge(A\vee B\vee C)=J$

$\Leftrightarrow(\bar{A}\wedge B\wedge\bar{C})\vee(A\wedge\bar{B}\wedge\bar{C})\vee(\bar{A}\wedge\bar{B}\wedge C)=J$

$\Leftrightarrow(A\wedge\bar{B}\wedge\bar{C})\vee(\bar{A}\wedge B\wedge\bar{C})\vee(\bar{A}\wedge\bar{B}\wedge C)=J.$ □

Proposition 5. $(S\wedge\overline{S'}\wedge\bar{T})\vee(\bar{S}\wedge S'\wedge\bar{T})\vee(\bar{S}\wedge\overline{S'}\wedge T)=J$

$\Leftrightarrow(S\wedge S')\vee(S'\wedge T)\vee(T\wedge S)=O, S\vee S'\vee T=J$

Proof. By Lemma 1. □

Proposition 6 (Krantz, 1967; Luce and Galanter, 1963). If $(S\wedge\overline{S'}\wedge\bar{T})\vee(\bar{S}\wedge S'\wedge\bar{T})\vee(\bar{S}\wedge\overline{S'}\wedge T)=J$, then $S\wedge S'=O$ and $I\leqq T$.

Proof. By Proposition 5, $(S\wedge S')\vee(S'\wedge T)\vee(T\wedge S)=O$ and $S\vee S'\vee T=J$, so $S\wedge S'=O$. Thus $S\wedge I=O$, and from $S\vee S'\vee T=J$ we get $I\leqq T$. □

Proposition 7 (Luce and Galanter, 1963). If $(S\wedge\overline{S'}\wedge\bar{T})\vee(\bar{S}\wedge S'\wedge\bar{T})\vee(\bar{S}\wedge\overline{S'}\wedge T)=J$ and $S\times T\times S\leqq S$, then $S\times S\leqq S$.

Proof. Using Proposition 6, we have $I\leqq T$. Thus from Proposition 4, $S\times S\leqq S$. □

Proposition 8. If $(S\wedge T)\vee(S'\wedge T)=O$ and $S\vee S'\vee T=J$, then $T=\bar{S}\wedge\overline{S'}$ and $T'=T$.

Proof. From $S\vee S'\vee T=J$, $\bar{S}\wedge\overline{S'}\wedge\bar{T}=O$, or $\bar{S}\wedge\overline{S'}\leqq T$. Moreover from $(S\wedge T)\vee(S'\wedge T)=O$, $(S\vee S')\wedge T=O$, or $T\leqq\bar{S}\wedge\overline{S'}$. Therefore $T=\bar{S}\wedge\overline{S'}$ and $T'=\overline{S'}\wedge\bar{S}=T$. □

Proposition 9 (Krantz, 1967; Luce and Galanter, 1963). If $(S\wedge\overline{S'}\wedge\bar{T})\vee(\bar{S}\wedge S'\wedge\bar{T})\vee(\bar{S}\wedge\overline{S'}\wedge T)=J$, then $T=\bar{S}\wedge\overline{S'}$ and $T'=T$.

Proof. It follows from Proposition 5 that $(S\wedge S')\vee(S'\wedge T)\vee(T\wedge S)=O$ and $S\vee S'\vee T=J$. Therefore using Proposition 8, we have $T=\bar{S}\wedge\overline{S'}$ and $T'=T$. □

Lemma 2 (Chipman, 1960; Luce, 1952; Schmidt and Ströhlein, 1993; Schröder, 1895).

For Boolean matrices $A\ (m \times n)$, $B\ (n \times p)$, and $C\ (m \times p)$, the following conditions are equivalent:

(1) $A \times B \leqq C$.

(2) $A \leqq C \diamond \overline{B'}$.

(3) $B \leqq \overline{A'} \diamond C$.

Proof. (1)⇒(2) Let $c_{ik} = 0$ and $b_{jk} = 1$. Then since $A \times B \leqq C$, $a_{ij} = 0$. Hence we obtain (2).

(2)⇒(1) Let $a_{ik} = 1$ and $b_{kj} = 1$. Then since $A \leqq C \diamond \overline{B'}$, $c_{ij} = 1$. Hence we obtain (1).

(1)⇒(3) Let $a_{ki} = 1$ and $c_{kj} = 0$. Then since $A \times B \leqq C$, $b_{ij} = 0$. Hence we obtain (3).

(3)⇒(1) Let $a_{ik} = 1$ and $b_{kj} = 1$. Then since $B \leqq \overline{A'} \diamond C$, $c_{ij} = 1$. Hence we obtain (1).□

Lemma 3. For Boolean matrices $A\ (m \times n)$, $B\ (n \times p)$, $C\ (m \times q)$, and $D\ (q \times p)$, the following conditions are equivalent:

(1) $(A \times B) \wedge (C \times D) = O$.

(2) $A \times B \leqq \overline{C \times D}$.

(3) $(A' \times C) \wedge (B \times D') = O$.

Proof. Clearly $(A \times B) \wedge (C \times D) = O \Leftrightarrow (A \times B) \leqq \overline{C \times D}$. Using Lemma 2, we have

$A \times B \leqq \overline{C \times D}$

$\Leftrightarrow A \times B \leqq \overline{C} \diamond \overline{D}$

$\Leftrightarrow A \times B \times D' \leqq \overline{C}$

$\Leftrightarrow B \times D' \leqq \overline{A'} \diamond \overline{C}$

$\Leftrightarrow B \times D' \leqq \overline{A' \times C}$

$\Leftrightarrow (B \times D') \wedge (A' \times C) = O$

$\Leftrightarrow (A' \times C) \wedge (B \times D') = O$.□

Proposition 10. If $T' = T$, then $S \times S \leqq \overline{T \times T} \Leftrightarrow (S \times T) \wedge (S' \times T) = O$.

Proof. From Lemma 3, $S \times S \leqq \overline{T \times T} \Leftrightarrow (S' \times T) \wedge (S \times T') = O \Leftrightarrow (S \times T') \wedge (S' \times T) = O$. Since $T' = T$, $S \times S \leqq \overline{T \times T} \Leftrightarrow (S \times T) \wedge (S' \times T) = O$.□

Proposition 11. If $S \vee S' \vee T = J$, $T' = T$, $I \leqq T$, and $S \times T \times S \leqq S$, then $(S \times T) \wedge (S' \times T) \leqq \overline{T} \Leftrightarrow S \times S \leqq \overline{T \times T}$.

Proof. (⇒) Suppose $s_{ik} \wedge s_{kj} = 1$. It will be shown that $\vee_l (t_{il} \wedge t_{lj}) = 0$. Assume by way of contradiction that $t_{il} \wedge t_{lj} = 1$ for some l. Since $T' = T$, $t_{li} = 1$ and $t_{jl} = 1$. Then from $(s_{kj} \wedge t_{jl}) \wedge (s_{ik} \wedge t_{il}) = 1$ it follows that $t_{kl} = 0$. Since $S \vee S' \vee T = J$, $s_{kl} \vee s_{lk} = 1$.

Case 1. $s_{kl} = 1$. Since $s_{kl} \wedge t_{li} \wedge s_{ik} = 1$ and $S \times T \times S \leqq S$, $s_{kk} = 1$. Furthermore since $s_{kk} \wedge t_{kk} = 1$ and $(S \times T) \wedge (S' \times T) \leqq \overline{T}$, $t_{kk} = 0$, which is a contradiction.

Case 2. $s_{lk} = 1$. Since $s_{kj} \wedge t_{jl} \wedge s_{lk} = 1$ and $S \times T \times S \leqq S$, $s_{kk} = 1$. Then since $s_{kk} \wedge t_{kk} = 1$ and $(S \times T) \wedge (S' \times T) \leqq \overline{T}$, $t_{kk} = 0$, which is a contradiction.

(⇐) Using Proposition 10 we have $(S \times T) \wedge (S' \times T) = O$. Hence $(S \times T) \wedge (S' \times T) \leqq \overline{T}$.□

Proposition 12. If $(S \wedge \overline{S'} \wedge \overline{T}) \vee (\overline{S} \wedge S' \wedge \overline{T}) \vee (\overline{S} \wedge \overline{S'} \wedge T) = J$ and $S \times T \times S \leqq S$, then $(S \times T) \wedge (S' \times T) \leqq \overline{T} \Leftrightarrow S \times S \leqq \overline{T \times T}$.

Proof. Using Propositions 5, 6, and 9, we have $S \vee S' \vee T = J$, $I \leqq T$, and $T' = T$. Therefore the result follows from Proposition 11. □

Proposition 13. Definition 3 ⇔ Definition 5.

Proof. By Proposition 12.□

Definition 6 (Krantz, 1967). A Boolean matrix system (S, T) represents a semiorder if

(1) $(S \wedge \overline{S'} \wedge \overline{T}) \vee (\overline{S} \wedge S' \wedge \overline{T}) \vee (\overline{S} \wedge \overline{S'} \wedge T) = J$,

(2) $S \times T \times S \leqq S$,

(3) $S \times S \leqq \overline{T \times T}$.

Proposition 14. Definition 5 ⇔ Definition 6.

Proof. By Proposition 6.□

Definition 7. A Boolean matrix S represents a semiorder if

(1) $S \wedge S' = O$,

(2) $S \times (\overline{S} \wedge \overline{S'}) \times S \leqq S$,

(3) $S \times S \leqq \overline{(S \vee S') \diamond (S \vee S')}$.

Proposition 15. Definition 6 ⇔ Definition 7 with T = $\overline{S} \wedge \overline{S'}$.

Proof. (⇒) By Proposition 6, from $(S \wedge \overline{S'} \wedge \overline{T}) \vee (\overline{S} \wedge S' \wedge \overline{T}) \vee (\overline{S} \wedge \overline{S'} \wedge T) = J$, it follows that $S \wedge S' = O$. Moreover using Proposition 9 we have $T = \overline{S} \wedge \overline{S'}$. Since $T = \overline{S} \wedge \overline{S'}$ and $S \times T \times S \leqq S$, $S \times (\overline{S} \wedge \overline{S'}) \times S \leqq S$. Furthermore since $T = \overline{S} \wedge \overline{S'}$ and $S \times S \leqq \overline{T \times T}$, $S \times S \leqq \overline{(S \vee S') \diamond (S \vee S')}$.

(⇐) $(S \wedge \overline{S'} \wedge \overline{T}) \vee (\overline{S} \wedge S' \wedge \overline{T}) \vee (\overline{S} \wedge \overline{S'} \wedge T)$

$= (S \wedge \overline{S'} \wedge (S \vee S')) \vee (\overline{S} \wedge S' \wedge (S \vee S')) \vee (\overline{S} \wedge \overline{S'} \wedge (\overline{S} \wedge \overline{S'}))$

$= (S \wedge \overline{S'}) \vee (\overline{S} \wedge S') \vee (\overline{S} \wedge \overline{S'})$

$= (S \wedge \overline{S'}) \vee \overline{S}$

$= \overline{S} \vee \overline{S'} = \overline{S \wedge S'} = J.$

$S \times T \times S = S \times (\overline{S} \wedge \overline{S'}) \times S \leqq S.$

$S \times S \leqq \overline{(S \vee S') \diamond (S \vee S')} = \overline{\overline{T} \diamond \overline{T}} = \overline{T \times T}$.□

Definition 8. A Boolean matrix S represents a semiorder if

(1) $S \wedge I = O$,

(2) $S \times (\overline{S} \wedge \overline{S'}) \times S \leqq S$,

(3) $S \times S \leqq \overline{(S \vee S') \diamond (S \vee S')}$.

Proposition 16 (Monjardet, 1988). If $S \wedge I = O$ and $S \times (\overline{S} \wedge \overline{S'}) \times S \leqq S$, then $S \times S \leqq S$.

Proof. Since $S \wedge I = O$, $I \leqq \overline{S}$; thus $I \leqq \overline{S} \wedge \overline{S'}$. Then $S \times S = S \times I \times S \leqq S \times (\overline{S} \wedge \overline{S'}) \times S \leqq S$.□

Proposition 17. If $S \times (\overline{S} \wedge \overline{S'}) \times S \leqq S$, then $S \wedge S' = O$ ⇔ $S \wedge I = O$.

Proof. (⇒) Clearly $S \wedge I = O$.

(⇐) Using Proposition 16 we have $S \times S \leqq S$. Since $S \times S \leqq S$ and $S \wedge I =$

O, $S \wedge S' = O$.□

Proposition 18. If $S \times S \leqq (S \vee S') \diamondsuit (S \vee S')$, then $S \wedge S' = O \Leftrightarrow S \wedge I = O$.

Proof. (⇒) Clearly $S \wedge I = O$.

(⇐) We assume that $s_{ij} \wedge s_{ji} = 1$. Since $S \times S \leqq (S \vee S') \diamondsuit (S \vee S')$, it follows that $\wedge_k (s_{ik} \vee s_{ki} \vee s_{ki} \vee s_{ik}) = 1$. For $k = i$, $s_{ii} \vee s_{ii} \vee s_{ii} \vee s_{ii} = 1$. This is a contradiction since $S \wedge I = O$. Hence $S \wedge S' = O$.□

Proposition 19. Definition 7 ⇔ Definition 8.

Proof. By Propositions 17 or 18.□

Definition 9. A Boolean matrix S represents a semiorder if

(1) $S \wedge I = O$,

(2) $S \times \overline{S}' \times S \leqq S$,

(3) $S \times S \leqq (S \vee S') \diamondsuit (S \vee S')$.

Proposition 20. If $S \wedge I = O$, then $S \times (\overline{S} \wedge \overline{S}') \times S \leqq S \Leftrightarrow S \times \overline{S}' \times S \leqq S$.

Proof. (⇒) Suppose $s_{ik} \wedge \overline{s_{lk}} \wedge s_{lj} = 1$. It will be shown that $s_{ij} = 1$.

Case 1. $s_{kl} = 1$. Since $S \wedge I = O$ and $S \times (\overline{S} \wedge \overline{S}') \times S \leqq S$, by Proposition 16 we have $S \times S \leqq S$. Therefore from $s_{ik} \wedge s_{kl} \wedge s_{lj} = 1$ it follows that $s_{ij} = 1$.

Case 2. $s_{kl} = 0$. Since $s_{ik} \wedge \overline{s_{kl}} \wedge \overline{s_{lk}} \wedge s_{lj} = 1$, $s_{ij} = 1$.

(⇐) Clearly $S \times (\overline{S} \wedge \overline{S}') \times S \leqq S \times \overline{S}' \times S \leqq S$.□

Proposition 21. Definition 8 ⇔ Definition 9.

Proof. The result follows from Proposition 20.□

Definition 10 (Scott and Suppes, 1958). A matrix S represents a semiorder if for every i, j, k, and l

(1) $(S)_{ii} = 0$,

(2) $(S)_{ij} = 1$, $(S)_{kl} = 1 \Rightarrow (S)_{il} = 1$ or $(S)_{kj} = 1$,

(3) $(S)_{ij} = 1$, $(S)_{ki} = 1 \Rightarrow (S)_{lj} = 1$ or $(S)_{kl} = 1$.

Definition 11. A matrix S represents a semiorder if for every i, j, k, and l

(1) $(S)_{ii} = 0$,

(2) $(S)_{ij} = 1,\ (\bar{S}')_{jk} = 1,\ (S)_{kl} = 1 \Rightarrow (S)_{il} = 1$,

(3) $(S)_{ki} = 1,\ (S)_{ij} = 1 \Rightarrow (S)_{kl} = 1$ or $(S)_{lj} = 1$.

Proposition 22. Definition 10 ⇔ Definition 11.

Proof. We can rewrite condition (2) of Definition 10 as follows:

$(S)_{ij} = 1,\ (S)_{kl} = 1 \Rightarrow (S)_{il} = 1$ or $(S)_{kj} = 1$

if and only if

$(S)_{ij} = 1,\ (S)_{kj} \neq 1,\ (S)_{kl} = 1 \Rightarrow (S)_{il} = 1$

if and only if

$(S)_{ij} = 1,\ (S)_{kj} = 0,\ (S)_{kl} = 1 \Rightarrow (S)_{il} = 1$

if and only if

$(S)_{ij} = 1,\ (\bar{S}')_{jk} = 1,\ (S)_{kl} = 1 \Rightarrow (S)_{il} = 1$.□

Definition 12 (Scott and Suppes, 1958). A Boolean matrix S represents a semiorder if

(1) $S \wedge I = O$,

(2) $S \times \bar{S}' \times S \leqq S$,

(3) $S \times S \leqq S \diamond S$.

Proposition 23. Definition 11 ⇔ Definition 12.

Proof. Definition 12 is a matrix representation of Definition 11.□

Proposition 24. If $S \times S \leqq S$, then $S \times S \leqq (S \vee S') \diamond (S \vee S')$ ⇔ $S \times S \leqq S \diamond S$.

Proof. (⇒) Let $s_{ik} \wedge s_{kj} = 1$. It will be shown that $\wedge_l (s_{il} \vee s_{lj}) = 1$. Then we assume by way of contradiction that $s_{il} \vee s_{lj} = 0$ for some l. Since $S \times S \leqq S$, $s_{ij} = 1$. Furthermore since $s_{ik} = 1$ and $s_{il} = 0$, $s_{kl} = 0$, and since $s_{kj} = 1$ and $s_{lj} = 0$, $s_{lk} = 0$. Similarly since $s_{kj} = 1$ and $s_{kl} = 0$, $s_{jl} = 0$, and since $s_{ik} = 1$ and $s_{lk} = 0$, $s_{li} = 0$. From $s_{ik} \wedge s_{kj} = 1$ and $S \times S \leqq (S \vee S') \diamond (S \vee S')$, it follows that $(s_{il} \vee s_{li}) \vee (s_{lj} \vee s_{jl}) = 1$. But this is a contradiction.

(⇐) Clearly $S \times S \leqq S \diamond S \leqq (S \vee S') \diamond (S \vee S')$.□

Proposition 25. If $S \wedge I = O$ and $S \times (\overline{S} \wedge \overline{S'}) \times S \leqq S$, then $S \times S \leqq (S \vee S') \diamond (S \vee S') \Leftrightarrow S \times S \leqq S \diamond S$.

Proof. If $S \wedge I = O$ and $S \times (\overline{S} \wedge \overline{S'}) \times S \leqq S$, then we have $S \times S \leqq S$ by Proposition 16. Thus the result follows from Proposition 24.□

Proposition 26. If $S \wedge I = O$ and $S \times \overline{S'} \times S \leqq S$, then $S \times S \leqq (S \vee S') \diamond (S \vee S') \Leftrightarrow S \times S \leqq S \diamond S$.

Proof. Since $S \times (\overline{S} \wedge \overline{S'}) \times S \leqq S \times \overline{S'} \times S \leqq S$, the result follows from Proposition 25.□

Proposition 27. Definition 9 ⇔ Definition 12.

Proof. The result follows from Proposition 26.□

Definition 13 (Ducamp and Falmagne, 1969). A Boolean matrix S represents a semiorder if

(1) $S \wedge I = O$,

(2) $S \times \overline{S'} \times S \leqq S$,

(3) $S \times S \times \overline{S'} \leqq S$ or $(S \times \overline{S'}) \wedge (S' \times \overline{S}) = O$.

Proposition 28 (Ducamp and Falmage, 1969). Definition 12 ⇔ Definition 13.

Proof. From Lemma 2, $S \times S \leqq S \diamond S \Leftrightarrow S \times S \times \overline{S'} \leqq S \Leftrightarrow S \times \overline{S'} \leqq \overline{S'} \diamond S \Leftrightarrow (S \times \overline{S'}) \wedge (\overline{\overline{S'} \diamond S}) = O \Leftrightarrow (S \times \overline{S'}) \wedge (S' \times \overline{S}) = O$.□

Proposition 29 (Fodor and Roubens, 1994). If $S \wedge I = O$ and $S \times S \leqq S \diamond S$, then $S \times S \leqq S$.

Proof. If $S \times S \leqq S \diamond S$, then from Lemma 2, $S \times S \times \overline{S'} \leqq S$. Since $S \wedge I = O$, $I \leqq \overline{S'}$; thus $S \times S = S \times S \times I \leqq S \times S \times \overline{S'} \leqq S$.□

Proposition 30. If $S \wedge S' \leqq I$ and $S \times S \leqq S \diamond S$, then $S \times S \leqq S$.

Proof. Suppose $s_{ik} \wedge s_{kj} = 1$. It will be shown that $s_{ij} = 1$. We then assume by way of contradiction that $s_{ij} = 0$. From $\wedge_l (s_{il} \vee s_{lj}) = 1$, it follows that $s_{ii} = 1$. Since $s_{ik} = 1$ and $s_{ij} = 0$, $k \neq j$; thus $s_{jk} = 0$. Furthermore, since $s_{ii} \wedge s_{ik} = 1$, $\wedge_l (s_{il} \vee s_{lk}) = 1$. For $l = j$, $s_{ij} \vee s_{jk} = 1$, which is a contradiction; hence $s_{ij}=1$.□

Proposition 31. If $S\vee S'\vee I=J$ and $S\times S\leqq S$, then $S\times S\leqq S\diamond S$.

Proof. Suppose $s_{ik}\wedge s_{kj}=1$. It will be shown that $\wedge_l(s_{il}\vee s_{lj})=1$. We then assume by way of contradiction that $s_{il}\vee s_{lj}=0$ for some l. Since $s_{ik}\wedge s_{kj}=1$, $s_{ij}=1$. Therefore, since $s_{lj}=0$, $i\neq l$. From $S\vee S'\vee I=J$ it then follows that $s_{li}=1$. Since $s_{li}\wedge s_{ij}=1$, $s_{lj}=1$, which is a contradiction; hence $s_{il}\vee s_{lj}=1$ for all l.□

Proposition 32. If $S\wedge S'\leqq I$ and $S\vee S'\vee I=J$, then $S\times S\leqq S\diamond S\Leftrightarrow S\times S\leqq S$.

Proof. By Propositions 30 and 31.□

Definition 14. A Boolean matrix S represents a semiorder if

(1) $S\wedge I=O$,

(2) $S\times\overline{S'}\leqq(S\diamond\overline{S'})\wedge(\overline{S'}\diamond S)$.

Proposition 33. Definition 12 $\Leftrightarrow$ Definition 14.

Proof. From Lemma 2, $S\times\overline{S'}\times S\leqq S\Leftrightarrow S\times\overline{S'}\leqq S\diamond\overline{S'}$,

$S\times S\leqq S\diamond S\Leftrightarrow S\times S\times\overline{S'}\leqq S\Leftrightarrow S\times\overline{S'}\leqq\overline{S'}\diamond S$.

Therefore $S\times\overline{S'}\times S\leqq S$, $S\times S\leqq S\diamond S$

$\Leftrightarrow S\times\overline{S'}\leqq(S\diamond\overline{S'})\wedge(\overline{S'}\diamond S)$.□

4 Conclusion

Using Boolean matrix theory we have shown that the definitions of semiorders given by Luce (1956, 1959) and Scott and Suppes (1958) are equivalent. Redundancy of a condition in the definition of semiorders given by Luce has been examined. Necessary conditions for properties of semiorders become clear by using Boolean matrices. Moreover generalization of well-known properties of semiorders is possible by Boolean matrix representation. Thus Boolean matrices are useful in the discussion of semiorders.

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