Elementary Proofs of Generalized Properties of Relational Traces Represented by Boolean Matrices

HASHIMOTO, Hiroshi

Abstract

We examine generalized properties of traces of a binary relation using Boolean matrices. Relational traces are relations obtained from a given relation, which are reflexive and transitive. There exist two traces for a relation, the right and left traces. Traces of a given relation can be represented by Boolean matrices. Starting with a well-known fact about relations or Boolean matrices we show some Boolean matrix inequalities and obtain Boolean matrices which represent transitive relations.

1 Introduction

Some properties of relational traces are examined by using Boolean matrices over the two element Boolean algebra [12]. Traces of a relation are binary relations which are associated with it. There exist two traces for a given relation, which are reflexive and transitive. Relational traces appear in various areas of application, and have some interesting properties [4-11, 14, 15]. We start with well-known results on binary relations or Boolean matrices, and show some inequalities related to Boolean matrices. Then using matrix representations of relational traces we construct Boolean matrices which represent transitive relations.

2 Notation and definitions

For $x, y \in \{0,1\}$ we define $x \vee y = \max\ (x, y)$, $x \wedge y = \min\ (x, y)$, and $\bar{x} = 1 - x$. Futhermore we define matrix operations and notation as follows : For Boolean matrices over $\{0,1\}$ $A = [a_{ij}]\ (m \times n)$, $B = [b_{ij}]\ (m \times n)$, $C = [c_{ij}]\ (n$

$\times p)$, $D = [d_{ij}]\ (n \times n)$,

$A \vee B = [a_{ij} \vee b_{ij}]$,

$A \wedge B = [a_{ij} \wedge b_{ij}]$,

$\overline{A} = [\overline{a_{ij}}]$,

$A' = [a_{ji}]$,

$A \times C = [\vee_k (a_{ik} \wedge c_{kj})]$,

$A \diamond C = [\wedge_k (a_{ik} \vee c_{kj})]$,

$D^2 = D \times D$,

$A \leqq B$ if and only if $a_{ij} \leqq b_{ij}$ for all i, j.

As special matrices, we denote the identity matrix by $I = [\delta_{ij}]$ (δ_{ij} is the Kronecker delta), and the zero matrix by O. For a Boolean matrix A, $\overline{A'} \diamond A$ and $A \diamond \overline{A'}$ correspond to the left and right traces of a relation represented by A, respectively [7].

3 Results

We show properties of relational traces of a binary relation using Boolean matrices. First from well-known results on relations or Boolean matrices we obtain some inequalities related to Boolean matrices. Then we construct Boolean matrices which represent transitive relations using relational traces. Throughout this paper we deal with Boolean matrices over {0,1}. We start with the following fact about Boolean matrix inequalities.

Lemma 1 [1-3, 13, 17, 18]. For any Boolean Matrices $A\ (m \times n)$, $B\ (n \times p)$, and $C\ (m \times p)$, the following conditions are equivalent.

(1) $A \times B \leqq C$.

(2) $A \leqq C \diamond \overline{B'}$.

(3) $B \leqq \overline{A'} \diamond C$.

Proposition 1 [18]. For any Boolean matrices $A\ (m \times n)$ and $B\ (p \times n)$, $(A \diamond \overline{B'}) \times B \leqq A$.

Proof. Since $A \diamond \overline{B'} \leqq A \diamond \overline{B'}$, using Lemma 1 we have $(A \diamond \overline{B'}) \times B \leqq A$. $\square$

Proposition 2 [18, 20]. For any Boolean matrices A $(m \times n)$ and B $(p \times n)$, $((A \diamond \overline{B'}) \times B) \diamond \overline{B'} = A \diamond \overline{B'}$.

Proof. Since $(A \diamond \overline{B'}) \times B \leqq (A \diamond \overline{B'}) \times B$, using Lemma 1 we have $A \diamond \overline{B'} \leqq ((A \diamond \overline{B'}) \times B) \diamond \overline{B'}$. Then by Proposition 1, $((A \diamond \overline{B'}) \times B) \diamond \overline{B'} \leqq A \diamond \overline{B'}$. Thus $((A \diamond \overline{B'}) \times B) \diamond \overline{B'} = A \diamond \overline{B'}$. $\square$

Proposition 3. For any Boolean matrices A $(m \times n)$, B $(p \times n)$, and C $(q \times n)$, $(A \diamond \overline{B'}) \times (B \diamond \overline{C'}) \times C \leqq A$.

Proof. By Proposition 1, $(A \diamond \overline{B'}) \times B \leqq A$ and $(B \diamond \overline{C'}) \times C \leqq B$. Therefore $(A \diamond \overline{B'}) \times (B \diamond \overline{C'}) \times C \leqq (A \diamond \overline{B'}) \times B \leqq A$. $\square$

Proposition 4. For any Boolean matrices A $(m \times n)$, B $(p \times n)$, and F $(n \times q)$, $(A \diamond \overline{B'}) \times (B \diamond F) \leqq A \diamond F$.

Proof. By Lemma 1 and Proposition 3 we have the result. $\square$

Proposition 5. For any Boolean matrices A $(m \times n)$, B $(p \times n)$, and F $(n \times n)$, $(A \diamond \overline{B'}) \times (B \wedge (B \diamond F)) \leqq A \wedge (A \diamond F)$.

Proof. By Proposition 1 we have

$$(A \diamond \overline{B'}) \times (B \wedge (B \diamond F)) \leqq (A \diamond \overline{B'}) \times B \leqq A.$$

By Proposition 4 we have

$$(A \diamond \overline{B'}) \times (B \wedge (B \diamond F)) \leqq (A \diamond \overline{B'}) \times (B \diamond F) \leqq A \diamond F.$$

Therefore

$$(A \diamond \overline{B'}) \times (B \wedge (B \diamond F)) \leqq A \wedge (A \diamond F). \quad \square$$

Proposition 6. For any Boolean matrices A $(m \times n)$, B $(p \times n)$, C $(p \times n)$, D $(m \times n)$, E $(p \times n)$, and F $(n \times n)$, if $A \leqq D$, $C \leqq B$, and $E \leqq B$, then

$$(A \diamond \overline{B'}) \times (C \wedge (E \diamond F)) \leqq D \wedge (A \diamond F).$$

Proof. By Proposition 5. $\square$

Proposition 7. For any Boolean matrices A $(m \times n)$, B $(p \times n)$, C $(p \times n)$, D $(m \times n)$, E $(p \times n)$, F $(n \times n)$, and G $(m \times p)$, if $A \leqq D$, $C \leqq B$, and E

$\leqq B$, then

$(G \wedge (A \diamond \overline{B'})) \times (C \wedge (E \diamond F)) \leqq D \wedge (A \diamond F)$.

Proof. By Proposition 6 we have

$(G \wedge (A \diamond \overline{B'})) \times (C \wedge (E \diamond F)) \leqq (A \diamond \overline{B'}) \times (C \wedge (E \diamond F)) \leqq D \wedge (A \diamond F)$. □

Proposition 8. For any $n \times n$ Boolean matrices A, B, and C, if $A \leqq C \leqq B$, then

(1) $(C \wedge (A \diamond \overline{B'}))^2 \leqq C \wedge (A \diamond \overline{B'})$,

(2) $(C \wedge (\overline{B'} \diamond A))^2 \leqq C \wedge (\overline{B'} \diamond A)$.

Proof. (1) By Proposition 7 we have the result.

(2) Since $A' \leqq C' \leqq B'$, using (1) above, we have $(C' \wedge (A' \diamond \overline{B}))^2 \leqq C' \wedge (A' \diamond \overline{B})$. Taking transposes on both sides of the above inequality, we obtain $(C \wedge (\overline{B'} \diamond A))^2 \leqq C \wedge (\overline{B'} \diamond A)$. □

Proposition 9. For any $n \times n$ Boolean matrices A and B, if $A \leqq B$, then

(1) $(A \wedge (A \diamond \overline{B'}))^2 \leqq A \wedge (A \diamond \overline{B'})$,

(2) $(A \wedge (\overline{B'} \diamond A))^2 \leqq A \wedge (\overline{B'} \diamond A)$.

Proof. By Proposition 8. □

Proposition 10. For any $n \times n$ Boolean matrices A and B, if $A \leqq B$, then

(1) $(B \wedge (A \diamond \overline{B'}))^2 \leqq B \wedge (A \diamond \overline{B'})$,

(2) $(B \wedge (\overline{B'} \diamond A))^2 \leqq B \wedge (\overline{B'} \diamond A)$.

Proof. By Proposition 8. □

Proposition 11 [9, 10]. For any $n \times n$ Boolean matrix A,

(1) $(A \wedge (A \diamond \overline{A'}))^2 \leqq A \wedge (A \diamond \overline{A'})$,

(2) $(A \wedge (\overline{A'} \diamond A))^2 \leqq A \wedge (\overline{A'} \diamond A)$.

Proof. By Propositions 9 or 10. □

Proposition 12. For any Boolean matrices A $(m \times n)$, B $(p \times n)$, C $(p \times n)$, and F $(n \times q)$, if $C \leqq B$, then $(A \diamond \overline{B'}) \times (C \diamond F) \leqq A \diamond F$.

Proof. By Proposition 4, we have $(A \diamond \overline{B'}) \times (C \diamond F) \leqq (A \diamond \overline{B'}) \times (B \diamond F)$

$\leqq A \Diamond F$. □

Proposition 13. For any Boolean matrices A $(m \times n)$, B $(m \times n)$, and F $(n \times q)$, if $A \leqq B$, then $(A \Diamond \overline{B'}) \times (A \Diamond F) \leqq A \Diamond F$.

Proof. By Proposition 12. □

Proposition 14. For any $m \times n$ Boolean matrices A and B, if $A \leqq B$, then $(A \Diamond \overline{B'})^2 \leqq A \Diamond \overline{B'}$.

Proof. From Proposition 13 we have the result by setting $F = \overline{B'}$. □

Proposition 15 [16]. For any $m \times n$ Boolean matrix A, $(A \Diamond \overline{A'})^2 = A \Diamond \overline{A'}$.

Proof. By Proposition 14, $(A \Diamond \overline{A'})^2 \leqq A \Diamond \overline{A'}$. Since $I \leqq A \Diamond \overline{A'}$, we have $(A \Diamond \overline{A'})^2 = A \Diamond \overline{A'}$. □

Proposition 16. For any Boolean matrices A $(m \times n)$, F $(n \times q)$,

(1) $(A \Diamond \overline{A'}) \times (A \Diamond F) = A \Diamond F$,

(2) $(A \times \overline{A'}) \Diamond (A \times F) = A \times F$.

Proof. (1) Setting $B = A$ in Proposition 13, we have $(A \Diamond \overline{A'}) \times (A \Diamond F) \leqq A \Diamond F$. Since $I \leqq A \Diamond \overline{A'}$, it follows that $(A \Diamond \overline{A'}) \times (A \Diamond F) = A \Diamond F$.

(2) Replacing A by $\overline{A}$ and F by $\overline{F}$ in (1) above, $(\overline{A} \Diamond A') \times (\overline{A} \Diamond \overline{F}) = \overline{A} \Diamond \overline{F}$. Then taking complements on both sides of the above equation, we get $(A \times \overline{A'}) \Diamond (A \times F) = A \times F$. □

Proposition 17. For any Boolean matrices A $(m \times n)$, F $(q \times m)$,

(1) $(F \Diamond A) \times (\overline{A'} \Diamond A) = F \Diamond A$,

(2) $(F \times A) \Diamond (\overline{A'} \times A) = F \times A$.

Proof. (1) By Proposition 16 (1), $(A' \Diamond \overline{A}) \times (A' \Diamond F') = A' \Diamond F'$. Taking transposes on both sides of the above equation, we get $(F \Diamond A) \times (\overline{A'} \Diamond A) = F \Diamond A$.

(2) Using an argument as in the proof of (1) above we get the result from Proposition 16 (2). □

Proposition 18 [18]. For any $m \times n$ Boolean matrix A,

(1) $(A \Diamond \overline{A'}) \times A = A$,

(2) $(A \times \overline{A'}) \diamond A = A$.

Proof. (1) Putting $F = \overline{I}$ in Proposition 16 (1), since $A \diamond \overline{I} = A$, we have $(A \diamond \overline{A'}) \times A = A$.

(2) Putting $F = I$ in Proposition 16 (2), since $A \times I = A$, we have $(A \times \overline{A'}) \diamond A = A$. □

Proposition 19 [18]. For any $m \times n$ Boolean matrix A,

(1) $A \times (\overline{A'} \diamond A) = A$,

(2) $A \diamond (\overline{A'} \times A) = A$.

Proof. By Proposition 17. □

Proposition 20. For any Boolean matrices A $(n \times n)$ and B $(n \times p)$, if $R = A \wedge (A \diamond \overline{A'}) \wedge (B \diamond \overline{B'})$, then $R^2 \leqq R$.

Proof. By Propositions 11 (1) and 15. □

Proposition 21. For any $n \times n$ Boolean matrix A, if $R = A \wedge (A \diamond \overline{A'}) \wedge (\overline{A} \diamond A')$, then $R^2 \leqq R$.

Proof. By Proposition 20. □

Proposition 22. For any $n \times n$ Boolean matrix A, if $R = \overline{A} \wedge (A \diamond \overline{A'}) \wedge (\overline{A} \diamond A')$, then $R^2 \leqq R$.

Proof. Replacing A by $\overline{A}$ in Proposition 21 we have the result. □

Proposition 23. For any $n \times n$ Boolean matrix A, if $A' = A$ and $R = \overline{A} \wedge (A \diamond \overline{A}) \wedge (\overline{A} \diamond A)$, then $R^2 = R$.

Proof. By Proposition 22 and symmetry of R. □

Proposition 24. For any $n \times n$ Boolean matrix S, if $R = \overline{S} \wedge \overline{S'} \wedge ((S \vee S') \diamond (\overline{S} \wedge \overline{S'})) \wedge ((\overline{S} \wedge \overline{S'}) \diamond (S \vee S'))$, then $R^2 = R$.

Proof. Letting $A = S \vee S'$ in Proposition 23 we have the result. □

Proposition 25 [19]. For any $n \times n$ Boolean matrix S, if $P = S \vee [(\overline{S} \wedge \overline{S'}) \wedge (((\overline{S} \wedge \overline{S'}) \times S) \vee (S \times (\overline{S} \wedge \overline{S'})))]$ and $R = \overline{P} \wedge \overline{P'}$, then $R^2 = R$.

Proof. Obviously

$\overline{P} = \overline{S} \wedge [(S \vee S') \vee (((S \vee S') \diamond \overline{S}) \wedge (\overline{S} \diamond (S \vee S')))]$,

$\overline{P'}=\overline{S'}\wedge[(S\vee S')\vee((\overline{S'}\diamond(S\vee S'))\wedge((S\vee S')\diamond\overline{S'}))]$.

Then

$R=\overline{P}\wedge\overline{P'}$

$=\overline{S}\wedge\overline{S'}\wedge[(S\vee S')\vee(((S\vee S')\diamond\overline{S})\wedge(\overline{S}\diamond(S\vee S')))]\wedge[(S\vee S')\vee((\overline{S'}\diamond(S\vee S'))\wedge((S\vee S')\diamond\overline{S'}))]$

$=\overline{S}\wedge\overline{S'}\wedge((S\vee S')\diamond\overline{S})\wedge(\overline{S}\diamond(S\vee S'))\wedge(\overline{S'}\diamond(S\vee S'))\wedge((S\vee S')\diamond\overline{S'})$

$=\overline{S}\wedge\overline{S'}\wedge((S\vee S')\diamond\overline{S})\wedge((S\vee S')\diamond\overline{S'})\wedge(\overline{S}\diamond(S\vee S'))\wedge(\overline{S'}\diamond(S\vee S'))$

$=\overline{S}\wedge\overline{S'}\wedge((S\vee S')\diamond(\overline{S}\wedge\overline{S'}))\wedge((\overline{S}\wedge\overline{S'})\diamond(S\vee S'))$.

Thus by Proposition 24 we have $R^2=R$. □

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