Some Generalized Properties of Preference Relations

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Abstract

By using Boolean matrices, well-known properties of preference relations are extended and the necessary conditions for the properties are clarified. In particular, properties of the asymmetric part and the symmetric part of a relation are examined. These properties are related to quasi transitivity.

Keywords : Preference relation; Quasi transitivity

1 Introduction

We generalize well-known properties of preference relations and clarify the necessary conditions for the properties. In particular, properties of the asymmetric part and the symmetric part of a relation are examined. These properties are related to quasi transitivity. In the following, binary relations are represented by Boolean matrices.

2 Definitions and results

Let $R=[r_{ij}]$, $S=[s_{ij}]$, and $T=[t_{ij}]$ be $n\times n$ Boolean matrices over $\{0, 1\}$. We define operations as follows.

$R\vee S=[r_{ij}\vee s_{ij}]=[\max(r_{ij}, s_{ij})]$

$R\wedge S=[r_{ij}\wedge s_{ij}]=[\min(r_{ij}, s_{ij})]$

$R\times S=[(r_{i1}\wedge s_{1j})\vee\cdots\vee(r_{in}\wedge s_{nj})]$

$\overline{R}=[\overline{r_{ij}}]=[1-r_{ij}]$

$R' = [r_{ji}]$ (transpose)

We denote the unit matrix by $I=[\delta_{ij}]$ (δ_{ij} is the Kronecker delta), the zero matrix by O, and the universal matrix, in which all elements are one, by J.

Proposition 1. $R \vee R' \vee S = J,\ R \wedge \overline{R'} \wedge S = O \Leftrightarrow R \wedge \overline{R'} = \overline{R'} \wedge \overline{S}$

Proof. The Proof is immediate.□

Proposition 2. If $R \vee R' \vee S = J,\ R \wedge \overline{R'} \wedge S = O$, then $(R \wedge \overline{R'}) \times R \leqq \overline{T'} \Leftrightarrow R \times T \leqq R \vee S'$

Proof. Since $(\overline{R'} \wedge \overline{S}) \times R \leqq \overline{T'} \Leftrightarrow R \times T \leqq R \vee S'$, we obtain the result by Proposition 1.□

Proposition 3. If $R \vee R' \vee S = J,\ R \wedge \overline{R'} \wedge S = O$, then $R \times (R \wedge \overline{R'}) \leqq \overline{T'} \Leftrightarrow T \times R \leqq R \vee S'$

Proof. The proof is similar to that of Proposition 2.□

Proposition 4. If $R \vee R' \vee I = J$, then $(R \wedge \overline{R'}) \times R \leqq \overline{R'} \Leftrightarrow R \times R \leqq R \vee I$

Proof. This proposition follows immediately from Proposition 2.□

Proposition 5. If $R \vee R' = J$, then $(R \wedge \overline{R'}) \times R \leqq \overline{R'} \Leftrightarrow R \times R \leqq R$

Proof. This proposition follows immediately from Proposition 4.□

Proposition 6. If $R \vee R' = J$, then $(R \wedge \overline{R'}) \times R \leqq R \wedge \overline{R'} \Leftrightarrow R \times R \leqq R$

Proof. Since $R \vee R' = J \Leftrightarrow \overline{R'} = R \wedge \overline{R'}$, we have the result by Proposition 5.□

Proposition 7. $(R \wedge \overline{R'}) \times R \leqq R \wedge \overline{R'} \Leftrightarrow (R \wedge \overline{R'}) \times (R \wedge \overline{R'}) \leqq R \wedge \overline{R'},\ (R \wedge \overline{R'}) \times (R \wedge R') \leqq R \wedge \overline{R'}$

Proof. Since $R = (R \wedge \overline{R'}) \vee (R \wedge R')$, the proof is immediate.□

Proposition 8 (Sen, 1969; Sen, 1970). $R \vee R' = J,\ (R \wedge \overline{R'}) \times (R \wedge \overline{R'}) \leqq R \wedge \overline{R'},\ (R \wedge \overline{R'}) \times (R \wedge R') \leqq R \wedge \overline{R'} \Rightarrow R \times R \leqq R$

Proof. This proposition follows immediately from Propositions 6 and 7.□

Proposition 9. $R \vee R' \vee I = J,\ (R \wedge \overline{R'}) \times (R \wedge R') \leqq \overline{R'} \Rightarrow (R \wedge R') \times (R \wedge \overline{R'}) \leqq R \wedge \overline{R'}$

Proof. Suppose that $r_{ik} \wedge r_{ki} \wedge r_{kj} \wedge \overline{r_{jk}} = 1$. Then $i \neq j$. It will be shown that

$r_{ij} \wedge \overline{r_{ji}} = 1$. Assume, by way of contradiction, $r_{ij} \wedge \overline{r_{ji}} = 0$.

Case 1. $r_{ij} = 0$. Then $r_{ji} = 1$. Since $r_{ji} \wedge \overline{r_{ij}} \wedge r_{ik} \wedge r_{ki} = 1$, we have $\overline{r_{kj}} = 1$, which is a contradiction.

Case 2. $r_{ji} = 1$, $r_{ij} = 1$. Since $r_{kj} \wedge \overline{r_{jk}} \wedge r_{ji} \wedge r_{ij} = 1$, we have $\overline{r_{ik}} = 1$, which is a contradiction.□

Proposition 10. $R \vee R' \vee I = J$, $(R \wedge R') \times (R \wedge \overline{R'}) \leqq \overline{R'} \Rightarrow (R \wedge \overline{R'}) \times (R \wedge R') \leqq R \wedge \overline{R'}$

Proof. The proof is similar to that of Proposition 9.□

Proposition 11. $(R \wedge \overline{R'}) \times (R \wedge R') \leqq \overline{R'} \Leftrightarrow (R \wedge R') \times (R \wedge \overline{R'}) \leqq \overline{R'}$

Proof. $(\Rightarrow)$. Suppose that $r_{ik} \wedge r_{ki} \wedge r_{kj} \wedge \overline{r_{jk}} = 1$. It will be shown that $r_{ji} = 0$. Assume, by way of contradiction, $r_{ji} = 1$.

Case 1. $r_{ij} = 0$. Since $r_{ji} \wedge \overline{r_{ij}} \wedge r_{ik} \wedge r_{ki} = 1$, we have $\overline{r_{kj}} = 1$, which is a contradiction.

Case 2. $r_{ij} = 1$. Since $r_{kj} \wedge \overline{r_{jk}} \wedge r_{ji} \wedge r_{ij} = 1$, we have $\overline{r_{ik}} = 1$, which is a contradiction.

Thus we have $(R \wedge R') \times (R \wedge \overline{R'}) \leqq \overline{R'}$

$(\Leftarrow)$. By the same arguments used in $(\Rightarrow)$, we have $(R \wedge \overline{R'}) \times (R \wedge R') \leqq \overline{R'}$.□

Proposition 12. If $R \vee R' \vee I = J$, then the following are equivalent.

(1) $(R \wedge \overline{R'}) \times (R \wedge R') \leqq \overline{R'}$

(2) $(R \wedge \overline{R'}) \times (R \wedge R') \leqq R \wedge \overline{R'}$

(3) $(R \wedge R') \times (R \wedge \overline{R'}) \leqq R \wedge \overline{R'}$

Proof. This proposition follows immediately from Propositions 9 and 10.□

Proposition 13 (Sonnenschein, 1965; Lorimer, 1967; Sen, 1970). If $R \vee R' \vee I = J$, then $(R \wedge \overline{R'}) \times (R \wedge R') \leqq R \wedge \overline{R'} \Leftrightarrow (R \wedge R') \times (R \wedge \overline{R'}) \leqq R \wedge \overline{R'}$

Proof. This proposition follows immediately from Proposition 12.□

Proposition 14. $R \vee R' \vee I = J$, $(R \wedge \overline{R'}) \times (R \wedge R') \leqq \overline{R'} \Rightarrow (R \wedge R') \times (R \wedge R') \leqq (R \wedge R') \vee I$

Proof. Suppose that $r_{ik} \wedge r_{ki} \wedge r_{kj} \wedge r_{jk} = 1$. It will be shown that $(r_{ij} \wedge r_{ji}) \vee$

$\delta_{ij}=1$. For $i=j$, the proof is trivial. We consider the case $i \neq j$. Assume, by way of contradiction, $r_{ij} \wedge r_{ji}=0$.

Case 1. $r_{ij}=0$. Then $r_{ji}=1$. Since $r_{ji} \wedge \overline{r_{ij}} \wedge r_{ik} \wedge r_{ki}=1$, we have $\overline{r_{kj}}=1$, which is a contradiction.

Case 2. $r_{ji}=0$. Then $r_{ij}=1$. Since $r_{ij} \wedge \overline{r_{ji}} \wedge r_{jk} \wedge r_{kj}=1$, we have $\overline{r_{ki}}=1$, which is a contradiction.□

Proposition 15. $R \vee R'=J,\ (R \wedge \overline{R'}) \times (R \wedge R') \leqq \overline{R'} \Rightarrow (R \wedge R') \times (R \wedge R')= R \wedge R'$

Proof. This proposition follows immediately from Proposition 14.□

Proposition 16 (Sen, 1969; Sen, 1970). $R \vee R'=J,\ (R \wedge \overline{R'}) \times (R \wedge R') \leqq R \wedge \overline{R'} \Rightarrow (R \wedge R') \times (R \wedge R')=R \wedge R'$

Proof. This proposition follows immediately from Proposition 15.□

Proposition 17. $R \vee R' \vee I=J,\ (R \wedge \overline{R'}) \times (R \wedge \overline{R'}) \leqq \overline{R'},\ (R \wedge R') \times (R \wedge R') \leqq R \Rightarrow R \times R \leqq R$

Proof. Suppose that $r_{ik} \wedge r_{kj}=1$. It will be shown that $r_{ij}=1$. If $i=k$ or $k=j$, then $r_{ij}=1$. We consider the case where $i \neq k$ and $k \neq j$. Assume, by way of contradiction, $r_{ij}=0$.

Case 1. $i \neq j$. Then $r_{ji}=1$.

Subcase 1.1. $r_{ki}=0$. Since $r_{ji} \wedge \overline{r_{ij}} \wedge r_{ik} \wedge \overline{r_{ki}}=1$, we have $\overline{r_{kj}}=1$, which is a contradiction.

Subcase 1.2. $r_{jk}=0$. Since $r_{kj} \wedge \overline{r_{jk}} \wedge r_{ji} \wedge \overline{r_{ij}}=1$, we have $\overline{r_{ik}}=1$, which is a contradiction.

Subcase 1.3. $r_{ki}=1$ and $r_{jk}=1$. Since $r_{ik} \wedge r_{ki} \wedge r_{kj} \wedge r_{jk}=1$, we have $r_{ij}=1$, which is a contradiction.

Case 2. $i=j$. Then $r_{ik} \wedge r_{ki}=1$, $r_{ii}=0$. Since $(r_{ik} \wedge r_{ki}) \wedge (r_{ki} \wedge r_{ik})=1$, we have $r_{ii}=1$, which is a contradiction.□

Proposition 18 (Sen, 1969; Sen, 1970; Roubens and Vincke, 1985). $R \vee R' \vee I=J,\ (R \wedge \overline{R'}) \times (R \wedge \overline{R'}) \leqq R \wedge \overline{R'},\ (R \wedge R') \times (R \wedge R') \leqq R \wedge R' \Rightarrow R \times R \leqq R$

Proof. This proposition follows immediately from Proposition 17.□

Proposition 19 (Sen, 1969; Sen, 1970). $R \vee R' \vee I = J$, $(R \wedge \overline{R'}) \times (R \wedge \overline{R'}) \leqq R \wedge \overline{R'}$, $(R \wedge R') \times (R \wedge R') \leqq R \wedge R' \Rightarrow (R \wedge \overline{R'}) \times (R \wedge R') \leqq R \wedge \overline{R'}$

Proof. Since $R \times R \leqq R \Rightarrow (R \wedge \overline{R'}) \times (R \wedge R') \leqq R \wedge \overline{R'}$, we have the result by Proposition 18.□

Proposition 20. If $R \vee S' \vee I = J$ and $T \leqq R$, then $(S \wedge \overline{R'}) \times (S \wedge \overline{R'}) \leqq \overline{T'} \Leftrightarrow (S \wedge \overline{R'}) \times T \leqq R$

Proof. ($\Rightarrow$). Suppose that $(s_{ik} \wedge \overline{r_{ki}}) \wedge t_{kj} = 1$. Then $i \neq j$. It will be shown that $r_{ij} = 1$. Assume, by way of contradiction, $r_{ij} = 0$. We have $s_{ji} = 1$. Since $s_{ji} \wedge \overline{r_{ij}} \wedge s_{ik} \wedge \overline{r_{ki}} = 1$, we have $\overline{t_{kj}} = 1$, which is a contradiction.

($\Leftarrow$). Suppose that $s_{ik} \wedge \overline{r_{ki}} \wedge s_{kj} \wedge \overline{r_{jk}} = 1$. It will be shown that $t_{ji} = 0$. Assume, by way of contradiction, $t_{ji} = 1$. Since $s_{kj} \wedge \overline{r_{jk}} \wedge t_{ji} = 1$, we have $r_{ki} = 1$, which is a contradiction.□

Proposition 21. If $R \vee S' \vee I = J$ and $T \leqq R$, then $(S \wedge \overline{R'}) \times (S \wedge \overline{R'}) \leqq \overline{T'} \Leftrightarrow T \times (S \wedge \overline{R'}) \leqq R$

Proof. The proof is similar to that of Proposition 20.□

Proposition 22. If $R \vee R' \vee I = J$, then $(R \wedge \overline{R'}) \times (R \wedge \overline{R'}) \leqq \overline{R'} \Leftrightarrow (R \wedge \overline{R'}) \times R \leqq R$

Proof. This proposition follows immediately from Proposition 20.□

Proposition 23. If $R \vee R' \vee I = J$, then $(R \wedge \overline{R'}) \times (R \wedge \overline{R'}) \leqq \overline{R'} \Leftrightarrow R \times (R \wedge \overline{R'}) \leqq R$

Proof. This proposition follows immediately from Proposition 21.□

Proposition 24. $(S \wedge \overline{R'}) \times S \leqq R \Rightarrow (S \wedge \overline{R'}) \times (S \wedge \overline{R'}) \leqq R \wedge \overline{S'}$

Proof. Suppose that $s_{ik} \wedge \overline{r_{ki}} \wedge s_{kj} \wedge \overline{r_{jk}} = 1$. It will be shown that $r_{ij} \wedge \overline{s_{ji}} = 1$. Since $s_{ik} \wedge \overline{r_{ki}} \wedge s_{kj} = 1$, we have $r_{ij} = 1$. Assume, by way of contradiction, $s_{ji} = 1$. Since $s_{kj} \wedge \overline{r_{jk}} \wedge s_{ji} = 1$, we have $r_{ki} = 1$, which is a contradiction.□

Proposition 25. $S \times (S \wedge \overline{R'}) \leqq R \Rightarrow (S \wedge \overline{R'}) \times (S \wedge \overline{R'}) \leqq R \wedge \overline{S'}$

Proof. The proof is similar to that of Proposition 24.□

Proposition 26. $(R \wedge \overline{R'}) \times R \leqq R \Rightarrow (R \wedge \overline{R'}) \times (R \wedge \overline{R'}) \leqq R \wedge \overline{R'}$

Proof. This proposition follows immediately from Proposition 24.□

Proposition 27. $R \times (R \wedge \overline{R'}) \leqq R \Rightarrow (R \wedge \overline{R'}) \times (R \wedge \overline{R'}) \leqq R \wedge \overline{R'}$

Proof. This proposition follows immediately from Proposition 25.□

Proposition 28. If $R \vee R' \vee I = J$, then the following are equivalent.

(1) $(R \wedge \overline{R'}) \times (R \wedge \overline{R'}) \leqq R \wedge \overline{R'}$

(2) $(R \wedge \overline{R'}) \times (R \wedge \overline{R'}) \leqq \overline{R'}$

(3) $(R \wedge \overline{R'}) \times R \leqq R$

(4) $R \times (R \wedge \overline{R'}) \leqq R$

Proof. This proposition follows immediately from Propositions 22, 23, 26, and 27.□

Proposition 29 (Sen, 1969). If $R \vee R' \vee I = J$, then $(R \wedge \overline{R'}) \times (R \wedge \overline{R'}) \leqq R \wedge \overline{R'} \Leftrightarrow (R \wedge \overline{R'}) \times R \leqq R$

Proof. This proposition follows immediately from Proposition 28.□

References

Lorimer, P., 1967. A note on orderings, Econometrica 35, 537-539.

Roubens, M., Vincke, Ph., 1985. Preference Modelling (Springer-Verlag, Berlin).

Sen, A., 1969. Quasi-transitivity, rational choice and collective decisions, Review of Economic Studies 36, 381-393.

Sen, A. K., 1970. Collective choice and social welfare (Holden-Day, San Francisco).

Sonnenschein, H., 1965. The relationship between transitive preference and the structure of the choice space, Econometrica 33, 624-634.