

INTEGRATION OF CHANDRASEKHAR'S INTEGRAL EQUATION

Tasuku Tanaka¹

*Earth Observation Research Center, National Space Development Agency of
Japan, 1-8-10 Harumi, Chuou-ku, Tokyo, 104-6023, Japan
e-mail:tanaka@eorc.nasda.go.jp*

Received 7 December 2001

Abstract

We solve Chandrasekhar's integration equation for radiative transfer in the plane-parallel atmosphere by iterative integration. The primary thrust in radiative transfer has been to solve the forward problem, i.e., to evaluate the radiance, given the optical thickness and the scattering phase function. In the area of satellite remote sensing, our problem is the inverse problem: to retrieve the surface reflectance and the optical thickness of the atmosphere from the radiance measured by satellites. In order to retrieve the optical thickness and the surface reflectance from the radiance at the top-of-the-atmosphere (TOA), we should express the radiance at TOA "explicitly" in the optical thickness and the surface reflectance. Chandrasekhar formalized radiative transfer in the plane-parallel atmosphere in a simultaneous integral equation, and he obtained the second approximation. Since then no higher approximation has been reported. In this paper we obtain the third approximation of the scattering function. We integrate functions derived from the second approximation in the integral interval from 1 to ∞ of the inverse of the *cos* of zenith angles. We can obtain the indefinite integral rather easily in the form of a series expansion. However the integrals at the upper limit, ∞ , are not yet known to us. We can assess the converged values of those series expansions at ∞ through calculus. For integration we choose coupling pairs to avoid unnecessary terms in the outcome of integral and discover that the simultaneous integral equation can be deduced to the mere integral equation. Through algebraic calculation, we obtain the third approximation as a polynomial of the third degree in the atmospheric optical thickness.

¹ Temporary address: c/o SIMBIOS Project, code 970.2, Goddard Space Flight Center, NASA, Greenbelt, Md. 20771, USA. E-mail:tanaka@simbios.gsfc.nasa.gov
Tel:301-286-5624

1 Introduction

Radiative transfer in a plane parallel atmosphere has been a major scientific and mathematical subject for many years. Chandrasekhar's historical work¹ in 1960 formulated the radiative transfer process and derived the simultaneous integral equation for the scattering function and transmitted functions.

Since then, there have been various methods to calculate scattered radiance at the top of the atmosphere. Hansen² and Gaudy³ summarized the pros and cons of those calculating schemes. The primary thrust for those method has been to solve the forward problem, that is, to evaluate the radiance at the top of atmosphere, given an optical thickness and a surface reflectance. In the forward problem we need algorithms, not an explicit expression of radiance.

From a space remote-sensing viewpoint, our concern is the inverse problem, i.e., to retrieve both the optical thickness and the surface reflectance from satellite observed data. The conventional method is to make a look-up table prepared with calculations of the forward problem and to interpolate the solutions using the look-up table.

Since the launch of the Coastal Zone Color Scanner(CZCS), aboard Nimbus-7, in 1987, many algorithms in the visible and near-infrared bands have been developed to retrieve chlorophyll concentration in the ocean⁴. Nearly ninety percent of satellite observed radiance comes from light scattered by the atmosphere and from the remaining ten percent comes from the ocean, from which chlorophyll concentration is retrieved.

Hence the atmospheric correction is the essential part of the retrieval algorithm for ocean color. The observed target is measured from only one direction in current satellite observations. From this single observed data we must retrieve two parameters: surface reflectance and atmospheric optical thickness.

Recently, multi-directional observation data from Polarization and Directionality of the Earth's Reflectances (POLDER)⁵ or Multi-angle Imaging Spectro Radiometer(MISR) have become available. Using these data we can, in principle retrieve the two unknown parameters i.e., the optical thickness and surface reflectance, in a single band from multi-directional observations. If we solve this problem with look-up tables and interpolations, the look-up tables might be enormously large.

So we have to seek the analytical solution, or explicit expressions of radiance at the top of the atmosphere in terms of both optical thickness and surface reflectance.

Chandrasekhar gave the second approximation to the simultaneous integral

equation in an isotropic, plane-parallel atmosphere. Since then no further work has been provided on higher-order approximations.

In this paper, as the first step, we solve the simultaneous integral equation in an isotropic, plane-parallel atmosphere up to the third order approximation. The solution entails iterative analytical integration (not numerical integration) of the scattered and transmitted functions on a unit semi-sphere surface. The scattered and transmitted functions are then expanded in a series expansion of the optical thickness.

Chandrasekhar's Simultaneous Integral Equation is introduced, and the iteration scheme for the scattering and transmitted functions are given in section two. The peculiar characteristics of integration in radiative transfer are discussed in section three. The second and third approximations are given in section four, and the conclusion is given in section five.

2 Chandrasekhar's Integral Equation

2.1 Scattering and Transmission Function

Radiance $I(0, i_1)$, emerging from the top of the atmosphere (TOA), in the direction i_1 is expressed below.

$$I(0, i_1) = \frac{1}{4\pi \cos \theta_1} \int S(\tau, i_1, i_0) I(0, i_0) d\Omega_0. \quad (1)$$

Here $I(0, i_0)$ is the incident radiance to TOA in the direction i_0 , 0 is the vertical coordinate of the TOA, θ_1 is the zenith angle of the direction of the scattered intensity, Ω_0 is the solid angle subtended by the incident radiance, τ is the optical thickness of the layer, $S(\tau, i_1, i_0)$ is the scattering function, and the integral domain is an upper half of a unit sphere. In the above equation, it is assumed that no radiance is given from the bottom. In the same manner, the radiance transmitted to the bottom of the atmosphere, $I(\tau, i_4)$ in the direction i_4 is expressed as

$$I(\tau, i_4) = \frac{1}{4\pi \cos \theta_4} \int T(\tau, i_4, i_0) I(0, i_0) d\Omega_0 + \exp\left(-\frac{\tau}{\cos \theta_4}\right) I(0, i_4). \quad (2)$$

here θ_2 is the zenith angle of the direction of the transmitted radiance, and $T(\tau, i_5, i_0)$ is the transmitted function. The scattering function $S(\tau, i_1, i_0)$ and transmitted function $T(\tau, i_4, i_0)$ satisfy a set of simultaneous integral equations that were first introduced by Chandrasekhar¹.

$$\begin{aligned}
S(\tau, i_1, i_0) &= \mu_{10} [1 - \exp(-\frac{\tau}{\mu_{10}})] P(i_1, i_0) \\
&+ \mu_{10} \int P(i_1, i_3) S(\tau, i_3, i_0) \frac{d\Omega_3}{4\pi\mu_3} - \mu_{10} e^{-\frac{\tau}{\mu_0}} \int T(\tau, i_1, i_3) P(i_3, i_0) \frac{d\Omega_3}{4\pi\mu_3} \\
&+ \mu_{10} \int S(\tau, i_1, i_2) P(i_2, i_0) \frac{d\Omega_2}{4\pi\mu_2} - \mu_{10} e^{-\frac{\tau}{\mu_1}} \int P(i_1, i_2) T(\tau, i_2, i_0) \frac{d\Omega_2}{4\pi\mu_2} \\
&+ \mu_{10} \int \int S(\tau, i_1, i_2) P(i_2, i_3) S(\tau, i_3, i_0) \frac{d\Omega_2}{4\pi\mu_2} \frac{d\Omega_3}{4\pi\mu_3} \\
&- \mu_{10} \int \int T(\tau, i_1, i_3) P(i_3, i_2) T(\tau, i_2, i_0) \frac{d\Omega_2}{4\pi\mu_2} \frac{d\Omega_3}{4\pi\mu_3} \tag{3}
\end{aligned}$$

$$\begin{aligned}
T(\tau, i_4, i_0) &= \bar{\mu}_{40} [\exp(-\frac{\tau}{\mu_0}) - \exp(-\frac{\tau}{\mu_4})] P(i_4, i_0) \\
&+ \bar{\mu}_{40} \int P(i_4, i_2) T(\tau, i_2, i_0) \frac{d\Omega_2}{4\pi\mu_2} - \bar{\mu}_{40} e^{-\frac{\tau}{\mu_4}} \int P(i_4, i_3) S(\tau, i_3, i_0) \frac{d\Omega_3}{4\pi\mu_3} \\
&+ \bar{\mu}_{40} e^{-\frac{\tau}{\mu_0}} \int S(\tau, i_4, i_3) P(i_3, i_0) \frac{d\Omega_3}{4\pi\mu_3} - \bar{\mu}_{40} \int T(\tau, i_4, i_2) P(i_2, i_0) \frac{d\Omega_2}{4\pi\mu_2} \\
&+ \bar{\mu}_{40} \int \int S(\tau, i_4, i_3) P(i_3, i_2) T(\tau, i_2, i_0) \frac{d\Omega_2}{4\pi\mu_2} \frac{d\Omega_3}{4\pi\mu_3} \\
&- \bar{\mu}_{40} \int \int T(\tau, i_4, i_2) P(i_2, i_3) S(\tau, i_3, i_0) \frac{d\Omega_2}{4\pi\mu_2} \frac{d\Omega_3}{4\pi\mu_3} \tag{4}
\end{aligned}$$

Here, $\frac{1}{\mu_{10}} = \frac{1}{\mu_1} + \frac{1}{\mu_0}$ and $\frac{1}{\bar{\mu}_{40}} = \frac{1}{\mu_4} - \frac{1}{\mu_0}$ and μ_n is cos of the zenith angle of the direction i_n .

2.2 Iteration Scheme and Approximation

The simultaneous integral equations of scattering and transmission can be solved by successive iteration. The first iterations are the first terms on the right-hand side in equations (3) and (4), expressed below.

$$S_1(\tau, i_1, i_0) = \mu_{10} (1 - \exp(-\frac{\tau}{\mu_{10}})) P(i_1, i_0) \tag{5}$$

$$T_1(\tau, i_4, i_0) = \bar{\mu}_{40} [\exp(-\frac{\tau}{\mu_0}) - \exp(-\frac{\tau}{\mu_4})] P(i_4, i_0) \tag{6}$$

These first iterations become the first approximation for the scattering and transmitted functions.

Inserting the first iterations into the integrals in the original integral equations, the second approximations, $S_2(\tau, i_1, i_0)$ and $T_2(\tau, i_4, i_0)$ are obtained.

$$\begin{aligned}
S_2(\tau, i_1, i_0) &= S_1(\tau, i_1, i_0) \\
&+ \mu_{10} \int P(i_1, i_3) S_1(\tau, i_3, i_0) \frac{d\Omega_3}{4\pi\mu_3} - \mu_{10} e^{-\frac{\tau}{\mu_0}} \int T_1(\tau, i_1, i_3) P(i_3, i_0) \frac{d\Omega_3}{4\pi\mu_3} \\
&+ \mu_{10} \int S_1(\tau, i_1, i_2) P(i_2, i_0) \frac{d\Omega_2}{4\pi\mu_2} - \mu_{10} e^{-\frac{\tau}{\mu_1}} \int P(i_1, i_2) T_1(\tau, i_2, i_0) \frac{d\Omega_2}{4\pi\mu_2} \\
&+ \mu_{10} \int \int S_1(\tau, i_1, i_2) P(i_2, i_3) S_1(\tau, i_3, i_0) \frac{d\Omega_2}{4\pi\mu_2} \frac{d\Omega_3}{4\pi\mu_3} \\
&- \mu_{10} \int \int T_1(\tau, i_1, i_3) P(i_3, i_2) T_1(\tau, i_2, i_0) \frac{d\Omega_2}{4\pi\mu_2} \frac{d\Omega_3}{4\pi\mu_3} \\
&= S_1(\tau, i_1, i_0) + \Delta S_2(\tau, i_1, i_0)
\end{aligned} \tag{7}$$

$$\begin{aligned}
T_2(\tau, i_4, i_0) &= T_1(\tau, i_4, i_0) + \\
&\mu_{40} \int P(i_4, i_2) T_1(\tau, i_2, i_0) \frac{d\Omega_2}{4\pi\mu_2} - \mu_{40} e^{-\frac{\tau}{\mu_4}} \int P(i_4, i_3) S_1(\tau, i_3, i_0) \frac{d\Omega_3}{4\pi\mu_3} \\
&+ \mu_{40} e^{-\frac{\tau}{\mu_0}} \int S_1(\tau, i_4, i_3) P(i_3, i_0) \frac{d\Omega_3}{4\pi\mu_3} - \mu_{40} \int T_1(\tau, i_4, i_2) P(i_2, i_0) \frac{d\Omega_2}{4\pi\mu_2} \\
&+ \mu_{40} \int \int S_1(\tau, i_4, i_3) P(i_3, i_2) T_1(\tau, i_2, i_0) \frac{d\Omega_2}{4\pi\mu_2} \frac{d\Omega_3}{4\pi\mu_3} \\
&- \mu_{40} \int \int T_1(\tau, i_4, i_2) P(i_2, i_3) S_1(\tau, i_3, i_0) \frac{d\Omega_2}{4\pi\mu_2} \frac{d\Omega_3}{4\pi\mu_3} \\
&= T_1(\tau, i_4, i_0) + \Delta T_2(\tau, i_4, i_0)
\end{aligned} \tag{8}$$

Here $\Delta S_2(\tau, i_1, i_0)$ and $\Delta T_2(\tau, i_1, i_0)$ are second iterations for the scattering and transmitted functions respectively.

We assume isotropic scattering or $P(i_1, i_0) = 1$ and introduce two intermediate functions $U_2(\tau, i_1)$ and $V_2(\tau, i_4)$ defined below.

$$U_2(\tau, i_1) = \int_0^1 S_1(\tau, i_3, i_1) \frac{d\mu_3}{\mu_3} = \int_0^1 S_1(\tau, i_1, i_2) \frac{d\mu_2}{\mu_2} \tag{9}$$

$$V_2(\tau, i_4) = \int_0^1 T_1(\tau, i_2, i_4) \frac{d\mu_2}{\mu_2} = \int_0^1 T_1(\tau, i_4, i_3) \frac{d\mu_3}{\mu_3} \tag{10}$$

Using $U_2(\tau, i_1)$ and $V_2(\tau, i_4)$, the second iterations are given below.

$$\begin{aligned}
\Delta S_2(\tau, i_1, i_0) &= \frac{\mu_{10}}{2} [U_2(\tau, i_0) - \exp(-\frac{\tau}{\mu_0}) V_2(\tau, i_1) + U_2(\tau, i_1) \\
&- \exp(-\frac{\tau}{\mu_1}) V_2(\tau, i_0) + U_2(\tau, i_0) U_2(\tau, i_1) - V_2(\tau, i_0) V_2(\tau, i_1)]
\end{aligned} \tag{11}$$

$$\begin{aligned} \Delta T_2(\tau, i_4, i_0) &= \frac{\mu_{40}}{2} [V_2(\tau, i_0) - V_2(\tau, i_4) + \exp(-\frac{\tau}{\mu_0})U(\tau, i_4) \\ &- \exp(-\frac{\tau}{\mu_4})U_2(\tau, i_0) + U_2(\tau, i_4)V_2(\tau, i_0) - V_2(\tau, i_4)U_2(\tau, i_0)] \end{aligned} \quad (12)$$

For the higher iterations, replacing 2 by n in the above equations, $\Delta S_n(\tau, i_1, i_0)$ and $\Delta T_n(\tau, i_4, i_0)$ are expressed by $U_n(\tau, i_0)$ and $V_n(\tau, i_0)$.

The functions $U_n(\tau, i_0)$ and $V_n(\tau, i_0)$ satisfy the recurrence relationships of the integral expressed below.

$$\begin{aligned} U_{n+1}(\tau, i_6) &= \int_0^1 [U_n(\tau, i_6) - \exp(-\frac{\tau}{\mu_6})V_n(\tau, i) + U_n(\tau, i) \\ &- \exp(-\frac{\tau}{\mu})V_n(\tau, i_6) + U_n(\tau, i_6)U_n(\tau, i) - V_n(\tau, i_6)V_n(\tau, i)] \frac{\mu_6}{\mu_6 + \mu} d\mu \end{aligned} \quad (13)$$

$$\begin{aligned} V_{n+1}(\tau, i_7) &= \int_0^1 [V_n(\tau, i_7) - V_n(\tau, i) + \exp(-\frac{\tau}{\mu_7})U_n(\tau, i) \\ &- \exp(-\frac{\tau}{\mu})U_n(\tau, i_7) + U_n(\tau, i)V_2(\tau, i_7) - V_n(\tau, i)U_n(\tau, i_7)] \frac{\mu_7}{\mu_7 - \mu} d\mu \end{aligned} \quad (14)$$

Owing to the iteration scheme, we obtain the iterations, $\Delta S_n(\tau, i_1, i_0)$. The scattering function $S(\tau, i_1, i_0)$ is approximated by the series of $\Delta S_n(\tau, i_1, i_0)$.

$$S(\tau, i_1, i_0) = S_1(\tau, i_1, i_0) + \Delta S_2(\tau, i_1, i_0) + \Delta S_3(\tau, i_1, i_0) + \dots \quad (15)$$

In this paper, we approximate the scattering function by the third approximation $S_3(\tau, i_1, i_0)$. Expanding the approximated scattering function into a series expansion in τ and truncating them up to the third degree, we obtain algebraic equations with τ as a variable. The truncated first approximation is given below.

$$S_1(\tau, i_1, i_0) = \tau - \left(\frac{1}{\mu_1} + \frac{1}{\mu_0}\right) \frac{\tau^2}{2!} + \left(\frac{1}{\mu_1} + \frac{1}{\mu_0}\right)^2 \frac{\tau^3}{3!} \quad (16)$$

The first term of the first approximation corresponds to the single scattering function. In the section four we derive the truncated second and third approximations for the scattering function.

In the following, the variable μ_i , i.e., \cos of zenith angle θ_i is replaced by $p_i = 1/\mu_i$.

3 Consideration on Integration

3.1 Form of Integration

The integration that we perform to solve Chandrasekhar's integration equation has the form given by,

$$F(\tau, p_8, p_9) = \int_1^{\infty} \frac{f(\tau p + \tau p_8)}{p(p + p_9)} dp. \quad (17)$$

The function $f(p)$ is expressed as a power series expansion in p as below.

$$f(p) = \sum_{n=1}^{\infty} \frac{(-1)^n a_n p^n}{n!} \quad (18)$$

If the integrand in equation (17) is $O(1/p^\beta)$ ($\beta > 1$) as $p = \infty$, the integration converges. All the functions in the second and third iterations satisfy this condition. Hence they converge. To evaluate the integration for $p_9 \neq 0$ we must decompose the equation (18) into partial fractions. The integrations of decomposed partial fractions shown below do not necessarily converge by themselves because the degrees in the denominators become 1 and they no more satisfy the condition for convergence of integration mentioned above. For these cases we must insert the identical upper limit into both integrals and make it approach ∞ synchronously.

$$F(\tau, p_8, p_9) = \lim_{p \rightarrow \infty} \frac{1}{p_9} \left[\int_1^p f(\tau p + \tau p_8) \frac{dp}{p} - \int_{1+p_9}^{p+p_9} f(\tau p + \tau p_8 - \tau p_9) \frac{dp}{p} \right] \quad (19)$$

As the integrals, shown above, have a same form, we use, at first, the second integral for further calculation. Substituting equation (18) into equation(17), the second integration is easily indefinitely integrated as below.

$$\begin{aligned} & \int_{1+p_9}^{\infty+p_9} f(\tau p + \tau(p_8 - p_9)) \frac{dp}{p} \quad (20) \\ &= \sum_{n=1}^{\infty} \frac{(-\tau)^n a_n}{n!} \left[\sum_{r=0}^{n-1} \frac{C_r^n p^{n-r} (p_8 - p_9)^r}{(n-r)} + (\log p)(p_8 - p_9)^n \right]_{1+p_9}^{\infty+p_9} \end{aligned}$$

Inserting $p = 1 + p_9$ we obtain the value at the lower limit. But at the upper limit the coefficient of τ^n , i.e, the term in the bracket does not converge as p approaches ∞ . Changing the order of summation, we obtain the form of the power series expansion in $(p_8 - p_9)$ with coefficients of $-\tau$ (refer to the Appendix 1):

$$\int_{1+p_9}^{\infty+p_9} f(\tau p + \tau(p_8 - p_9)) \frac{dp}{p} = \lim_{p \rightarrow \infty} \left[\sum_{r=0}^{\infty} F_r(\tau, p + p_9) \frac{(-\tau(p_8 - p_9))^r}{r!} \right] - \sum_{n=1}^{\infty} f_n(1 + p_9, p_8 - p_9) \frac{a_n(-\tau)^n}{n!} - \log(1 + p_9) f(\tau(p_8 - p_9)) \quad (21)$$

Here, the two coefficients are expressed below.

$$f_n(p_1, p_2) = \sum_{r=0}^{n-1} \frac{C_r^n p_1^{n-r} p_2^r}{(n-r)} = \sum_{r=1}^n \frac{C_r^n p_1^r p_2^{n-r}}{r} \quad (22)$$

$$F_r(\tau, p) = \sum_{n=1}^{\infty} \frac{(-1)^n a_{n+r} (\tau p)^n}{nn!} + a_r \log(p). \quad (23)$$

Here a_0 is conventionally designated 0. The problem for the integration is to evaluate the value of the function $F_r(\tau, p)$ as p approaches ∞ .

$$F_r(\tau, \infty) = \lim_{p \rightarrow \infty} \left[\sum_{n=1}^{\infty} \frac{a_{n+r} (-\tau p)^n}{nn!} + a_r \log(p) \right]. \quad (24)$$

$(r = 0, 1, 2, \dots)$

$F_r(\tau, \infty)$ except for $r = 0$ in the second and third approximations converge and their converged values at ∞ are given in the following subsection.

Substituting $p_9 = 0$ in equation (21) and then subtracting equation (21), we obtain the function $F(\tau, p_8, p_9)$.

$$\begin{aligned} & F(\tau, p_8, p_9) \\ &= \frac{1}{p_9} \left[\sum_{r=0}^{\infty} F_r(\tau, \infty) \frac{(-\tau p_8)^r}{r!} - \sum_{r=0}^{\infty} [F_r(\tau, \infty + p_9) \frac{(-\tau(p_8 - p_9))^r}{r!}] \right] \\ & - \sum_{n=1}^{\infty} \{f_n(1, p_8) - f_n(1 + p_9, p_8 - p_9)\} \frac{a_n(-\tau)^n}{n!} \\ & + \log(1 + p_9) f(\tau(p_8 - p_9)) \end{aligned} \quad (25)$$

As the $F_r(\tau, \infty)$, except for $r = 0$, converges in the second and third approximation, $F_r(\tau, \infty) = F_r(\tau, \infty + p_9)$ holds. The first term, $F_0(\tau, \infty)$, does not necessarily converge by itself. However combined $F_0(\tau, p) - F_0(\tau, p + p_9)$ in the synchronized approach of p to ∞ becomes 0, as shown below.

$$F_0(\tau, p) - F_0(\tau, p + p_9) = \sum_{n=1}^{\infty} \frac{(-1)^n a_n (\tau p)^n}{n n!} \Big|_{p=\infty}^{p=\infty+p_9} = - \int_{\infty}^{\infty+p_9} \frac{f(\tau p) dp}{p} \quad (26)$$

This term becomes finite, even though the integration $\int_1^{\infty} \frac{f(\tau p) dp}{p}$ does not converge. Because the integral interval is p_9 , or bounded, in the synchronous approach of infinite integration, and the integrand $f(\tau p)/p$ is also bounded. In most cases $f(\tau p)/p$ approaches to 0 as p approaches 0. Then the function $F(\tau, p_8, p_9)$ has a reduced form shown below.

$$\begin{aligned} F(\tau, p_8, p_9) &= \frac{1}{p_9} \left[\sum_{r=1}^{\infty} F_r(\tau, \infty) (p_8^r - (p_8 - p_9)^r) \frac{(-\tau)^r}{r!} \right. \\ &\quad \left. - \sum_{n=1}^{\infty} \left\{ \sum_{r=1}^n \frac{C_r^n}{r} (p_8^{n-r} - (1 + p_9)^r (p_8 - p_9)^{n-r}) \right\} \frac{a_n (-\tau)^n}{n!} \right] \\ &\quad + \frac{\log(1 + p_9)}{p_9} f(\tau(p_8 - p_9)) \end{aligned} \quad (27)$$

We can eliminate the log terms in the equation above, by coupling terms in the integrand. If the two terms $f_a(p, p_7)$ and $f_b(p, p_8)$ satisfy $f_a(-p_9, p_7) = f_b(-p_9, p_8)$, the coupling terms integration do not yield the log terms, as shown below.

$$\begin{aligned} F(\tau, p_8, p_9) &= \int_1^{\infty} \frac{f_a(\tau p + \tau p_7)}{p(p + p_9)} dp - \int_1^{\infty} \frac{f_b(\tau p + \tau p_8)}{p(p + p_9)} dp \\ &= \frac{1}{p_9} [\dots - \dots] + \frac{\log(1 + p_9)}{p_9} f_a(\tau(p_7 - p_9)) - \frac{\log(1 + p_9)}{p_9} f_b(\tau(p_8 - p_9)) \\ &= \frac{1}{p_9} [\dots - \dots] \end{aligned} \quad (28)$$

The the coefficient of $(-\tau)^n$ is a finite polynomial in p_9 , the degree of which is $n - 1$. Note that τ is included in the the coefficient $F_r(\tau, \infty)$. The function $F(\tau, p_8, p_9)$ has a form below.

$$F(\tau, p_8, p_9) = \sum_{n=1}^{\infty} \left\{ \sum_{r=0}^{n-1} K_n^r(p_8, \log \tau) p_9^r \right\} \frac{a_n (-\tau)^n}{n!}$$

$$+ \frac{\log(1 + p_9)}{p_9} f(\tau(p_8 - p_9)) \quad (29)$$

For the case of $p_9 = 0$, the integration is reduced to integration by parts.

$$\begin{aligned} \int_1^{\infty} f(\tau(p_8 + p)) \frac{dp}{p^2} &= \left. \frac{f(\tau(p_8 + p))}{-p} \right]_1^{\infty} - \int_1^{\infty} f'(\tau(p_8 + p)) \frac{dp}{-p\tau} \\ &= f(\tau(p_8 + 1)) - \int_1^{\infty} \left(\sum_{n=0}^{\infty} \frac{a_{n+1}(-p + p_8)\tau)^n}{p} \right) dp \end{aligned} \quad (30)$$

The first term can be evaluated with knowledge of $f(p)$, and the derivative of $f(p)$ has a easy form of integral.

3.2 Integrated Functions and Their Converged Values at Infinity

In this section several integrated functions, which are necessary for the second and third iterations and their converged values at infinity, are discussed^{6 7 8}.

(1) Function $\text{eax}(p)$

We define the function $\text{eax}(p)$ as,

$$\text{eax}(p) = \sum_{n=1}^{\infty} \frac{(-1)^n p^n}{nn!}. \quad (31)$$

The function $\text{eax}(p)$ is equal to the indefinite integral of $(1 - \exp(-p))/p$.

$$\int \frac{\exp(-p) - 1}{p} dp = \int \left(\sum_{n=1}^{\infty} \frac{(-1)^n p^n}{n!} \right) \frac{dp}{p} = \sum_{m=1}^{\infty} \frac{(-1)^m p^m}{mm!} \quad (32)$$

The derivative of $\text{eax}(p)$ is given below.

$$\text{eax}'(p) = \left(\sum_{n=1}^{\infty} \frac{(-1)^n p^{n-1}}{n!} \right) = \frac{\exp(-p) - 1}{p} \quad (33)$$

In order to evaluate $\text{eax}(p_1)$ at $p_1 = \infty$, we evaluate $1/1 + 1/2 + \dots + 1/n$.

$$\begin{aligned}
\sum_{r=1}^n \frac{1}{r} &= \int_0^1 (1 + x + x^2 + \dots + x^{n-1}) dx = \int_0^1 \frac{1 - x^n}{1 - x} dx \\
&= \int_{+0}^1 \frac{1 - (1 - y)^n}{y} dy = \int_{+0}^n \frac{1 - (1 - \frac{p}{n})^n}{p} dp
\end{aligned} \tag{34}$$

Where variables are changed: $y = 1 - x$, $p = ny$. Subtracting $\int_1^n \frac{dp}{p}$ from both sides in the equation above, we obtain below.

$$\sum_{r=1}^n \frac{1}{r} - \int_1^n \frac{dp}{p} = \int_{+0}^n \frac{1 - (1 - \frac{p}{n})^n}{p} dp - \log n. \tag{35}$$

As $n \rightarrow \infty$, the term $(1 - \frac{p}{n})^n$ approaches to $\exp -p$ and the left hand side of the above equation is γ . Here $\gamma = 0.577216$ is Euler's constant.

$$\gamma = \lim_{n \rightarrow \infty} \left[\int_{+0}^n \frac{1 - \exp(-p)}{p} dp - \log n \right] \tag{36}$$

The right hand side of equation above is equal to $-\text{mathrmeax}(n) - \log n$. Thus we obtain the value of $\text{eax}(p)$ at $p = \infty$.

$$\lim_{p \rightarrow \infty} (\text{eax}(p) + \log p) = -\gamma \tag{37}$$

Using the function $\text{eax}(p)$, we can evaluate the exponential integral function $E_1(\tau)$,

$$\begin{aligned}
E_1(\tau) &= \int_0^1 \frac{\exp(-\frac{\tau}{\mu})}{\mu} d\mu = \int_{\tau}^{\infty} \frac{\exp(-p)}{p} dp = \int_{\tau}^{\infty} \text{eax}'(p) dp + \int_{\tau}^{\infty} \frac{dp}{p} \\
&= -\gamma - \log \tau - \sum_{n=1}^{\infty} \frac{(-1)^n \tau^n}{nn!}
\end{aligned} \tag{38}$$

Henceforce, we designate $C = \gamma + \log \tau$.

(2)Function $\text{ebx}(p)$ ⁹

Similarly we define a function $\text{ebx}(p)$ as

$$\text{ebx}(p) = \int \frac{\text{eax}(p)}{p} dp = \sum_{m=1}^{\infty} \frac{(-1)^m p^m}{nnn!}. \tag{39}$$

To evaluate $\text{ebx}(p)$ at $p = \infty$, we begin the integration following, which is analogous to the case of $a_n = 1$

$$\int_0^1 \left[\int_0^t \frac{1 - (1 - t')^n}{t'} dt' \right] \frac{dt}{t} = \int_0^n \left[\int_0^s \frac{1 - (1 - s'/n)^n}{s'} ds' \right] \frac{ds}{s} \quad (40)$$

As $n \rightarrow \infty$, the integration above approaches to the $-\text{ebx}(\infty)$, for the same reason as in the case of $a_n = 1$. The integration is performed by expanding the integral (refer to appendix 2 for the derivation).

$$\lim_{p \rightarrow \infty} (\text{ebx}(p) + \frac{1}{2}(\log p + \gamma)^2) = -\frac{1}{2} \left(\sum_{r=1}^{\infty} \frac{1}{r^2} \right) = -\frac{\pi^2}{12} \quad (41)$$

The function $E_1^{(2)}(\tau)$ is defined as

$$E_1^{(2)}(\tau) = \int_{\tau}^{\infty} \frac{E_1(p)}{p} dp. \quad (42)$$

Using the function $\text{ebx}(p)$, we can evaluate the function $E_1^{(2)}(\tau)$ (refer to appendix 2 for the derivation).

$$E_1^{(2)}(\tau) = \frac{(\gamma + \log \tau)^2}{2} + \frac{\pi^2}{12} + \sum_{m=1}^{\infty} \frac{(-1)^m \tau^m}{n n n!} \quad (43)$$

(3) Function $A_r(p)$

We define a function, $A_r(p)$, as below,

$$A_r(p) = \sum_{n=1}^{\infty} \frac{(-p)^n}{(r+n)n!} \quad (44)$$

$A_1(p)$ is easily evaluated.

$$A_1(p) = \sum_{n=1}^{\infty} \frac{(-p)^n}{(n+1)!} = \frac{1}{(-p)} \sum_{n=2}^{\infty} \frac{(-p)^n}{n!} = \frac{\exp(-p) - 1 - (-p)}{(-p)} \quad (45)$$

$A_2(p)$ is evaluated below,

$$\begin{aligned}
A_2(p) &= \sum_{n=1}^{\infty} \frac{(-p)^n}{(n+2)n!} - A_1(p) + A_1(p) \\
&= - \sum_{n=1}^{\infty} \frac{(-p)^n}{(n+2)(n+1)n!} + A_1(p) = - \frac{A_1(p) - A_{1,1}(p)}{(-p)} + A_1(p) \quad (46)
\end{aligned}$$

Here $A_{1,1}(p)$ is the first term of $A_1(p)$ in the power series expansion in $-p$ and is equal to $(-p)/2$. Similarly we obtain $A_m(p)$ as below.

$$\begin{aligned}
A_r(p) &= \sum_{n=1}^{\infty} \frac{(-p)^n}{(n+r)n!} = - \sum_{n=1}^{\infty} \frac{(r-1)(-p)^n}{(n+r)(n+1)n!} + A_1(p) \\
&= - \frac{r-1}{(-p)} \sum_{n=2}^{\infty} \frac{(-p)^n}{(n+r-1)n!} + A_1(p) = - \frac{r-1}{(-p)} (A_{r-1}(p) - \frac{(-p)}{r}) + A_1(p) \quad (47)
\end{aligned}$$

Thus we obtain a recursive formula for $A_r(p)$. As p approaches to ∞ , $A_r(\infty)$ is given below.

$$\begin{aligned}
A_r(\infty) &= \lim_{p \rightarrow \infty} \left[- \frac{r-1}{(-p)} (A_{r-1}(p) - \frac{(-p)}{r}) \right] + A_1(\infty) \\
&= \frac{(r-1)}{r} - 1 = - \frac{1}{r} \quad (48)
\end{aligned}$$

(4) Function eix(p)

We define a function eix(p) as below.

$$\text{eix}(p) = \exp(-p)\text{eax}(-p). \quad (49)$$

The function is a polynomial of p and given as

$$\text{eix}(p) = \left\{ \sum_{n=0}^{\infty} \frac{(-1)^n p^n}{n!} \right\} \left\{ \sum_{n=1}^{\infty} \frac{p^n}{nn!} \right\} = \sum_{n=1}^{\infty} \left(\sum_{r=1}^n \frac{-1}{r} \right) \frac{(-1)^n p^n}{n!} \quad (50)$$

The derivative of eix(p) is given below.

$$\text{eix}'(p) = -\text{eix}(p) + (-1) \exp(-p)\text{eax}'(p) = -\text{eix}(p) + \frac{1 - \exp(-p)}{p} \quad (51)$$

The value of $\text{eix}(p)$ at $p = \infty$ is given below.

$$\lim_{p \rightarrow \infty} \text{eix}(p) = \lim_{p \rightarrow \infty} \frac{\text{eax}(-p)}{e^p} = \lim_{p \rightarrow \infty} \frac{(\text{eax}(-p))'}{(e^p)'} = \lim_{p \rightarrow \infty} \frac{e^p - 1}{pe^p} = 0 \quad (52)$$

(5)Function $\text{ejx}(p)$

We define a function $\text{ejx}(p)$ as the indefinite integration of the function $\text{eix}(p)$, divided by p .

$$\text{ejx}(p) = \int \frac{\text{eix}(p)}{p} dp. \quad (53)$$

The function is a polynomial of p and is given as

$$\text{ejx}(p) = \sum_{n=1}^{\infty} \left(\sum_{r=1}^n \frac{1}{r} \right) \frac{(-1)^{n+1} p^n}{nn!} \quad (54)$$

The function is further derived below.

$$\begin{aligned} \text{ejx}(p) &= - \sum_{n=1}^{\infty} \left(\sum_{r=1}^{\infty} \left(\frac{1}{r} - \frac{1}{r+n} \right) \right) \frac{(-p)^n}{nn!} = - \sum_{n=1}^{\infty} \left(\sum_{r=1}^{\infty} \frac{1}{r(r+n)} \right) \frac{(-p)^n}{n!} \\ &= - \sum_{r=1}^{\infty} \frac{1}{r} \left(\sum_{n=1}^{\infty} \frac{(-p)^n}{(r+n)n!} \right) = - \sum_{r=1}^{\infty} \frac{1}{r} A_r(p) \end{aligned} \quad (55)$$

Finally we obtain $\text{eax}(\infty)$.

$$\text{ejx}(\infty) = \sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{6} \quad (56)$$

(6)Function $G_r(\tau, p)$

We define a function $G_r(\tau, p)$ as below.

$$G_r(\tau, p) = \sum_{n=1}^{\infty} \left\{ \frac{(-\tau)^n p^n}{(n+r)nn!} \right\} + \frac{\log p}{r} \quad (57)$$

$(r = 1, 2, \dots)$

Decomposing the fraction into two fractions, the function $G_r(\tau, p)$ is further derived as below.

$$\begin{aligned} G_r(\tau, p) &= \frac{1}{r} \left[\sum_{n=1}^{\infty} \frac{(-\tau)^n p^n}{nn!} - \sum_{n=1}^{\infty} \frac{(-\tau)^n p^n}{(n+r)n!} \right] + \frac{\log p}{r} \\ &= \frac{1}{r} (\text{eax}(\tau p) - A_r(\tau p) + \log p) \end{aligned} \quad (58)$$

Finally we obtain $G_r(\tau, \infty)$.

$$G_r(\tau, \infty) = \frac{1}{r} (-\gamma - \log \tau + \frac{1}{r}) = \frac{1}{r^2} - \frac{C}{r} \quad (59)$$

(7)Function $H_r(\tau, p)$

We define a function $H_r(\tau, p)$ as below.

$$\begin{aligned} H_r(\tau, p) &= - \sum_{n=1}^{\infty} \left\{ \left(\sum_{q=1}^{n+r} \frac{1}{q} \right) \frac{(-\tau)^n p^n}{nn!} \right\} - \left(\sum_{q=1}^r \frac{1}{q} \right) \frac{\log p}{r} \\ &\quad (r = 1, 2, \dots) \end{aligned} \quad (60)$$

The function $H_r(\tau, p)$ is further derived as below(refer to Appendix 3 for derivation).

$$\begin{aligned} H_r(\tau, p) &= - \sum_{n=1}^{\infty} \left[\left\{ \sum_{q=1}^{\infty} \left(\frac{1}{q} - \frac{1}{n+r+q} \right) \right\} \frac{(-\tau)^n p^n}{nn!} \right] - \left(\sum_{q=1}^r \frac{1}{q} \right) \frac{\log p}{r} \\ &= - \sum_{q=1}^{\infty} \frac{1}{q} \left[A_{r+q}(\tau p) + r G_{r+q}(\tau p) - \frac{r}{r+q} \log p \right] - \left(\sum_{q=1}^r \frac{1}{q} \right) \frac{\log p}{r} \end{aligned} \quad (61)$$

Finally we obtain $H_r(\tau, \infty)$ (refer to Appendix 3 for derivation).

$$\begin{aligned} &H_r(\tau, \infty) \\ &= \sum_{q=1}^{\infty} \frac{1}{(r+q)^2} + C \sum_{q=1}^r \frac{1}{q} = \frac{\pi^2}{6} - \sum_{q=1}^r \frac{1}{q^2} + C \sum_{q=1}^r \frac{1}{q} \end{aligned} \quad (62)$$

4 Integration of Iteration

4.1 Second Iteration

$U_2(\tau, i_6)$ is evaluated by the integral below.

$$U_2(\tau, i_6) = \int_0^1 (1 - \exp(-\frac{\tau}{\mu} - \frac{\tau}{\mu_6})) \frac{\mu_6 d\mu}{\mu_6 + \mu} = \int_1^\infty \frac{1 - e^{-\tau(p+p_6)}}{(p+p_6)p} dp \quad (63)$$

Since the integrand of the above equation approaches $O(1/p^2)$ as p approaches ∞ , the integration converges. To perform the integration above, we apply equation (27) for the case $a_n = 1$ and $p_8 = p_9 = p_6$. The coefficient $F_r(\tau, \infty)$ is calculated below,

$$F_r(\tau, \infty) = \lim_{p \rightarrow \infty} \left[\sum_{n=1}^{\infty} \frac{(-\tau p)^n}{nn!} + \log p \right] = \lim_{p \rightarrow \infty} (e^{-\tau p} + \log p) = -C. \quad (64)$$

$$a_r = 1 (r = 1, 2, \dots)$$

Substituting $F_r(\tau, \infty)$ into equation (27) we obtain

$$\begin{aligned} U_2(\tau, i_6) &= \frac{-1}{p_6} \left[\sum_{r=1}^{\infty} F_r(\tau, \infty) p_6^r \frac{(-\tau)^r}{r!} \right. \\ &- \sum_{n=1}^{\infty} \left\{ \sum_{r=1}^n \frac{C_r^n}{r} p_6^{n-r} - \frac{(1+p_6)^n}{n} \right\} \frac{(-\tau)^n}{n!} \left. \right] \\ &= \frac{1}{p_6} \left[\sum_{n=1}^{\infty} \left\{ C p_6^n + \left(\sum_{r=1}^n \frac{C_r^n}{r} p_6^{n-r} \right) - \frac{(1+p_6)^n}{n} \right\} \frac{(-\tau)^n}{n!} \right]. \end{aligned} \quad (65)$$

There exists no further reduced, or concise expression for $U_2(\tau, i_6)$ as the function of the polynomial in τ . We can obtain the truncated $U_2(\tau, i_6)$ up to the third degree.

$$\begin{aligned} U_2(\tau, i_6) &= (C-1)(-\tau) + \left(1 + \left(C - \frac{1}{2}\right)p_6\right) \frac{(-\tau)^2}{2!} \\ &+ \left(\frac{1}{2} + 2p_6 + \left(C - \frac{1}{3}\right)p_6^2\right) \frac{(-\tau)^3}{3!} \end{aligned} \quad (66)$$

The summations in the above equation are expressed with a newly defined function $\text{eax}(p)$ (refer to subsection 3.2).

$$\begin{aligned} U_2(\tau, i_6) &= \frac{1}{p_6} \left[\sum_{n=1}^{\infty} \left\{ C p_6^n + \left(\sum_{r=1}^{\infty} \frac{(-\tau)^r}{r r!} \right) p_6^n - \frac{1 - (1 + p_6)^n}{n} \right\} \frac{(-\tau)^n}{n!} \right] \\ &= \frac{1}{p_6} [C(\exp(-p_6\tau) - 1) + \exp(-p_6\tau)\text{eax}(\tau) - \text{eax}(1 + p_6)\tau] \end{aligned}$$

$V_2(\tau, i_7)$ is evaluated below.

$$\begin{aligned} V_2(\tau, i_7) &= \int_0^1 T_1(\tau, i, i_7) \frac{d\mu}{\mu} = \int_0^1 \frac{1}{\frac{1}{\mu\tau} - \frac{1}{\mu}} [\exp(-\frac{\tau}{\mu}) - \exp(-\frac{\tau}{\mu\tau})] \frac{d\mu}{\mu} \\ &= e^{-\tau p_7} \left[\int_1^{p_7-\epsilon} \oplus \int_{p_7+\epsilon}^{\infty} \frac{1 - \exp(-\tau(p - p_7))}{(p - p_7)p} dp = \exp(-\tau p_7) U_2(\tau, -p_7) \right] \\ &= \frac{1}{p_7} [(\exp(-\tau p_7) - 1)C - \text{eax}(\tau) + \exp(-\tau p_7)\text{eax}(\tau(1 - p_7))] \quad (67) \end{aligned}$$

Here the mark \oplus designates the principal value integral. The integration of $V_2(\tau, i_7)$ is a coupling integration described in the section three. The first function is $f_a(p, p_8) = 1$ and the second function $f_b(p, p_7) = \exp(-\tau(p - p_7))$. Since these two functions satisfy $f_a(p_7, p_8) = f_b(p_7, p_7)$, the outcome integration does not have the log term. We can obtain the remaining terms by inserting $-p_7$ into p_6 in the outcome integration of $U_2(p_6)$. We call this relationship between the function $U_2(p_6)$ and $V_2(p_7)$ as 'dependence of V function'. Hence, we do not need to discuss the V functions independently of the U functions.

As $U_2(\tau, i_6)$ varies in the interval of $1 < p_6 < \infty$, we must know its value at $p_6 = \infty$.

$$\lim_{p_6 \rightarrow \infty} U_2(\tau, i_6) = \lim_{p_6 \rightarrow \infty} \frac{-\text{eax}(\tau(1 + p_6))}{p_6} = \lim_{p_6 \rightarrow \infty} \frac{\log(1 + p_6)}{p_6} = 0 \quad (68)$$

Substituting $U_2(\tau, p)$ and $V_2(\tau, p)$ into equation (11) we obtain the second iteration of the scattering function $S(\tau, p_1, p_0)$.

$$\begin{aligned} \Delta S_2(\tau, i_1, i_0) &= \left(\frac{3}{4} - \frac{C}{2} \right) \tau^2 \\ &+ \left(-\frac{C}{4} + \frac{5}{12} + \frac{C-1}{2} (p_1 + p_0) \right) \tau^3 \quad (69) \end{aligned}$$

4.2 Third Iteration

For the third iteration, we need $U_3(\tau, i_6)$ and $V_3(\tau, i_7)$ functions.

$$U_3(\tau, i_6) = \int_1^{\infty} [U_2(p_6) - \exp(-\tau p)V_2(p_6) + U_2(p) - \exp(-\tau p_6)V_2(p) + U_2(p_6)U_2(p) - V_2(p_6)V_2(p)] \frac{dp}{(p + p_6)p} \quad (70)$$

$$V_3(\tau, i_7) = \int_1^{\infty} [V_2(p_7) - V_2(p) - \exp(-\tau p)U_2(p_7) + \exp(-\tau p_7)U_2(p) + U_2(p)V_2(p_7) - V_2(p)U_2(p_7)] \frac{dp}{(p - p_7)p} \quad (71)$$

Here we ignore the argument τ . All the terms in the bracket in the integrand above approach $0(p^0)$ and the denominator becomes $0(p^2)$, as p approaches ∞ . Hence the integration above converges and, furthermore, each term converges by itself.

Taking account of dependence of V function in the second iteration, we can obtain dependence of V function in the third iteration.

$$V_3(\tau, i_7) = \exp(-\tau p_7) \int_1^{\infty} [U_2(-p_7) - \exp(\tau p_7)V_2(p) - \exp(-\tau p)V_2(-p_7) + U_2(p) + U_2(p)U_2(-p_7) - V_2(p)V_2(-p_7)] \frac{dp}{(p - p_7)p} = \exp(-\tau p_7)U_2(-p_7)$$

To integrate U_3 , we make pairs of terms in the integrand shown as below.

$$U_3(\tau, i_6) = \int_1^{\infty} [U_2(p_6) - \exp(-\tau p_6)V_2(p)] \frac{dp}{(p + p_6)p} + \int_1^{\infty} [U_2(p) - \exp(-\tau p)V_2(p_6)] \frac{dp}{(p + p_6)p} + \int_1^{\infty} [U_2(p_6)U_2(p) - V_2(p_6)V_2(p)] \frac{dp}{(p + p_6)p} \quad (73)$$

The first pairing of the above equation satisfies the condition for coupling shown below.

$$\begin{aligned} U_2(p_6) - \exp(-\tau p_6)V_2(p) &= U_2(p_6) - \exp(-\tau(p_6 + p))U_2(-p) \\ &= U_2(p_6) - U_2(-p) - (\exp(-\tau(p_6 + p)) - 1)U_2(-p) \end{aligned} \quad (74)$$

Inserting $p = -p_6$ in the above equation, the first coupling term becomes 0. Similarly the second and the third couplings also satisfy the condition.

The first integral is performed by substituting $U_2(p)$ into the integration and decomposing the fraction.

$$\begin{aligned} &\int_1^\infty [U_2(p) - \exp(-\tau p)V_2(p_6)] \frac{dp}{(p + p_6)p} \\ &= \frac{1}{p_6} [(C + \text{eax}(\tau(1 - p_6)))U_2(p_6) \\ &+ \int_1^\infty \{\text{eax}(\tau(1 + p)) - \text{eax}(\tau(1 - p_6))\} \frac{dp}{(p + p_6)p} \\ &+ \int_1^\infty (\exp(-\tau p) - 1)(C + \text{eax}(\tau)) + \text{eax}(\tau) - \text{eax}(\tau(1 + p)) \frac{dp}{p^2}] \end{aligned} \quad (75)$$

To perform the first integral in the above equation, we apply the method given in the section three.

$$\begin{aligned} &\int_1^\infty \frac{\text{eax}(\tau(p + 1)) - \text{eax}(\tau(1 - p_6))}{p(p + p_6)} dp \\ &= \frac{1}{p_6} \left[\sum_{r=1}^\infty G_r(\tau, \infty) (1 - (1 - p_6)^r) \frac{(-\tau)^r}{r!} \right. \\ &\left. - \sum_{n=1}^\infty (f_n(1, 1) - f_n(1 + p_6, 1 - p_6)) \frac{(-\tau)^n}{nn!} \right] \end{aligned} \quad (76)$$

Here, $G_r(\tau, \infty)$ is given below (refer to section 3.2).

$$G_r(\tau, \infty) = \frac{1}{r^2} - \frac{C}{r} \quad (77)$$

The second term in equation (76) is further rearranged as below. (refer to Appendix 2 for derivation)

$$\begin{aligned}
f_n(1, 1) - f_n(1 + p_6, 1 - p_6) &= \sum_{r=1}^n \frac{2^r (1 - (1 - p_6)^{n-r})}{r} \\
- \left(\sum_{r=1}^n \frac{1}{r} \right) (1 - (1 - p_6)^n) & \quad (78)
\end{aligned}$$

Gathering the coefficient of the power series expansion in τ , we obtain the following.

$$\begin{aligned}
& \int_1^{\infty} \frac{\text{eax}(\tau(p+1)) - \text{eax}(\tau(1-p_6))}{p(p+p_6)} dp \\
&= \sum_{n=1}^{\infty} \frac{g_n(1 - (1 - p_6)^n) - D_n(1 - p_6)}{p_6} \frac{(-\tau)^n}{n!} \quad (79)
\end{aligned}$$

Here g_n and $D_n(p)$ are given below.

$$g_n = \frac{1}{n^2} - \frac{C}{n} + \left(\sum_{r=1}^n \frac{1}{r} \right) \frac{1}{n} \quad (80)$$

$$D_n(p) = \frac{1}{n} \sum_{r=1}^n \frac{2^r (1 - p^{n-r})}{r} \quad (81)$$

The second integration in equation (75) is performed below.

$$\begin{aligned}
& \int_1^{\infty} \frac{(\exp(-\tau p) - 1)(C + \text{eax}(\tau)) + \text{eax}(\tau) - \text{eax}(\tau(1+p))}{p^2} dp \\
&= [-U_2(p)]_1^{\infty} + \int_1^{\infty} \left\{ -\tau(\exp(-\tau p))(C + \text{eax}(\tau)) \right. \\
&\quad \left. - \frac{\exp(-\tau(1+p)) - 1}{1+p} \right\} \frac{dp}{p} = 2U_2(1) + \tau(C + \text{eax}(\tau))^2 \quad (82)
\end{aligned}$$

Finally we obtain the integration in equation (75).

$$\begin{aligned}
& \int_1^{\infty} [U_2(p) - \exp(-\tau p)V_2(p_6)] \frac{dp}{(p+p_6)p} \\
&= \frac{1}{p_6} [\{C + \tau(1 - p_6)\}U_2(p_6) + 2U_2(1) + \tau(C + \text{eax}(\tau))^2]
\end{aligned}$$

$$+ \sum_{n=1}^{\infty} \frac{g_n(1 - (1 - p_6)^n) - D_n(1 - p_6) \frac{(-\tau)^n}{n!}}{p_6} \quad (83)$$

In the same manner we integrate the second integration in equation(73).

$$\begin{aligned} & \int_1^{\infty} [U_2(p_6) - \exp(-\tau p_6)V_2(p)] \frac{dp}{(p + p_6)p} = \frac{1}{p_6} [-CU_2(p_6) \\ & + \int_1^{\infty} \exp(-\tau(p + p_6))\text{eax}(\tau(1 - p)) - \text{eax}(\tau(1 + p_6))] \frac{dp}{(p + p_6)p} \\ & - e^{-p_6\tau} \int_1^{\infty} [(\exp(-\tau p) - 1)C - \text{eax}(\tau) + \exp(-\tau p)\text{eax}(\tau(1 - p))] \frac{dp}{p^2} \end{aligned} \quad (84)$$

To perform the first integration in above equation, we also apply the method in the section three.

$$\begin{aligned} & \int_1^{\infty} \exp(-\tau(p + p_6))\text{eax}(\tau(1 - p)) - \text{eax}(\tau(1 + p_6))] \frac{dp}{(p + p_6)p} \\ & = \frac{\exp(-\tau(1 + p_6))}{p_6} \left[\sum_{r=1}^{\infty} H_r(\tau, \infty) \{(-1)^r - (-1 - p_6)^r\} \frac{(-\tau)^r}{r!} \right. \\ & \left. - \sum_{n=1}^{\infty} (f_n(1, -1) - f_n(1 + p_6, -(1 + p_6))) \left(-\sum_{r=1}^n \frac{1}{r}\right) \frac{(-\tau)^n}{n!} \right] \end{aligned} \quad (85)$$

Here $H_r(\tau, \infty)$ and $(f_n(1, -1) - f_n(1 + p_7, -(1 + p_7)))$ are given below.

$$H_r(\tau, \infty) = \frac{\pi^2}{6} - \sum_{q=1}^r \frac{1}{q^2} + C \sum_{q=1}^r \frac{1}{q} \quad (86)$$

$$(r = 1, 2, \dots)$$

$$f_n(1, -1) - f_n(1 + p_7, -(1 + p_7)) = (-1)^{n+1} \left(\sum_{r=1}^n \frac{1}{r} \right) (1 - (1 + p_7)^n) \quad (87)$$

Gathering the coefficients of the power series expansion of the integration above in τ , we obtain following.

$$\int_1^{\infty} \frac{\exp(-\tau(p + p_6))\text{eax}(\tau(1 - p)) - \text{eax}(\tau(1 + p_6))}{p(p + p_6)} dp$$

$$= \exp(-\tau(1 + p_6)) \sum_{n=1}^{\infty} \frac{h_n(1 - (1 + p_6)^n) (-\tau)^n}{p_6 n!} \quad (88)$$

Here h_n is given below.

$$h_n = (-1)^n \left[\frac{\pi^2}{6} - \sum_{r=1}^n \frac{1}{r^2} + C \sum_{r=1}^n \frac{1}{r} - \left(\sum_{r=1}^n \frac{1}{r} \right)^2 \right] \quad (89)$$

The second integration in equation(84) is performed as below.

$$\begin{aligned} & \int_1^{\infty} [(\exp(-\tau p) - 1)C - \text{eax}(\tau) + \exp(-\tau p)\text{eax}(\tau(1 - p))] \frac{dp}{p^2} \\ &= [-V_2(p)]_1^{\infty} + \int_1^{\infty} \left\{ -\tau \exp(-\tau p)C + \exp(-\tau) \frac{d}{dp} (\text{eix}(\tau(p - 1))) \right\} \frac{dp}{p} \quad (90) \\ &= V_2(1) + \tau C(C + \text{eax}(\tau)) + \exp(-\tau) \left\{ U_2(\tau, -1) - \tau \int_1^{\infty} \text{eix}(p - \tau) \frac{dp}{p} \right\} \\ &= 2V_2(1) + \tau C(C + \text{eax}(\tau)) - \tau \exp(-\tau) \left(\frac{\pi^2}{6} - \text{ejx}(\tau) + \sum_{n=1}^{\infty} h_n \frac{(-\tau)^n}{n!} \right) \end{aligned}$$

Finally we obtain the second integration in equation(73)

$$\begin{aligned} & \int_1^{\infty} [U_2(p_6) - \exp(-\tau p_6)V_2(p)] \frac{dp}{(p + p_6)p} \quad (91) \\ &= \frac{1}{p_6} [-CU_2(p_6) - \exp(-\tau p_6)(2V_2(1) + \tau C(C + \text{eax}(\tau))) \\ &+ e^{-\tau(p_6+1)} \left\{ \sum_{n=1}^{\infty} \frac{h_n(1 - (1 + p_6)^n) (-\tau)^n}{p_6 n!} + \tau \left(\frac{\pi^2}{6} + \sum_{n=1}^{\infty} h_n \frac{(-\tau)^n}{n!} \right) \right\}] \end{aligned}$$

The third integration in equation (73) can be expressed by the first and second integration in equation(73).

$$\int_1^{\infty} [U_2(p_6)U_2(p) - V_2(p_6)V_2(p)] \frac{dp}{(p + p_6)p}$$

$$\begin{aligned}
&= \exp(\tau p_6) V_2(p_6) \int_1^\infty [U_2(p_6) - \exp(-\tau p_6) V_2(p)] \frac{dp}{(p + p_6)p} \\
&+ U_2(p_6) \int_1^\infty [U_2(p) - \exp(-\tau p) V_2(p_6)] \frac{dp}{(p + p_6)p} - \exp(\tau p_6) V_2(p_6) U_2(p_6)^2
\end{aligned} \tag{92}$$

Thus we finish the integration $U_3(\tau, p_6)$.

$$\begin{aligned}
U_3(\tau, p_6) &= (1/p_6)[\text{eax}(\tau(1 - p_6))U_2(p_6) + 2(U_2(1) - V_2(1)) \\
&+ \tau(C + \text{eax}(\tau))\text{eax}(\tau) - (\exp(-\tau p_6) - 1)(2V_2(1) + \tau C(C + \text{eax}(\tau))) \\
&+ U_2(-p_6)\{-CU_2(p_6) - \exp(-\tau p_6)(2V_2(1) + \tau C(C + \tau(C + \text{eax}(\tau))))\} \\
&+ U_2(p_6)\{CU_2(p_6) + \tau(1 - p_6)U_2(p_6) + 2U_2(1) + \tau(C + \text{eax}(\tau))^2\} \\
&+ (1 + U_2(p_6)) \sum_{n=1}^\infty \frac{g_n(1 - (1 - p_6)^n) - D_n(1 - p_6)}{p_6} \frac{(-\tau)^n}{n!} + e^{-\tau(p_6+1)} \\
&\times (1 + U_2(-p_6)) \left\{ \sum_{n=1}^\infty \frac{h_n(1 - (1 + p_6)^n)}{p_6} \frac{(-\tau)^n}{n!} + \tau \left(\frac{\pi^2}{6} + \sum_{n=1}^\infty h_n \frac{(-\tau)^n}{n!} \right) \right\} \\
&- \exp(\tau p_6) V_2(p_6) U_2(p_6)^2
\end{aligned} \tag{93}$$

The truncated $U_3(\tau, p_6)$ up to the third degree in τ is calculated.

$$\begin{aligned}
U_3(\tau, p_6) &= (2C^2 - 5C + \frac{9}{2} - \frac{\pi^2}{6}) \frac{(-\tau)^2}{2!} \\
&+ \left\{ \dots + (3C^2 - \frac{15C + 7}{2} - \frac{\pi^2}{3}) p_0 \right\} \frac{(-\tau)^3}{3!}
\end{aligned} \tag{94}$$

The third approximation is given below.

$$\begin{aligned}
\Delta S_3(\tau, i_1, i_0) &= \frac{1}{2(p_1 + p_0)} [U_3(p_0) - \exp(-\tau p_1) V_3(p_0) \\
&+ U_3(p_1) - \exp(-\tau p_0) V_3(p_1) + U_3(p_0) U_3(p_1) - V_3(p_0) V_3(p_1)]
\end{aligned} \tag{95}$$

Substituting $U_3(\tau, p_6)$ into equation above, we obtain the third approximation.

$$\Delta S_3(\tau, p_1, p_0) = \left(\frac{C^2}{2} - \frac{5C}{4} + \frac{5}{3} - \frac{\pi^2}{36} \right) \tau^3 \tag{96}$$

We conclude the third approximation of the scattering function $S_3(\tau, p_1, p_0)$.

$$\begin{aligned}
S_3(\tau, p_1, p_0) = & \tau + \left(-\frac{\log \tau}{2} + \frac{3}{4} - \frac{\gamma}{2} - \frac{p_1 + p_0}{2}\right)\tau^2 \\
& + \left[\left(\frac{(\log \tau)^2}{2} + \left(\gamma - \frac{3}{2} + \frac{p_1 + p_0}{2}\right)\log \tau + \left(\frac{\gamma^2}{2} - \frac{3\gamma}{2} + \frac{35}{12} - \frac{\pi^2}{36}\right)\right.\right. \\
& \left.\left.+ \frac{(\gamma - 1)(p_1 + p_0)}{2} + \frac{(p_1 + p_0)^2}{6}\right]\tau^3
\end{aligned} \tag{97}$$

5 Conclusion

Chandrasekhar's integral equation for the isotropic atmosphere is solved by iterative integration up to the third approximation. The explicit form of the scattered function, $S_3(\tau, p_1, p_0)$, is given in equation (97), up to the third power of τ . For the third approximation, we should integrate two new functions, in equation (76) and (84). The possibility of the integration, or convergence of integration, is obvious, because the integrand in both equations are $O(1/p^\beta)$ ($\beta > 1$) as $p = \infty$. Decomposing the fraction in the prototype integral into partial fractions, we can easily obtain the indefinite integral. The key problem is to assess the value of the integration at the upper limit, i.e., at ∞ . With changing the order of summation and rearranging series expansions, we can obtain the converged value of the two integrations in equation (79) and (88). To obtain the converged value of the integrations mentioned above, we must assess several power series expansions at ∞ . The traditional 'Exponential Integral Function' $E_1(\tau)$ is modified as $\text{eax}(p)$ and its converged value is assessed in section 3.2. We can give a new proof of the converged value of the function $E_1^{(2)}(\tau)$.

In the third approximation, the scattering function is expressed as a "quasi-power series expansion" in τ , the coefficients of which include $\log \tau$. The term $\tau^3(\log \tau)^2$ is more significant than τ^3 but less significant than τ^2 , which is expressed below.

$$\lim_{\tau \rightarrow 0} \frac{\tau^3(\log \tau)^2}{\tau^3} = \infty \tag{98}$$

$$\lim_{\tau \rightarrow 0} \frac{\tau^3(\log \tau)^2}{\tau^2} = \lim_{\tau \rightarrow 0} \frac{(\log \tau)^2}{1/\tau} = \lim_{\tau \rightarrow 0} \frac{2(\log \tau)}{-1/\tau} = \lim_{\tau \rightarrow 0} \tau = 0 \tag{99}$$

Therefore the term $\tau^3 \log \tau$ does not affect the main term of the the second approximation, τ^2 . In the third approximation the order of significance, as τ approaches to 0, is given as below.

$$\tau, \tau^2 \log \tau, \tau^2, \tau^3(\log \tau)^2, \tau^3 \log \tau, \tau^3$$

It is noted that the approximated solution of the scattering function can be expanded not by a power series expansion in τ but by a series expansion in $\tau^m(\log \tau)^n$, ($m > n$), which we called "quasi-power series expansion".

In the second iteration the lowest degree of the power in τ is 2, more precisely $\tau^2 \log \tau$ and in the third approximation it is 3, or $\tau^3(\log \tau)^2$. It suggests that the Nth iteration might be expressed as a power series in τ , whose lowest degree is N, with coefficients which are a polynomial of $\log \tau$. This fact gives us the validity of the iterative integral solution, but does not, by itself, give us the mathematically rigorous proof that the approximation thus obtained with the iterative integral could converge.

In the second and third approximations, we can select coupling pairs of terms and, by integrating those pairs, we can avoid unnecessary terms of logarithm in the outcome of integral. Owing to this coupling integral, the function $V_n(\tau, p)$ is proved to become equal to $e^{-\tau P} U_n(\tau, -p)$. We do not need to integrate $V_n(\tau, p)$ independently to $U_n(\tau, p)$. Therefore the simultaneous integral equation is deduced to a mere integral equation. We can avoid complicated and tedious algebra calculations, based on the coupling integral and dependency of the function V , and then obtain the truncated approximation of the scattering function.

Acknowledgment The author expresses his sincere appreciation to Prof. M. Nakamura, Prof. Y. Yasuoka, Prof. T. Nakajima and Prof. Sumi for their invaluable discussions and suggestions. This work was completed during the author's tour of duty at the NASA Goddard Space Flight Center. The author expresses his gratitude to NASA personnel and Dr. R. Barnes for his review of the manuscript.

References

- [1] S. Chandrasekhar, *RADIATIVE TRANSFER*, Dover Publishing, Inc., New York, (1960).
- [2] J. E. Hansen and L. D. Travis, Light scattering in planetary atmosphere, *Space Science Review* **16** (1974) 527–610
- [3] R. M. Gudy and Y. L. Yung, *Atmospheric Radiation*, Oxford University Press, New York Oxford, (1989)
- [4] H. R. Gordon, Atmospheric correction of ocean color imagery in the Earth Observing System era, *J. of Geophysical Research* **Vol. 102, No. D14** (1997) 17,081–17,106
- [5] P. Deshampe, F. Breon, M. Leroy, A. Podaire, A. Bricaud, J. Buriez, and G. Seze, *The Polder Mission: Instrument Characteristics and Scientific Objectives*,

- [6] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, U. S. Department of Commerce, (1964)
- [7] S. Moriguchi, K. Udagawa and S. Hitotumatu, Iwanami Suugaku Koushiki, Iwanami Shoten, (1987)
- [8] Y. Komatu, Tokushu Kansuu(Kindai Suugaku Kouza ,9) Asakura Shoten, (1967)
- [9] I. W. Busbridge, The mathematics of radiative transfer, Cambridge University Press, (1960)

Appendix 1, Derivation of Series Expansion of Function $F_m(\tau \cdot \infty)$

$$\begin{aligned}
 & \int f(\tau q + \tau q_1) \frac{dq}{q} = \int \sum_{n=1}^{\infty} a_n (-\tau)^n \frac{(q + q_1)^n}{n!} \frac{dq}{q} \\
 &= \sum_{n=1}^{\infty} \left[\frac{a_n (-\tau)^n}{n!} \int \frac{(q + q_1)^n}{q} dq \right] = \sum_{n=1}^{\infty} \left[\frac{a_n (-\tau)^n}{n!} \int \sum_{r=0}^n \frac{C_r^n q^{n-r} q_1^r}{q} dq \right] \\
 &= \sum_{n=1}^{\infty} \frac{a_n (-\tau)^n}{n!} \left[\sum_{r=0}^{n-1} \frac{C_r^n q^{n-r} q_1^r}{(n-r)} + (\log q) q_1^n \right] \\
 &= \sum_{n=1}^{\infty} \left\{ \frac{a_n (-\tau)^n q^n}{nn!} \right\} + \sum_{r=1}^{\infty} \left[\sum_{n=r+1}^{\infty} \left\{ \frac{a_n (-\tau)^n C_r^n q^{n-r}}{(n-r)n!} \right\} + \frac{a_r (-\tau)^r}{r!} (\log q) \right] q_1^r \\
 &= \sum_{n=1}^{\infty} \left\{ \frac{a_n (-\tau)^n q^n}{nn!} \right\} + \sum_{r=1}^{\infty} \left[\sum_{n=r+1}^{\infty} \left\{ \frac{a_n (-\tau)^n q^{n-r}}{(n-r)(n-r)!} \right\} + a_r (-\tau)^r (\log q) \right] \frac{q_1^r}{r!} \\
 &= \sum_{n=1}^{\infty} \frac{a_n (-\tau)^n q^n}{nn!} + \sum_{r=1}^{\infty} \left[\sum_{m=1}^{\infty} \left\{ \frac{a_{m+r} (-\tau)^{m+r} q^m}{mm!} \right\} + a_r (-\tau)^r (\log q) \right] \frac{q_1^r}{r!} \\
 &= \sum_{n=1}^{\infty} \frac{a_n (-\tau)^n q^n}{nn!} + \sum_{r=1}^{\infty} \left[\sum_{m=1}^{\infty} \left\{ \frac{a_{m+r} (-\tau)^m q^m}{mm!} \right\} + a_r \log q \right] \frac{(-\tau q_1)^r}{r!} \\
 &= \sum_{n=1}^{\infty} \frac{a_n (-\tau)^n q^n}{nn!} + \sum_{r=1}^{\infty} \left\{ \sum_{m=1}^{\infty} \frac{a_{m+r} (-\tau)^m q^m}{mm!} \right\} \frac{(-\tau q_1)^r}{r!} + f(\tau q_1) \log q
 \end{aligned}$$

Appendix 2, Derivation of $eax(\infty)$ and $E_1^{(2)}(\tau)$

$$\begin{aligned}
 & \int_0^1 \left[\int_0^t \frac{1 - (1-t')^n}{t'} dt' \right] \frac{dt}{t} = \int_0^1 \left[\int_{1-t}^1 (1 + t' + t'^2 + \dots + t'^{n-1}) dt' \right] \frac{dt}{t} \\
 &= \int_0^1 \left[\sum_{r=1}^n \frac{1}{r} - \sum_{r=1}^n \frac{(1-t)^r}{r} \right] \frac{dt}{t} = \left(\sum_{r=1}^n \frac{1}{r} \right) \int_0^1 \frac{dt}{t} + \sum_{r=1}^n \frac{1}{r} \int_0^1 \left[\frac{1-t^r}{1-t} - \frac{1}{1-t} \right] dt
 \end{aligned}$$

$$\begin{aligned}
&= \left(\sum_{r=1}^n \frac{1}{r} \right) \int_{+0}^1 \frac{dt}{t} + \sum_{r=1}^n \frac{1}{r} \left(\sum_{q=1}^r \frac{1}{q} - \int_{+0}^1 \frac{dt}{t} \right) = \sum_{r=1}^n \frac{1}{r} \left(\sum_{q=1}^r \frac{1}{q} \right) \\
&= \frac{1}{2} \left(\left(\sum_{r=1}^n \frac{1}{r} \right)^2 + \sum_{r=1}^n \frac{1}{r^2} \right) \tag{100}
\end{aligned}$$

$$\begin{aligned}
E_1^{(2)}(\tau) &= \int_{\tau}^{\infty} \frac{E_1(p)}{p} dp = \int_{\tau}^{\infty} \frac{-\gamma - \log p - \text{eax}(p)}{p} dp \\
&= -\left[\gamma \log p + \frac{(\log p)^2}{2} + \text{ebx}(p) \right]_{\tau}^{\infty} \\
&= \frac{(\gamma + \log \tau)^2}{2} + \frac{\pi^2}{12} + \sum_{m=1}^{\infty} \frac{(-1)^m \tau^m}{n n m!} \tag{101}
\end{aligned}$$

Appendix 3, Derivation of $H_r(\tau, p)$

$$\begin{aligned}
H_r(\tau, p) &= - \sum_{n=1}^{\infty} \left[\left\{ \sum_{q=1}^{\infty} \left(\frac{1}{q} - \frac{1}{n+r+q} \right) \right\} \frac{(-\tau)^n p^n}{n n!} \right] - \left(\sum_{q=1}^r \frac{1}{q} \right) \frac{\log p}{r} \\
&= - \sum_{n=1}^{\infty} \left[\left\{ \sum_{q=1}^{\infty} \left(\frac{n+r}{q(n+r+q)} \right) \right\} \frac{(-\tau)^n p^n}{n n!} \right] - \left(\sum_{q=1}^r \frac{1}{q} \right) \frac{\log p}{r} \\
&= - \sum_{n=1}^{\infty} \left[\left\{ \sum_{q=1}^{\infty} \left(\frac{1}{q(n+r+q)} \right) \left(1 + \frac{r}{n} \right) \right\} \frac{(-\tau)^n p^n}{n!} \right] - \left(\sum_{q=1}^r \frac{1}{q} \right) \frac{\log p}{r} \\
&= - \sum_{q=1}^{\infty} \frac{1}{q} \left[\sum_{n=1}^{\infty} \left(\frac{1}{n+r+q} \right) \left(1 + \frac{r}{n} \right) \frac{(-\tau)^n p^n}{n!} \right] - \left(\sum_{q=1}^r \frac{1}{q} \right) \frac{\log p}{r} \\
&= - \sum_{q=1}^{\infty} \frac{1}{q} \left[A_{r+q}(\tau p) + r G_{r+q}(\tau p) - \frac{r}{r+q} \log p \right] - \left(\sum_{q=1}^r \frac{1}{q} \right) \frac{\log p}{r} \tag{102}
\end{aligned}$$

$H_r(\tau, \infty)$.

$$\begin{aligned}
&H_r(\tau, \infty) \\
&= - \sum_{q=1}^{\infty} \frac{1}{q} \left[A_{r+q}(\tau p) + r G_{r+q}(\tau p) - \frac{r}{r+q} \log p \right] - \left(\sum_{q=1}^r \frac{1}{q} \right) \frac{\log p}{r} \\
&= - \sum_{q=1}^{\infty} \frac{1}{q} \left[-\frac{1}{r+q} + r \left(\frac{1}{(r+q)^2} - \frac{C}{r+q} \right) - \frac{r}{r+q} \log p \right] - \left(\sum_{q=1}^r \frac{1}{q} \right) \frac{\log p}{r} \\
&= - \sum_{q=1}^{\infty} \left[-\frac{1}{(r+q)^2} - \frac{Cr}{q(r+q)} - \frac{r}{q(r+q)} \log p \right] - \left(\sum_{q=1}^r \frac{1}{q} \right) \frac{\log p}{r}
\end{aligned}$$

$$= \sum_{q=1}^{\infty} \frac{1}{(r+q)^2} + C \sum_{q=1}^r \frac{1}{q} = \frac{\pi^2}{6} - \sum_{q=1}^r \frac{1}{q^2} + C \sum_{q=1}^r \frac{1}{q} \quad (103)$$

Appendix 4, Derivation of Coefficient $f_n(p_1, p_2)$

$$\begin{aligned} f_n(p_1, p_2) &= \sum_{r=1}^n \frac{C_r^n p_1^r p_2^{n-r}}{r} = \left[\sum_{r=1}^n \frac{C_r^n y^r p_2^{n-r}}{r} \right]_{y=0}^{y=p_1} \\ &= \int_0^{p_1} \frac{(y+p_2)^n - p_2^n}{y} dy = \int_{p_2}^{p_2+p_1} \frac{x^n - p_2^n}{x-p_2} dx = \int_{p_2}^{p_2+p_1} \sum_{r=0}^{n-1} x^r p_2^{n-r-1} dx \\ &= \sum_{r=1}^n \frac{(p_1+p_2)^r p_2^{n-r}}{r} - \left(\sum_{r=1}^n \frac{1}{r} \right) p_2^n \end{aligned}$$