THE GEOMETRIC SHAPES OF POLYGONAL KNOTS

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(Received September 28, 2001)

1. Introduction

Knots are simple closed curves in the Euclidean 3-space $\mathbb{R}^3$ and are often used to represent nature : circular DNA , other polymers and so on. When knots are used to describe macromolecules in chemistry and biology, it is convenient to regard knots as ones constructed from a sequence of straight line segments rather than smooth and flexible ones. Such a knot obtained from straight line segments is a polygon linearly embedded into $\mathbb{R}^3$, called a polygonal knot. Usually, in the knot theory one studies the topological placement problem of (topological or flexible) knots. When we consider polygonal knots, we are interested in their geometric shapes and their polygonal structures. There are two kinds of canonical shapes for polygonal knots. In this paper, we consider two basic questions about polygonal knots associated with canonical shapes. The first question is the minimal number of straight line segments required among polygonal representations for a given knots. This number is a knot invariant, called the polygon index of a knot [5]. The polygon index gives the elementary measure of complexity for knots, and it is also known as the broken line number [9], the edge number [7, 10] or the stick number [1]. For example, the polygon indices of the unknot, the trefoil knot and the figure-eight knot are 3, 6 and 7 respectively. However, since the polygon index is very geometric, it is difficult to determine the polygon index of every knot as well as the crossing number. In this paper, we consider how to determine the polygon indices of knots with small crossing numbers.

The second question about polygonal knots is the topology of the space of polygonal knots obtained from n-sided polygons embedded into $\mathbb{R}^3$. If two polygonal knots with n segments are connected by a path in this conformation space of n-sided polygonal knots, then the one can be deformed to the other via a piecewise-linear deformation preserving the number of segments. Then such two polygonal knots are said to be geometrically equivalent. That is to say, polygonal knots in a connected component of the conformation space are geometrically equivalent. On the other hand, if there is a topological deformation between two (topological) knots, then we consider that they are topologically equivalent. It is clear that if polygonal knots are geometrically equivalent, then they are topologically equivalent. However, not much is known about the converse. Even if two polygonal knots are topologically equivalent, these knots may not be geometrically equivalent.

We can think that each connected component of the conformation space stands for the geometric shape of polygonal knots. It is important to investigate the number of connected components of the conformation space. In this paper, we investigate the conformation spaces for polygon knots with small number of straight line segments.

In §2, we define knots, polygonal knots and polygon index. There are some types of polygon indices, and we discuss them. In §3, we discuss the relation between the polygon index and other knot invariants. We calculate the polygon indices by making use of the relation. In §4, we prove that the number of straight line segments required for polygonal representations of a knot restricts the form of its Conway polynmial. We determine the polygon indices of knots with crossing number ≤ 6. In §5, we investigate the topologies of the conformation spaces.

2. Polygonal knots and the polygon indices

This section is devoted to definitions of knots (or links), polygonal knots (or links), the polygon indices and so on.

A *link* is a disjoint union of a finite number of simply closed curves in $\mathbb{R}^3$, and in particular, a link with only one component is called a *knot*. In the knot theory, we usually treat a *tame* knot (or link), which is a knot (or link) with regular neighborhood and which is a natural object. Suppose that every knot in this paper is tame. For two knots K_1 and K_2, an *ambient isotopy* of $\mathbb{R}^3$ between K_1 and K_2 is a continuous map $H : \mathbb{R}^3 \times I \to \mathbb{R}^3$ such that H_t is ahomeomorphism on $\mathbb{R}^3$ for any $t \in I$, H_0 is the identity map of $\mathbb{R}^3$ and $H_1(K_1) = K_2$. Here I stands for the unit interval $[0,1]$ and $H_t : \mathbb{R}^3 \to \mathbb{R}^3$ is a map defined by $H_t(x) = H(x,t)$. Two knots K_1 and K_2 are said to be *ambient isotopic* if we find an ambient isotopy of $\mathbb{R}^3$ between K_1 and K_2. This implies that we can deform K_1 to K_2 continuously without crossing itself. It is a basic result of the knot theory that K_1 is ambient isotopic to K_2 if and only if there is an orientation-preserving homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ with $h(K_1) = K_2$. The set of all knots ambient isotopic to a knot K is called the *knot type* of K. The set of all knots is the disjoint union of the family consisting of all the knot types. Thus we can classify knots by topological equivalence.

A knot K is said to be a *trivial* knot or an *unknot* if K is ambient istopic to the unit circle in the xy-plane $\{(x,y,0) \in \mathbb{R}^3 \mid x,y \in \mathbb{R}\}$ and equivalently if K bounds a 2-disk in $\mathbb{R}^3$. (See [11] for the terminology of the knot theory.) A *polygonal knot* (or *link*) is a knot (or link) obtained by joining a finite number of points in $\mathbb{R}^3$, called *vertices*, with straight line segments, called *edges*. We think of a polygonal knot with n edges as an n-sided polygon embedded into $\mathbb{R}^3$ via a linear embedding. In particular, a polygonal knot with n edges is called an *n-sided polygonal knot.* Furthermore, for a knot K, we say that a polygonal knot K' ambient isotopic to K is a *polygonal representation* of K. Two n-sided polygonal knots K_1 and K_2 are *geometrically equivalent* if the one can be deformed to the other via a piecewise-linear deformation preserving the number of edges and the polygonal structure, without crossing itself. It is clear that geometrically equivalent polygonal knots are ambient isotopic. However, it is not known much whether there are polygonal knots which are ambient isotopic but not geometrically equivalent.

Triangles are planar, and so triangular knots are only unknots. Quadrilateral are not always on a plane but any polygonal knot with four edges is the unknot. Pentagonal knots are also only unknots. Of course, if the number of edges is increased, then we can construct polygonal knots of other knot type. In fact, by choosing sufficiently many points on K and by connecting these points with straight line segments we obtain a polygonal representation of K. For example, there are a hexagonal representation of the trefoil knot and heptagonal representation of the figure-eight knot. See Fig. 2.1.

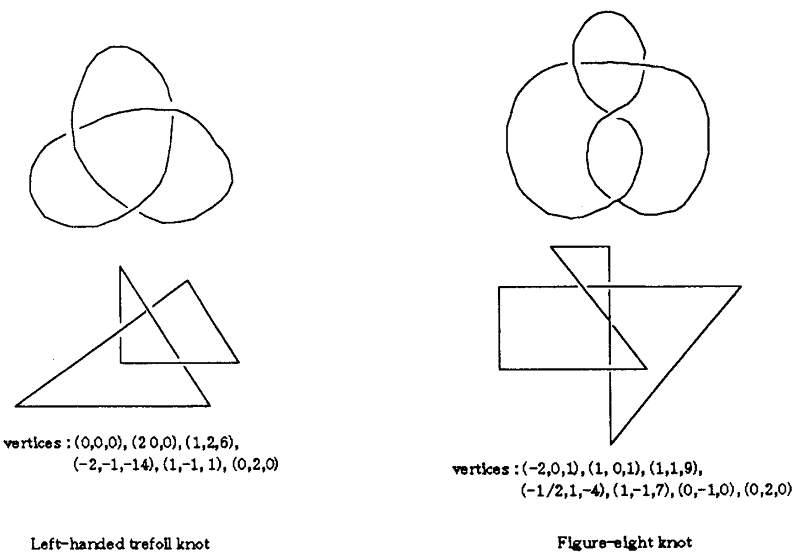

Fig. 2.1

However, it is not always possible to find an n-sided polygonal representation for a given number n and a knot K. In fact, as we discuss in §§3 and 4, the crossing numbers of knots restrict knot types with n-sided polygonal representations.

Thus we pay attention to the number of edges required to construct polygonal representations of a given knot. If a knot K has an n-sided polygonal representation, then it has an m-sided polygonal representation for any number m ($m \geq n$) by adding some vertices on edges of the n-sided polygonal representation. See Fig. 2.2.

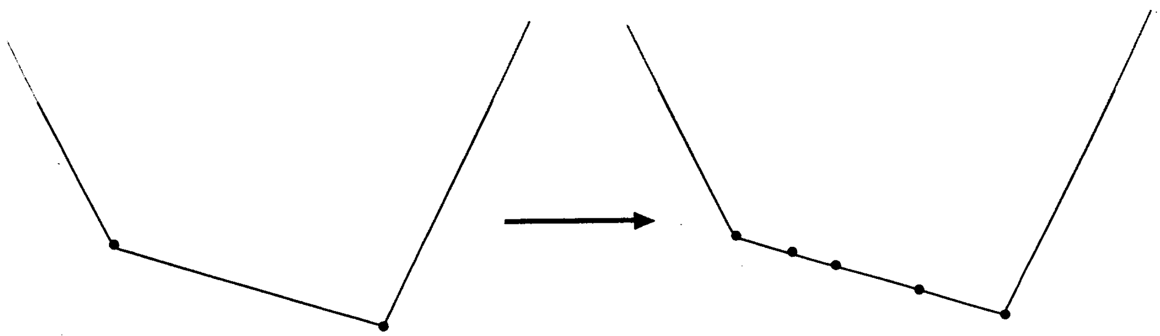

Fig. 2.2

Therefore, it is significant to determine the minimal number of edges among all polygonal representations of a given knot K. The minimal number of edges is called the *polygon index* of K, denoted by $\mathfrak{p}(K)$. We may think that the polygonal representation realizing the polygon index $\mathfrak{p}(K)$ is the most economical and it is a canonical shape of the knot type of K. The lower and upper bounds of the crossing number are given by the polygon index as follows (See §3.) :

$$\frac{5+\sqrt{9+8c(K)}}{2} \leq \mathfrak{p}(K) \leq 2c(K),$$

where $c(K)$ is the crossing number of K. The polygon index of a knot with small crossing number will be not so big. We can determine the polygon indices of knots with small crossing numbers as follows :

Theorem 2.1. 1. *A knot with polygon index 3 is only the unknot.*

2. *Knots with polygon index 6 are only the right-handed trefoil knot and the left-handed*

trefoil knot.

3. *A knot with polygon index 7 is only the figure-eight knot.*
4. *All prime knots with crossing number 5 or 6 have the polygon index 8.*

In §§3 and 4, we state the relation between the polygon index and other knot invariants and show this theorem.

We can restrict our attention to some classes ; One is the class of equilateral knots which are polygonal knots obtained from a linear embedding of a regular polygon with unit-length edges. One of the others is the class of polygonal knots in a cubic lattice where edges lie along the edges of the lattice. The polygon index will vary, depending on the restriction placed on each class. Some kinds of polygon indices are treated in [1] : $s_=(K), s_\perp(K), e_\perp(K)$ and so on. The index $s_=(K)$ is the minimal number of edges among all equilateral representations of K, the index $s_\perp(K)$ is the minimal number of edges among all polygonal representations in a cubic lattice of K and the index $e_\perp(K)$ is the minimal number of edges among all equilateral representations in a unit-length cubic lattice of K. Clearly, $\mathfrak{p}(K)$ is less than or equal to all of the others. For the trefoil knot, we have that $\mathfrak{p}(\text{trefoil knot}) = s_=(\text{trefoil knot}) = 6$, but it is unknown whether the indices $\mathfrak{p}(K)$ and $s_=$ are equal in general or not. It seems that an equilateral representation realizing $s_=$ is another canonical shape.

Next we discuss the properties of polygon indices about connected sums and mirror images. We can obtain a new knot from some knots by taking a connected sum or the mirror image ; Let K_1 and K_2 be knots. Remove a small arc from each of K_1 and K_2. By joining the resultant four endpoints with two "parallel" arcs we can obtain a new knot, called a *connected sum* of K_1 and K_2. The new knot is denoted by $K_1 \sharp K_2$. Notice that a connected sum is not always decided uniquely. If we give an orientation on each knots, then these orientations decide a connected sum uniquely. A knot K is said to be *prime* if for every connected sum decomposition $K = K_1 \sharp K_2$, either K_1 or K_2 is trivial.

Choose a plane Π away from a given knot K and take the orthogonal coordinates such that Π is the xy-plane. Then the reflection map $r : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $r(x, y, z) = (x, y, -z)$ is an orientation-reversing homeomorphism of $\mathbb{R}^3$. The image, rK, of K via the reflection map r is a knot, called the *mirror image* of K.

A knot K is said to be *amphicheiral* if the mirror image rK is ambient isotopic to K. Using the terminology of chemistry, we say also that K is *achiral* if K is amphicheiral, and

K is *chiral* elsewhere. For example, the trefoil knot is chiral, that is, the right-handed trefoil knot is not ambient isotopic to the left-handed trefoil knot which is the mirror image of the right-handed one. But the figure-eight knot is achiral.

We have the followings about the polygon indices of a connected sum and the mirror image.

Theorem 2.2 ([5], [1]). *The inequality*

$$\mathfrak{p}(K_1 \sharp K_2) \leq \mathfrak{p}(K_1) + \mathfrak{p}(K_2) - 3$$

holds for any connected sum of two knots K_1 and K_2.

Outline of the proof. If K_1 or K_2 is trivial, then the equality holds.

Let K_1 and K_2 be nontrivial polygonal knots realizing the polygon indices. Let $\mathfrak{p}(K_1) = n_1$ and $\mathfrak{p}(K_2) = n_2$. Let e_1 and e_2 be edges of K_1 and K_2 respectively which we will use to take a connected sum of K_1 and K_2. We begin with the case where the edge e_1 is contained in the boundary of the convex hull of K_1, called an *external* edge in [5]. Let d_i, e_i, f_i be three consecutive edges of K_i for $i = 1, 2$. By $\vec{d_i}, \vec{e_i}, \vec{f_i}$ we denote the corresponding vectors with their directions determined by an orientation of K_i for $i = 1, 2$. Assume that $\det(\vec{d_1}\ \vec{e_1}\ \vec{f_1}) > 0$ and $\det(\vec{d_2}\ \vec{e_2}\ \vec{f_2}) > 0$. We deform K_1 and K_2 via orientation-preserving linear homeomorphisms of $\mathbb{R}^3$ obtained by Gram-Schmidt orthonormalizations on $\{\vec{d_1}, \vec{e_1}, \vec{f_1}\}$ and $\{\vec{d_2}, \vec{e_2}, \vec{f_2}\}$ respectively. Let the resultant polygonal knots be still denoted by K_1 and K_2. Using a parallel transformation and/or a rotation on $\mathbb{R}^3$, move K_1 to K_2 so that e_1 and e_2 match and so that either $d_1 \cup d_2$ or $d_1 \cup f_2$ becomes one straight line segment. See Fig. 2.3.

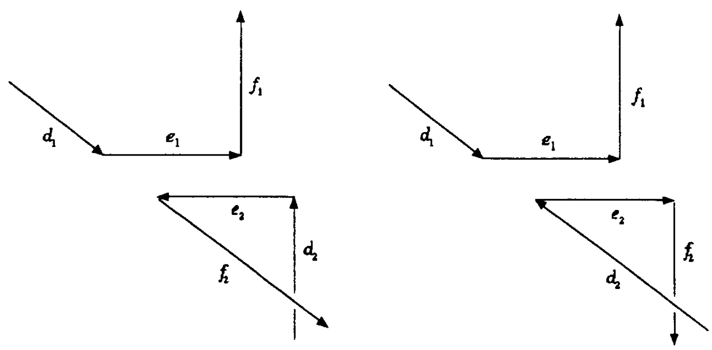

Fig. 2.3

Remove e_i from K_i for $i = 1, 2$. Take a connected sum of K_1 and K_2. Then the number of edges of $K_1 \sharp K_2$ is $n_1 + n_2 - 4$. Hence, we have that $\mathfrak{p}(K_1 \sharp K_2) \leq n_1 + n_2 - 4 = \mathfrak{p}(K_1) + \mathfrak{p}(K_2) - 4$.

We turn back to the general case. Since K_2 is not trivial, we can find an edge e_2 of K_2 such that three consecutive edges of K_2 containing e_2 can not lie on a common plane. There is not always an external edge of K_1 but we can choose an external vertex of K_1, say v. We consider the straight line segment obtained by joining a point on one of two edges incident to v to a point on the other edge. Then we choose these points so that the straight line segment obtained by joining them lies in a sufficiently small neighborhood of v. Although we increase the number of edges of K_1 by one, we can get an external edge e_1 on a polygonal knot ambient isotopic to K_1. See Fig. 2.4.

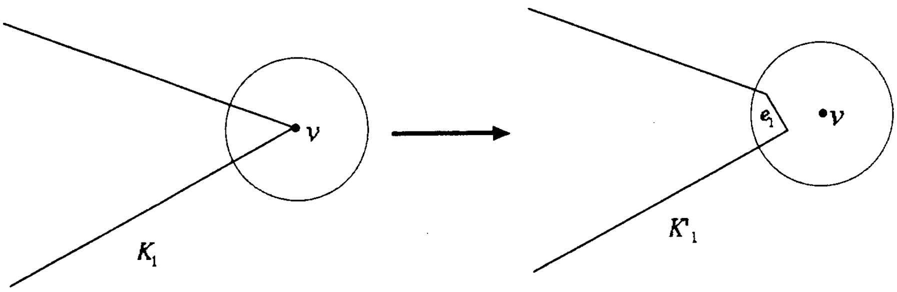

Fig. 2.4

Thus it follows from the same argument as the preceding that $\mathfrak{p}(K_1 \sharp K_2) \leq (n_1 + 1) + n_2 - 4 = n_1 + n_2 - 3 = \mathfrak{p}(K_1) + \mathfrak{p}(K_2) - 3$. □

It is shown in [1] that the similar inequalities hold for the indices $s_=$, $s_\perp$ and $e_\perp$.

Example 2.1. The square knot is the connected sum $3_1 \sharp r3_1$ of the left-handed trefoil knot 3_1 and the right-handed trefoil knot $r3_1$. The granny knot is the connected sum $3_1 \sharp 3_1$ of two copies of the left-handed trefoil knot 3_1. Since two types of the trefoil knot have hexagonal representations with external edges, it follows from the proof of Theorem 2.2 that the square knot $3_1 \sharp r3_1$ and the granny knot $3_1 \sharp 3_1$ satisfy the inequalities

$$\mathfrak{p}(3_1 \sharp r3_1) \leq 6 + 6 - 4 = 8 \text{ and } \mathfrak{p}(3_1 \sharp 3_1) \leq 8.$$

In fact, it is shown in §4 that $\mathfrak{p}(3_1 \sharp r3_1) = 8$ and $\mathfrak{p}(3_1 \sharp 3_1) = 8$.

Theorem 2.3. *For any knot K, $\mathfrak{p}(rK) = \mathfrak{p}(K)$. Furthermore, this property also holds for other indices $s_=$, $s_\perp$ and $e_\perp$.*

Proof. Let K be a polygonal representation realizing the polygon index $\mathfrak{p}$ and let $\mathfrak{p}(K) = n$. Since the reflection map $r : \mathbb{R}^3 \to \mathbb{R}^3$ is linear, $r(K)$ is an n-sided polygonal representation of the mirror image rK of K. Hence we have that $\mathfrak{p}(rK) \leq n = \mathfrak{p}(K)$. Noting that $r(rK) = K$, we have that $\mathfrak{p}(K) \leq \mathfrak{p}(rK)$ and so that $\mathfrak{p}(rK) = \mathfrak{p}(K)$. □

3. The relation between the polygon index and other knot invariants

In this section, we consider the relation between the polygon index and other knot invariants : the crossing number, the superbridge index and the bridge index. We will introduce the way to estimate the polygon index, argumented in [5] and [1].

Recall that the *crossing number* of a knot K, denoted by $c(K)$, is defined as the minimal number of crossings taken over all the general position projections of K into a plane or sphere. The crossing number is the most fundamental knot invariant and gives the most elementary measure of complexity for knots. For example, the knot table in [11] gives us the classification of prime knots with respect to their crossing numbers. In [9], Negami proved that the following inequality holds for any nontrivial knot :

$$\frac{5+\sqrt{9+8c(K)}}{2} \leq \mathfrak{p}(K) \leq 2c(K).$$

In [3], Calvo improved this inequality and showed the following :

Theorem 3.1 ([3]). *For any knot K,*

$$\frac{7+\sqrt{1+8c(K)}}{2} \leq \mathfrak{p}(K)$$

Next we define the bridge indices and the superbridge indices of knots. Given a knot K or a knot type, let K' be a knot ambient isotopic to K. For a unit vector z in $\mathbb{R}^3$, let $b_z(K')$ denote the number of local maxima of the orthogonal projection of K' into the axis $\mathbb{R}z$. Then we define the *bridge index* of K, denoted by $\mathfrak{b}(K)$, and the *superbridge index* of K, denoted by $\mathfrak{sb}(K)$, as follows (see [12], [11], [6]) :

$$\mathfrak{b}(K) = \min_{K' \sim K} \min_{z \in S^2} b_z(K') \quad \text{and} \quad \mathfrak{sb}(K) = \min_{K' \sim K} \max_{z \in S^2} b_z(K'),$$

where the symbol "$\sim$" implies that two knots are ambient isotopic.

Proposition 3.1. *For any knot K, $2\mathfrak{sb}(K) \leq \mathfrak{p}(K)$.*

Proof. Let K' be any polygonal representation of K. Suppose that K' consists of n edges. Then K' has n vertices. Since each edge is a straight line segment and each vertex is incident to edges, we have that $b_z(K') \le n/2$ for any $z \in S^2$, and so $\max_{z \in S^2} b_z(K') \le n/2$. Hence, we have that

$$\mathfrak{sb}(K) = \min_{K'' \sim K} \max_{z \in S^2} b_z(K'') \le \max_{z \in S^2} b_z(K') \le \frac{n}{2}.$$

Since $\mathfrak{sb}(K)$ is a constant independent of polygonal representations of K, we have that $\mathfrak{sb}(K) \le (1/2) \min \{n \mid n \text{ is the number of edges of a polygonal representation of } K\} = \mathfrak{p}(K)/2$. □

In [5], the polygon indices of torus knots are treated. A *torus knot* is a knot which is ambient isotopic to a simple closed curve on the unknotted torus $T^2 = \{(x,y,z) \in \mathbb{R}^3 \mid (\sqrt{x^2+y^2}-2)^2 + z^2 = 1\}$. In particular, a knot K is called the *torus knot of type* (p,q), denoted by $T_{p,q}$, if a simple closed curve on T^2 which is ambient isotopic to K represents the homology class $p\mathbf{l} + q\mathbf{m} \in H_1(T^2;\mathbb{Z})$. Here, $\mathbf{l}$ is the homology class represented by the longitude $\{(x,y,0) \in T^2 \mid x^2+y^2 = 9\}$ and $\mathbf{m}$ is the homology class represented by the meridian $\{(x,0,z) \in T^2 \mid (x-2)^2 + z^2 = 1\}$. For example, the trefoil knot is the torus knot of type $(2,3)$ but the figure-eight knot is not a torus knot. Notice that if K is the torus knot of type (p,q), then integers p and q must be relatively prime and that the torus knot $T_{(p,q)}$ is ambient isotopic to $T_{(q,p)}$.

For relatively prime integers p and q satisfying $2 \le p < q$, we will construct the torus knot $T_{(p,q)}$ as a polygonal knot. Let C_- be the unit circle centered at $(0,0,-1)$ in the plane with equation $z = -1$ and let C_+ be the unit circle centered at $(0,0,1)$ in the plane with equation $z = 1$. We choose a number α with $\pi p/q < \alpha < \min(\pi, 2\pi p/q)$, and we define $X_k = (\cos(2\pi pk/q), \sin(2\pi pk/q), -1)$, $X'_k = (\cos(2\pi pk/q), \sin(2\pi pk/q), 1)$ and $Y_k = (\cos(2\pi pk/q + \alpha), \sin(2\pi pk/q + \alpha), 1)$ for $k = 0,1,\ldots,q-1$. Then the collection of straight line segments, $\{\overline{X_0Y_0},\ \overline{X_1Y_1},\ \ldots,\ \overline{X_{q-1}Y_{q-1}}\}$, is obtained from q vertical line segments $\overline{X_0X'_0},\ \overline{X_1X'_1},\ \ldots,\ \overline{X_{q-1}X'_{q-1}}$ on the cylinder $\{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 = 1\}$ by rotating C_+ and q points $X'_0, X'_1, \ldots, X'_{q-1}$ about the z-axis through the angle α and by carrying these q vertical lines segments along while fixing their bottom endpoints $X_0, X_1, \ldots, X_{q-1}$ on C_-. Thus the collection $\{\overline{X_0Y_0},\ \overline{X_1Y_1},\ \ldots,\ \overline{X_{q-1}Y_{q-1}}\}$ lies on a hyperboloid centered about the z-axis. Let H_α be the annulus obtained from the part $|z| \le 1$ of this hyperboloid. Then $\overline{X_kY_k} \subset H_\alpha$ for $k = 0,1,\ldots,q-1$. In the same manner, it is seen that the second collection $\{\overline{Y_0X_1},\ \overline{Y_1X_2},\ \dots,\ \overline{Y_{q-1}X_0}\}$ lies on another hyperboloid centered about the z-axis. If we let H_β be the annulus obtained from the part $|z| \leq 1$ of this hyperboloid, then $\overline{Y_kX_{k+1}} \subset H_\beta$ for $k = 0, 1, \dots, q-1$. Here $X_q = X_0$. Thus the spatial polygon $K_{p,q} = \cup_{k=0}^{q-1}\left(\overline{X_kY_k} \cup \overline{Y_kX_{k+1}}\right)$ lies on the torus $H_\alpha \cup H_\beta$. The circle C_- is a longitude of the torus $H_\alpha \cup H_\beta$ and the polygon $K_{p,q}$ intersects C_- at q points $X_0, X_1, \dots, X_{q-1}$ transversely. Furthermore, the polygon $K_{p,q}$ intersects the half-plane $\{(x, 0, z) \in \mathbb{R}^3 \mid x > 0\}$ at p points, and so $K_{p,q}$ intersects a meridian $(H_\alpha \cup H_\beta) \cap \{(x, 0, z) \in \mathbb{R}^3 \mid x > 0\}$ of the torus $H_\alpha \cup H_\beta$ at p points transversely. Therefore, the polygon $K_{p,q}$ is a polygonal representation of the torus knot $T_{p,q}$ of type (p, q), which has $2q$ edges. Thus we have the following :

Proposition 3.2 ([5]). *For relatively prime integers p and q with $2 \leq p < q$, $\mathfrak{p}(T_{p,q}) \leq 2q$.*

Kuiper showed in [6] that $\mathfrak{sb}(T_{p,q}) = \min(2p, q)$ for relatively prime integers p and q with $2 \leq p < q$. Hence it follows from Proposition 3.1 that $\mathfrak{p}(T_{p,q}) \geq \min(4p, 2q)$. Thus by combining with Proposition 3.2 we have the following :

Theorem 3.2 ([5]). *If p and q are relatively prime integers with $2 \leq p < q < 2p$, then $\mathfrak{p}(T_{p,q}) = 2q$.*

Next we will consider the way developed in [1], which uses the concept of the total curvature of a knot. The *total curvature* of a polygonal knot K, denoted by $\kappa(K)$, is defined as follows (see [8]) ; Let $v_0, v_1, \dots, v_n = v_0$ be the vertices of K and let $e_1, e_2, \dots, e_n$ be the edges of K such that e_i and e_{i+1} are the adjacent edges joined at v_i for $i = 1, 2, \dots, n$. Furthermore, let θ_i be the exterior angle at the vertex v_i. See Fig. 3.1.

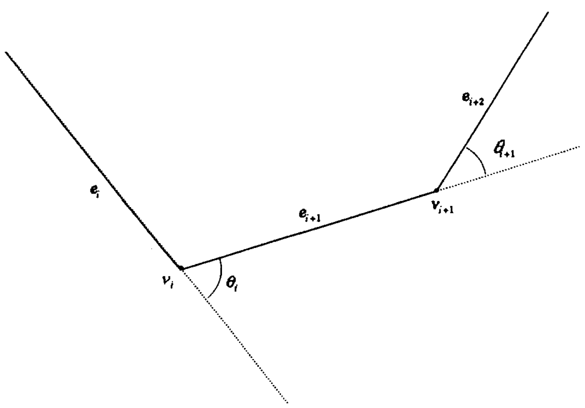

Fig. 3.1

We define the total curvature of K by

$$\kappa(K) = \sum_{i=1}^{n} \theta_i.$$

Then Milnor proved in [8] that

$$\kappa(K) \geq 2\pi \mathfrak{b}(K).$$

We generalize the result in [1] as follows. By the definitions of the bridge index and the superbridg index it is clear that $\mathfrak{sb}(K) > \mathfrak{b}(K)$, and so Theorem 3.3 follows from Proposition 3.1. However, we give another proof using the total curvature.

Theorem 3.3. *For a knot K, $\mathfrak{p}(K) \geq 2(\mathfrak{b}(K)+1)$.*

Proof. Let $\mathfrak{b}(K) = n$ and suppose that K is a polygonal knot with $2n+1$ edges. We choose an edge e of K and consider a plane Π perpendicular to e. Let $f : \mathbb{R}^3 \to \Pi$ be the orthogonal projection. Notice that f is linear and the image $f(K)$ of K is a polygon in Π which may intersect itself. Since the edge e projects to a single point, $f(K)$ consists of $2n$ edges. Since the exterior angle at each vertex of $f(K)$ is less than π, we have that $\kappa(f(K)) < 2n\pi$. Let m be the number of self-intersections in $f(K)$ and let ε be an arbitrary positive number. Then at each self-intersection we choose the edge which should be overpass and we bend it to get another representation of K. Cross it over up by an angle of $\varepsilon/(8m)$ a small distance of d before the crossing, down by an angle of $\varepsilon/(4m)$ just over the crossing, and then back up by angle of $\varepsilon/(8m)$ a distance of d beyond the crossing. See Fig. 3.2.

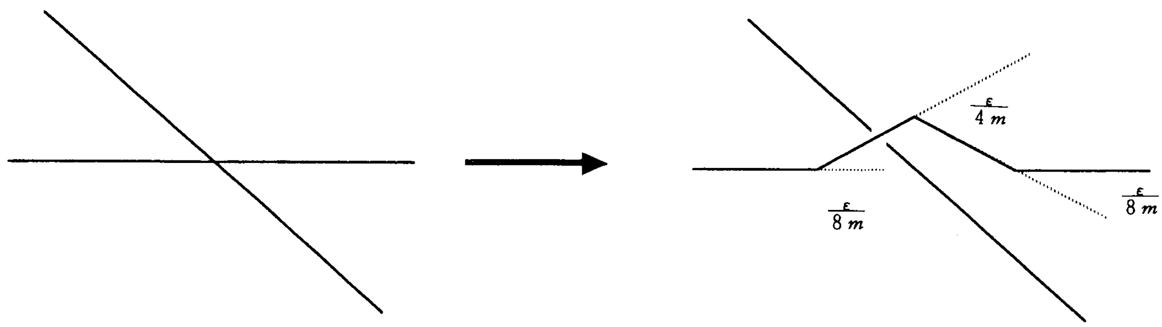

Fig. 3.2

The resultant knot K' is another polygonal representation of K. This local operation at each self-intersection contributes $\varepsilon/(2m)$ to the total curvature. Hence, we have that

$$\kappa(K') = \kappa(f(K)) + \sum_{x:\text{self-intersection}} \left(\frac{\varepsilon}{2m}\right) = \kappa(f(K)) + \frac{\varepsilon}{2} < \kappa(f(K)) + \varepsilon.$$

By a Milnor's theorem, it holds that $\kappa(K') \geq 2\pi\mathfrak{b}(K') = 2n\pi$ and so $2n\pi < \kappa(f(K))+\varepsilon$. If $\varepsilon \longrightarrow 0$, then we have that $\kappa(f(K)) \geq 2n\pi$. This contradicts the fact that $\kappa(f(K)) < 2n\pi$. Therefore, we need $2n+2$ straight line segments to construct a polygonal representation of K. □

Schubert calculated the bridge indices of torus knots in [12] and proved that $\mathfrak{b}(T_{p,q}) = \min(p,q)$. Hence it follows from Proposition 3.2 and Theorem 3.3 that $\mathfrak{p}(T_{q-1,q}) = 2q$. More generally, we have

Corollary 3.1. *If K is an $(n-1)$-bridge knot and has a polygonal representation with $2n$ edges, then $\mathfrak{p}(K) = 2n$.*

4. Polygon indices and Conway polynomials

Polynomial invariants of knots are important ones to classify knots and they center in the knot theory. The first polynomial invariant is the *Alexander polynomial.* It is known that Alexander polynomials are defined in the vary ways using Fox's free calculus on the funadamental group of the knot complement, the homology group of the universal abelian cover of the knot complement, the Seifert form or the recursive formula. Conway found the way to calculate the Alexander polynomial by the recursive formula, called the *skein relation.* We call the representation of the Alexander polynomial via the skein relation the *Conway polynomial.* The Conway polinomial $\nabla_K(z)$ of K is a polynomial in $\mathbb{Z}[z]$ and defined by the following relations :

1. If K is ambient isotopic to K', then $\nabla_K(z) = \nabla_{K'}(z)$.
2. If K is the unknot, then $\nabla_K(z) = 1$.
3. $\nabla_{K_+}(z) - \nabla_{K_-}(z) = z\nabla_{K_0}(z)$.

Here, by K_+, K_-, K_0 we denote the links differing only locally as shown :

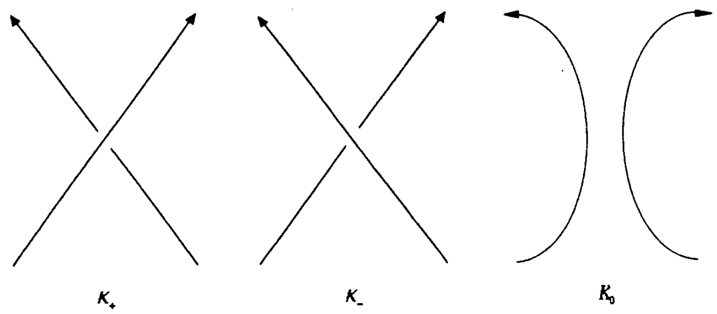

Fig. 4.1

In [10], Randell showed that the Alexander polynomial of a polygonal knot with six or fewer edges has a restricted form. Then we have the following :

Theorem 4.1. *A polygonal knot with seven or fewer edges has the Conway polynomial of the unknot, a trefoil knot or the figure-eight knot.*

Proof. Let K be a polygonal knot with vertices $v_1, v_2, \dots, v_7$ and with edges $e_1, e_2, \dots, e_7$ such that the edge e_i connects v_i and v_{i+1} for $i = 1, 2, \dots, 7$, where $v_8 = v_1$. We give K an orientation according to the order of a sequence $v_1, v_2, \dots, v_7$. Let T be the triangle spanned by three vertices v_1, v_2, v_3 and let S be the quadrilateral spanned by four vertices v_4, v_5, v_6, v_7.

Case 1 : $T \cap S = \emptyset$.

We define the map $h : I \to \mathbb{R}^3$ by

$$h(t) = (1-t)v_2 + \frac{t}{2}(v_1 + v_3).$$

Then the map h gives the isotopy which moves v_2 in a straight line path across T to the midpoint of the straight line segment $\overline{v_1 v_3}$ without changing the other six vertices. The resultant polygonal knot K' is ambient isotopic to K and we may assume that K' consists of five edges. Since a pentagonal knot is the unknot, K' is the unknot and K is so. Hence $\nabla_K(z) = 1$. See Fig. 4.2.

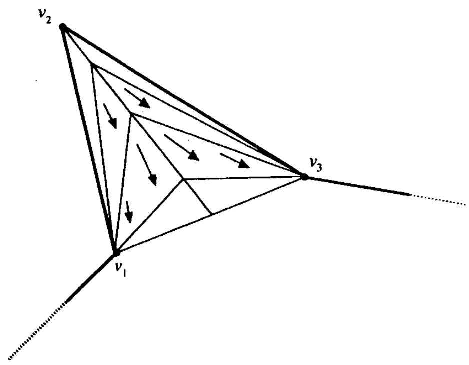

Fig. 4.2

Case 2 : $T \cap S \neq \emptyset$. Let T_1 be the triangle spanned by three vertices v_4, v_5, v_6 and let T_2 be the triangle spanned by three vertices v_4, v_6, v_7. The quadrilateral S consists of two triangles T_1 and T_2 hinged along the straight line segment $\overline{v_4 v_6}$. Then it is possible to consider the following five cases essentially.

(2-i) $e_i \cap T_j \neq \emptyset$ $(i, j = 1, 2)$.

(2-ii) $e_2 \cap T_1 = \emptyset, e_2 \cap T_2 \neq \emptyset$ and $e_1 \cap T_i = \emptyset$ $(i = 1, 2)$.

(2-iii) $e_1 \cap T_1 \neq \emptyset, e_1 \cap T_2 = \emptyset$ and $e_2 \cap T_i \neq \emptyset$ $(i = 1, 2)$.

(2-iv) $e_1 \cap T_i \neq \emptyset$ $(i = 1, 2)$ and $e_2 \cap T_j = \emptyset$ $(j = 1, 2)$.

(2-v) $e_2 \cap T_1 \neq \emptyset, e_2 \cap T_2 = \emptyset, e_1 \cap T_1 = \emptyset$ and $e_1 \cap T_2 \neq \emptyset$.

See Fig. 4.3.

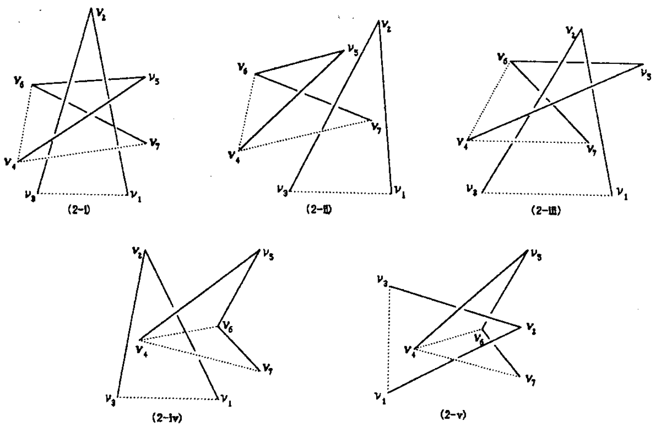

Fig. 4.3

Subcase (2-i) : In the same manner as Case 1, the isotopy obtained from the triangle T deforms K to a pentagonal knot. Hence K is the unknot and $\nabla_K(z) = 1$.

Subcase (2-ii) : We consider the skein relation. We assume that K is shown as Fig. 4.4 and $K = K_+$.

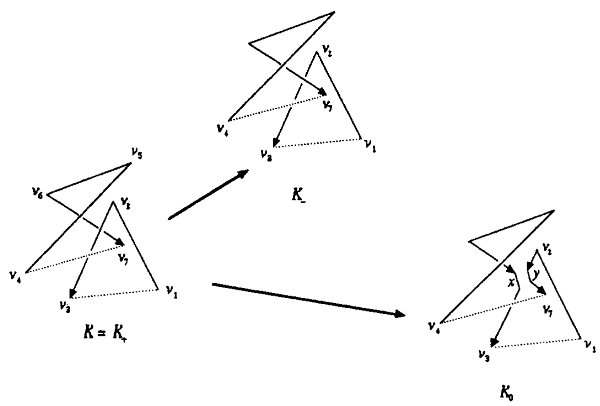

Fig. 4.4

It follows from Case 1 that K_- is the unknot and $\nabla_{K_-}(z) = 1$. By shrinking the new edges x, y of K_0 and by modifying K_0, we can regard K_0 as a two-component polygonal link which consists of a component with four edges and a component with five edges. Hence K_0 is either the trivial link or the Hopf link. If K_0 is the trivial link, then $\nabla_{K_0}(z) = 0$ and it follows from the skein relation that $\nabla_K(z) = \nabla_{K_+}(z) = \nabla_{K_-}(z) + z\nabla_{K_0}(z) = 1$. If K_0 is the Hopf link, then $\nabla_{K_0}(z) = z$ and it follows from the skein relation that $\nabla_K(z) = \nabla_{K_-}(z) + z\nabla_{K_0}(z) = 1 + z^2$.

Next we assume that K is shown as below and $K = K_-$. See Fig. 4.5.

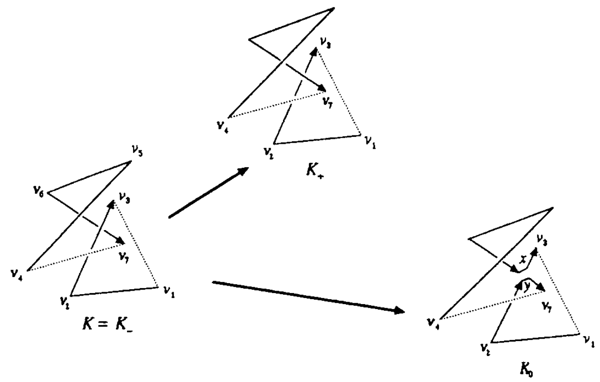

Fig. 4.5

Since K_+ is the unknot, $\nabla_{K_+}(z) = 1$. Since K_0 is either the trivial link or the Hopf link, we have that either $\nabla_{K_0}(z) = 0$ or $\nabla_{K_0}(z) = z$. Hence, by the skein relation we have that $\nabla_K(z) = \nabla_{K_-}(z) = \nabla_{K_+}(z) - z\nabla_{K_0}(z) = 1$ or $\nabla_K(z) = \nabla_{K_+}(z) - z\nabla_{K_0}(z) = 1 - z^2$.

Subcase (2-iii) : Since the triangle T_1 intersects e_1 and e_2, let w_i be the intersection between T_1 and e_i for $i = 1, 2$. Slide v_2 on the triangle spanned by three vertices v_2, w_1, w_2 and push v_2 through the triangle T_1. See Fig. 4.6.

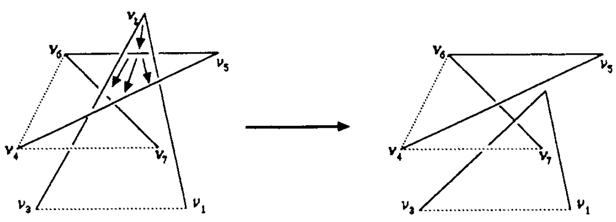

Fig. 4.6

Thus we can result in Subcase (2-ii).

Subcase (2-iv) : We may assume that K is shown as below and $K = K_-$, for we can prove also the case where K is shown as $K = K_+$ in the same manner. See Fig. 4.7.

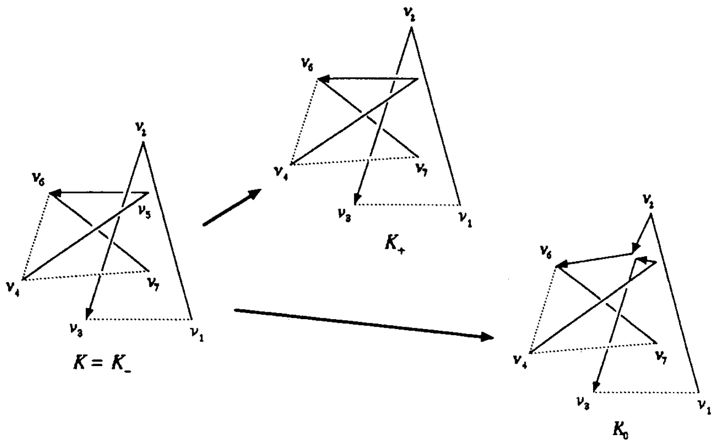

Fig. 4.7

It follows from Subcase (2-ii) that $\nabla_{K_+}(z) = 1$ or $1 + z^2$. Since it is seen that K_0 is either the trivial link or the Hopf link, the Conway polynomial of $K = K_-$ has the form of $1, 1 + z^2$ and $1 - z^2$.

Subcase (2-v) : By modifying K we can result in Subcase (2-iv). See Fig. 4.8.

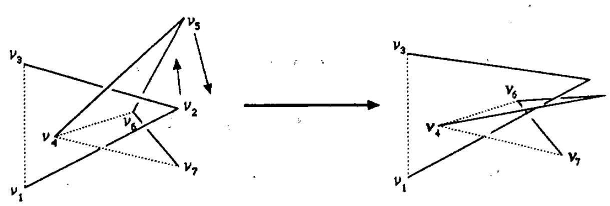

Fig. 4.8

Since the Conway polynomials of the unknot, a trefoil knot and the figure-eight knot have the forms of $1, 1 + z^2$ and $1 - z^2$ respectively, this completes the proof. □

It follows from Theorem 4.1 that if K dose not have the Conway polynomial of the unknot, a trefoil knot or the figure-eight knot, then $\mathfrak{p}(K) \geq 8$. Therefore, if K is a prime knot with crossing number $c(K) \geq 5$, then $\mathfrak{p}(K) \geq 8$. A prime knot K with crossing number 5 or 6 has an octagonal representation, indicated in [7]. Thus Theorem 2.1(4) follows from Theorem 4.1. Since the square knot $3_1 \sharp r 3_1$ and the granny knot $3_1 \sharp 3_1$ have octagonal representation and their Conway polynomial have the form of $(1 + z^2)^2$, $\mathfrak{p}(3_1 \sharp r 3_1) = \mathfrak{p}(3_1 \sharp 3_1) = 8$.

5. Conformation spaces of polygonal knots

We will consider the set of all n-sided polygonal knots in $\mathbb{R}^3$. We may view an n-sided polygonal knot K as a point of $(\mathbb{R}^3)^n = \mathbb{R}^{3n}$ by listing the triple of coordinates for each of its n vertices. Given a point $(v_1, v_2, \ldots, v_n) \in (\mathbb{R}^3)^n = \mathbb{R}^{3n}$, we can construct a spatial polygon P by connecting vertices $v_1, v_2, \ldots, v_n$ with straight line segments, in turn. We denote this polygon P by $P = < v_1, v_2, \ldots, v_n >$. All points in $\mathbb{R}^{3n}$ do not always correspond to n-sided polygonal knots. If $P = < v_1, v_2, \ldots, v_n >$ is an n-sided polygon knot, then the point $(v_1, v_2, \ldots, v_n) \in \mathbb{R}^{3n}$ have to satisfy the followings :

(1) no two vertices v_i and v_j are equal, and

(2) no two edges $\overline{v_i v_{i+1}}$ and $\overline{v_j v_{j+1}}$ meet except at one common endpoint.

Let $\Sigma^{(n)}$ be the set of all points in $\mathbb{R}^{3n}$ corresponding to spatial polygons with self-intersections, called the *discriminant.* Set $\mathfrak{Geo}^{(n)} = \mathbb{R}^{3n} - \Sigma^{(n)}$ and give $\mathfrak{Geo}^{(n)}$ and $\Sigma^{(n)}$ the topologies coming from $\mathbb{R}^{3n}$. A point $(v_1, v_2, \ldots, v_n) \in \mathfrak{Geo}^{(n)}$ corresponds to an n-sided polygonal knot, and so we call $\mathfrak{Geo}^{(n)}$ the *conformation space* of n-sided polygon knots. A path $\omega : I \to \mathfrak{Geo}^{(n)}$ connecting two points $(v_1, v_2, \ldots, v_n)$ and $(w_1, w_2, \ldots, w_n)$ corresponds to an isotopy between $K_1 = < v_1, v_2, \ldots, v_n >$ and $K_2 = < w_1, w_2, \ldots, w_n >$. Hence, if polygonal knots lie on the same path-component of $\mathfrak{Geo}^{(n)}$, then they are geometrically equivalent. Thus it is important to investigate the topology of $\mathfrak{Geo}^{(n)}$, in particular the connected components of $\mathfrak{Geo}^{(n)}$. If we do so, then we will understand the shape of polygonal knots.

Notice that every point of $\mathfrak{Geo}^{(n)}$ depends on a choice of a sequential labeling $v_1, v_2, \ldots, v_n$ for the vertices, that is , a different choice of labels lead to a different points in $\mathfrak{Geo}^{(n)}$ corresponding to the same polygonal knot. Furthermore, rigid motions of $\mathbb{R}^3$ which are parallel transformations or rotations give also a different points in $\mathfrak{Geo}^{(n)}$. Hence the dihedral group $\mathbb{D}_n$ of order $2n$ and the group $\mathbb{R}^3 \rtimes SO(3)$ of rigid motions on $\mathbb{R}^3$ act on $\mathfrak{Geo}^{(n)}$. Here the dihedral group $\mathbb{D}_n$ is the symmetry group of regular n-sided polygon in a plane and is generated by the shift $\mathbf{s}$ of its vertices and the reflection $\mathbf{r}$ along a symmetry axis. The action of $\mathbb{D}_n$ on $\mathfrak{Geo}^{(n)}$ is defined by

$$\mathbf{s} : < v_1, v_2, \ldots, v_n > \mapsto < v_n, v_1, \ldots, v_{n-1} >$$

$$\mathbf{r} : < v_1, v_2, \ldots, v_n > \mapsto < v_1, v_n, v_{n-1}, \ldots, v_2 >$$

These actions do not change the geometric equivalences of polygonal knots. Therefore, the quotient space $\mathbb{D}_n \backslash \mathfrak{Geo}^{(n)} / \mathbb{R}^3 \rtimes SO(3)$ gives the real shapes of n-sided polygonal knots. However, in this paper we consider the conformation space $\mathfrak{Geo}^{(n)}$.

We can find the followings in [2].

Theorem 5.1 ([2]). *The conformation space $\mathfrak{Geo}^{(n)}$ is a dense open subset of $\mathbb{R}^{3n}$. If $n > 1$, then the discriminant $\Sigma^{(n)}$ is the union of the closure of some semi-algebraic varieties of codimension 1.*

It follows from Theorem 3.1 that for a given integer n, the crossing number of a knot K with $\mathfrak{p}(K) = n$ is bounded above and below. Hence there are knots of only finitely many knot types which have n-sided polygonal representations, that is, $\mathfrak{Geo}^{(n)}$ consists of knots of only finitely many knot types. Moreover, one can prove the following theorem by using a Whitney's theorem about rel algebraic varieties :

Theorem 5.2 ([2]). *The conformation space $\mathfrak{Geo}^{(n)}$ has only finitely many components.*

Since every triangle in $\mathbb{R}^3$ lies on a plane, any two triangles are geometrically equivalent. Furthermore, by reversing triangles if necessary, it is seen that $\mathfrak{Geo}^{(3)}$ is path-connected. The discriminant $\Sigma^{(3)}$ is defined by

$$\Sigma^{(3)} = \{(v_1, v_2, v_3) \in \mathbb{R}^9 \mid \text{three vertices } v_1, v_2, v_3 \text{ lie on a straight line in } \mathbb{R}^3\}$$

By parallel transformations on $\mathbb{R}^3$, it is seen that $\Sigma^{(3)}$ is homeomorphic to the space $\mathbb{R}^3 \times \{(w_1, w_2) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid$ two points w_1, w_2 lie on a straight line through the origin$\}$. Recall that the total space τ of the tautological line bundle over $\mathbb{R}P^2$ is defined by

$$\begin{aligned} \tau &= \{(\ell, v) \in \mathbb{R}P^2 \times \mathbb{R}^3 \mid v \in \ell\} \\ &= \{([w], v) \in \mathbb{R}P^2 \times \mathbb{R}^3 \mid v \in [w]\}, \end{aligned}$$

where $[w]$ sdands for the straight line connecting the origin and w. The second factor of that space is homeomorphic to $\mathbb{R} \times \tau$. Hence the discriminant $\Sigma^{(3)}$ is homeomorphic to $\mathbb{R}^4 \times \tau$. A quadrilateral seems to be a burrerfly with two triangles , and we can fatten wings out. Thus since every quadrilaterale is planar, $\mathfrak{Geo}^{(4)}$ is path-connected. Itis also seen that $\mathfrak{Geo}^{(5)}$ is path-connected, and that the spaces $\mathfrak{Geo}^{(3)}, \mathfrak{Geo}^{(4)}$ and $\mathfrak{Geo}^{(5)}$ consist of unknots.

Since knots whose polygon indices are less than equal to 6 are unknots and trefoil knots. Notice that the trefoil knot is chiral and so the left-handed trefoil knot and the right-handed trefoil knot are not geometrically equivalent. Hence $\mathfrak{Geo}^{(6)}$ includes at least three connected components consisting of unknots, left-handed trefoil knots and right-handed trefoil knots. Noticing that the figure-eight knot is achiral, it follows from Theorem 1.1 and Theorem 4.1 that $\mathfrak{Geo}^{(7)}$ includes at least five connected components. In the same manner, $\mathfrak{Geo}^{(8)}$ contains connected components consisting of all prime knots with crossing number ≤ 6, the square knot and the granny knot, and consisting of the mirror images of these knots except achiral knots like the unknot, the figure-eight knot, the square knot and so on.

We have the following about the fundamental groups of the conformation spaces.

Theorem 5.3. *The fundamental group of each component $\mathfrak{K}$ of $\mathfrak{Geo}^{(n)}$ is isomorphi to $G \times \mathbb{Z}_2$ for some group G.*

Proof. Let $\mathfrak{K}$ be a connected component of $\mathfrak{Geo}^{(n)}$. The group $\mathbb{R}^3 \rtimes SO(3)$ acts on $\mathfrak{K}$ freely. Hence $\mathfrak{K}$ is a principal bundle with fiber $\mathbb{R}^3 \rtimes SO(3)$. For each equivalence class in $\mathfrak{K}/\mathbb{R}^3 \rtimes SO(3)$, we can choose a unique representative by placing the first vertex v_1 on the origin, v_2 on the positive x-axis and v_3 on the upper-half xy-plane. This gives us a section $\sigma : \mathfrak{K}/\mathbb{R}^3 \rtimes SO(3) \to \mathfrak{K}$. Hence, the principal bundle $\mathfrak{K} \to \mathfrak{K}/\mathbb{R}^3 \rtimes SO(3)$ is trivial, and so $\mathfrak{K}$ is homeomorphic to $(\mathfrak{K}/\mathbb{R}^3 \rtimes SO(3)) \times (\mathbb{R}^3 \rtimes SO(3))$. Therefore, we have that $\pi_1(\mathfrak{K}) \cong \pi_1(\mathfrak{K}/\mathbb{R}^3 \rtimes SO(3)) \times \pi_1(\mathbb{R}^3 \rtimes SO(3)) \cong \pi_1(\mathfrak{K}/\mathbb{R}^3 \rtimes SO(3)) \times \mathbb{Z}_2$. □

Remark 5.1. Calvo proved in [4] that for a component $\mathfrak{T}$ of $\mathfrak{Geo}^{(6)}$ consisting of right-handed trefoil knots, $\pi_1(\mathfrak{T}/\mathbb{R}^3 \rtimes SO(3))$ is trivial and so $\pi_1(\mathfrak{T})$ is isomorphic to $\mathbb{Z}_2$.

6. A CONCJUDING REMARK AND OPEN QUESTIONS

Although we do not know whether there is a pair of geometrically distinct n-sided polygonal representations of a given knot or not, there is not always only one component of $\mathbb{D}_n \backslash \mathfrak{Geo}^{(n)}/\mathbb{R}^3 \rtimes SO(3)$ consisting of n-sided polygonal representations of a given knot. By $\mathfrak{gs}(K,n)$, we denote the number of components of $\mathbb{D}_n \backslash \mathfrak{Geo}^{(n)}/\mathbb{R}^3 \rtimes SO(3)$ consisting of n-sided polygonal representations of K. Since the number $\mathfrak{gs}(K,n)$ implies the number of "geometric shapes" of n-sided polygonal representations of K, we call $\mathfrak{gs}(K,n)$ the *n-th geometric shape number* of K. Here are some open questions about geometric shapes of knots :

1.(Geometric unknottedness) It follows clearly from the argument in §5 that $\mathfrak{gs}(\text{unknot}, n) = 1$ for $n = 3, 4, 5$. Is there an integer n with $\mathfrak{gs}(\text{unknot}, n) \geq 2$? More generally, there is a knot K such that $\mathfrak{gs}(K,n) \geq 2$ for some integer n ? Notice that knots can always be approximated by polygonal knots consisting of many edges, and that any topological deformation between ambient isotopic knots can always be approximated by a piecewise-linear deformation of polygonal knots, as long as the number of edges is allowed to increase. If so, then does the sequence $\{\mathfrak{gs}(K,n)\}_{n\in\mathbb{N}}$ converge into 1 ?

2. In [3, 4], one can find that $\mathfrak{gs}(\text{trefoil knot}, 6) = 1$ and $\mathfrak{gs}(\text{figure-eight knot}, 7) = 1$. This suggests us the following question ; For any knot K and $n = \mathfrak{p}(K)$, is the geometric shape number $\mathfrak{gs}(K,n)$ equal to 1 ? That is to say, can we choose an economical polygonal representation of K uniquely ?

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