A Solution of Radiative Transfer In Isotropic Plane-Parallel Atmosphere By Integrating Milne Equation

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Abstract

In this paper we solve the inversion problem of the radiative transfer process in the isotropic plane-parallel atmosphere by iterative integrations of the Milne integral equation. As a result, we obtain the scattering function in the form of a cubic polynomial in optical thickness. The author has already solved the same problem by iterative integrations of Chandrasekhar's integral equation. In the Milne integral equation, both the cosines of the viewing angles and the optical thickness are integral variables, while in Chandrasekhar's integral equation the cosines of the viewing angles are variables but the optical thickness is not. We derive several series of exponential-like functions as intermediate derivations. Their convergences are evaluated by the author's previous work in the solution of Chandrasekhar's integral equation. The truncated scattering function up to the third order in optical thickness thus obtained is identical to that obtained from Chandrasekhar's integral equation, though their apparent forms are different. Chandrasekhar pointed out that the solution of Chandrasekhar's integral equation does not have a uniqueness of solution. The Milne equation, in contrast, has been proven to have a unique solution. We discuss the uniqueness of the solution by these two methods.

1 Introduction

In the area of satellite remote sensing, we observe the radiance at the top of the atmosphere (TOA) by instruments aboard satellites, and we retrieve the surface reflectance and the optical thickness from the observed radiance (inversion problem). We assume the homogeneous plane-parallel atmosphere and the isotropic scattering by matters within the atmosphere.

We show the geometry of the radiative transfer process in Fig. 1. The atmospheric layer has the vertical optical thickness τ and the optical depth x is measured from the top. The input solar irradiance comes to the surface in the direction $\vec{i}_0$ and the scattered radiance that we observe from satellites gets out from the surface in the direction $\vec{i}_1$. At the bottom the transmitted radiation gets out in the direction $\vec{i}_2$ and we assume no incident radiance from the bottom. $\vec{i}_z$ is the zenith direction. We introduce a variable μ_n, $\mu_n = |\cos(\vec{i}_z \wedge \vec{i}_n)|$, for the direction $\vec{i}_n$. The sign of the cosines of the zenith angle for both the upper and lower bound directions is positive .

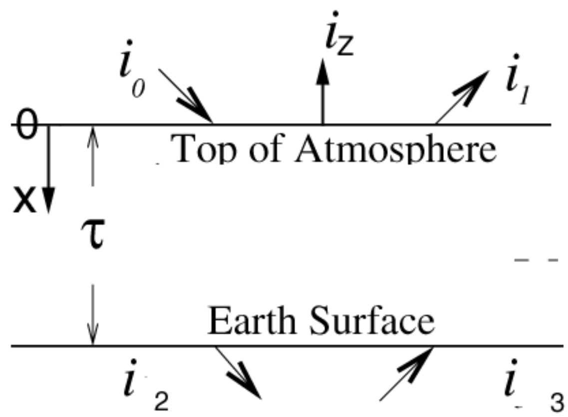

Fig. 1. Geometry of Radiative Transfer

We introduce two functions: the scattering function S and the transmission function T. The scattered radiance $I(0, \vec{i}_1)$ and transmitted radiance $I(\tau, \vec{i}_2)$, are given, as

$$I(0, \vec{i}_1) = \frac{1}{4\pi\mu_1} \int_L S(\tau, \vec{i}_1, \vec{i}_0) I(0, \vec{i}_0) d\Omega_0, \tag{1}$$

$$I(\tau, \vec{i}_2) = \frac{1}{4\pi\mu_2} \int_L T(\tau, \vec{i}_2, \vec{i}_0) I(0, \vec{i}_0) d\Omega_0 + \exp(-\frac{\tau}{\mu_2}) I(0, \vec{i}_2). \tag{2}$$

where $I(0, i_0)$ is the incident intensity, Ω_0 is the solid angle subtended around the input direction $\vec{i}_0$ and the integral domain is the lower half of the unit sphere. The incident radiance to the layer is the solar radiance from the TOA. It is given as $F_0 \delta(\vec{i} - \vec{i}_0)$, where $\delta(\vec{i})$ is Dirac's delta function. Inserting the incident radiance into (1), we obtain $I(0, \vec{i}_1)$,

$$I(0, \vec{i}_1) = \frac{F_0}{4\pi\mu_1} S(\tau, \vec{i}_1, \vec{i}_0).$$

Our problem is to obtain the scattering function S as a polynomial in τ with coefficients that are derived from $\vec{i}_1, \vec{i}_0$.

Introducing the source function $J(x, \mu)$, the radiative transfer process is governed by the Milne integral equation [4] [2].

$$J(x, \mu_0) = \exp(-\frac{x}{\mu_0}) + \int_0^1 \int_0^\tau \exp(-\frac{|y-x|}{\mu}) J(y, \mu_0) dy \frac{d\mu}{2\mu} \tag{3}$$

We solve the Milne equation by iterative integration with the initial solution as $\exp(-\frac{x}{\mu_0})$. Using $J(x, \mu_0)$, the scattering and transmission functions are expressed as below [4],

$$S(\tau, \mu_1, \mu_0) = \int_0^\tau \exp(-\frac{y}{\mu_1}) J(y, \mu_0) dy, \tag{4}$$

$$T(\tau, \mu_1, \mu_0) = \int_0^\tau \exp(-\frac{\tau - y}{\mu_1}) J(y, \mu_0) dy \tag{5}$$

Chandrasekhar derived an integral equation that governs the radiative transfer process [1]. It is a non-linear, simultaneous integral equation with the scattering and transmission functions S and T as two unknown functions to be solved [1] as below.

$$\begin{aligned}
(\frac{1}{\mu_1} + \frac{1}{\mu_0}) S(\tau, \mu_1, \mu_0) &= [1 - \exp(-\frac{\tau}{\mu_1} - \frac{\tau}{\mu_0})] \qquad (6) \\
&+ \int_0^1 [S(\tau, \mu, \mu_1) - \exp(-\frac{\tau}{\mu_1}) T(\tau, \mu, \mu_0) + S(\tau, \mu, \mu_0) - \exp(-\frac{\tau}{\mu_0}) T(\tau, \mu, \mu_0)] \frac{d\mu}{2\mu} \\
&+ [\int_0^1 S(\tau, \mu, \mu_1) \frac{d\mu}{2\mu}][\int_0^1 S(\tau, \mu, \mu_0) \frac{d\mu}{2\mu}] - [\int_0^1 T(\tau, \mu, \mu_1) \frac{d\mu}{2\mu}][\int_0^1 T(\tau, \mu, \mu_0) \frac{d\mu}{2\mu}] \\
(\frac{1}{\mu_1} - \frac{1}{\mu_0}) T(\tau, \mu_1, \mu_0) &= [\exp(-\frac{\tau}{\mu_0}) - \exp(-\frac{\tau}{\mu_1})] \qquad (7) \\
&+ \int_0^1 [T(\tau, \mu, \mu_0) - \exp(-\frac{\tau}{\mu_0}) S(\tau, \mu, \mu_1) - T(\tau, \mu, \mu_1) - \exp(-\frac{\tau}{\mu_1}) S(\tau, \mu, \mu_0)] \frac{d\mu}{2\mu} \\
&+ [\int_0^1 S(\tau, \mu, \mu_1) \frac{d\mu}{2\mu}][\int_0^1 T(\tau, \mu, \mu_0) \frac{d\mu}{2\mu}] - [\int_0^1 T(\tau, \mu, \mu_1) \frac{d\mu}{2\mu}][\int_0^1 S(\tau, \mu, \mu_0) \frac{d\mu}{2\mu}].
\end{aligned}$$

It is noted that the optical thickness τ is not an integral variable in Chandrasekhar's equation. We obtained the solution by iteratively integrating Chandrasekhar's integral equation, [5], [6] that is described in Section 4.

We have to integrate the Milne equation with respect to the optical thickness in addition to integrating it with respect to the zenith angle. Changing the sequence of integration, we integrate with respect to the optical thickness and then with respect to the zenith angle. We derive several series of exponential-like functions that are evaluated by the author's previous work, [5] [6]. Truncating the series expansions of the scattering function S up to the third power in τ, we obtain the third approximation of the Milne equation in Section 2. The crucial derivations for the iterative integration is discussed in Section 3.

We then compare the solution by the Milne equation with that of Chandrasekhar's integral equation (Section 4). Because of the linearities and the the condition of the boundednes, it is proved that the Milne equation has a unique solution [2]. It is also proved that Chandrasekhar's integral equation can be derived from the Milne equation [2]. The family of solutions for Chandrasekhar's integral equation is derived based on the uniqueness of the Milne equation [3]. We discuss the uniqueness of the solutions in Section 5.

2 Integration of the Milne Equation

2.1 Iteration of Integration

The first iteration is the first term of the Milne equation (3),

$$J_1(x, \mu_0) = \exp(-\frac{x}{\mu_0}). \tag{8}$$

The higher iteration integrations are given as below.

$$\begin{aligned} J_{n+1}(x, \mu_0) &= \int_0^1 \int_0^\tau \exp(-\frac{|y-x|}{\mu}) J_n(y, \mu_0) dy \frac{d\mu}{2\mu} \\ &= \int_0^1 \{ \int_0^x \exp(-\frac{x-y}{\mu}) J_n(y, \mu_0) dy + \int_x^\tau \exp(-\frac{y-x}{\mu}) J_n(y, \mu_0) dy \} \frac{d\mu}{2\mu}. \end{aligned} \tag{9}$$

$J(x, \mu_0)$ is the summation of $J_n(x, \mu_0)$

$$J(x, \mu_0) = \sum_{n=1}^{\infty} J_n(x, \mu_0). \tag{10}$$

Substituting $J(x, \mu_0)$ into equation (4), we obtain $S(\tau, \mu_1, \mu_0)$,

$$\Delta S(\tau, \mu_1, \mu_0) = \int_0^\tau \exp(-\frac{y}{\mu_1})(\sum_{n=1}^{\infty} J_n(y, \mu_0)) dy = \sum_{n=1}^{\infty} \Delta S_n(\tau, \mu_1, \mu_0) \tag{11}$$

$$\Delta T(\tau, \mu_2, \mu_0) = \int_0^\tau \exp(-\frac{\tau - y}{\mu_2})(\sum_{n=1}^{\infty} J_n(y, \mu_0)) dy = \sum_{n=1}^{\infty} \Delta T_n(\tau, \mu_2, \mu_0) \tag{12}$$

where $\Delta S_n(\tau,\mu_1,\mu_0)$ is given as

$$\Delta S_n(\tau,\mu_1,\mu_0) = \int_0^{\tau} \exp(-\frac{y}{\mu_1})J_n(y,\mu_0)dy. \tag{13}$$

Similarly, we define $\Delta T_n(\tau,\mu_2,\mu_0)$ as

$$\Delta T_n(\tau,\mu_2,\mu_0) = \int_0^{\tau} \exp(-\frac{\tau-y}{\mu_2})J_n(y,\mu_0)dy. \tag{14}$$

We obtain the relation between $\Delta S_n(\tau,\mu_1,\mu_0)$ and $\Delta T_n(\tau,\mu_2,\mu_0)$, by a simple algebraic operation on the above two equations,

$$\Delta T_n(\tau,\mu_2,\mu_0) = \exp(-\frac{\tau}{\mu_2})\Delta S_n(\tau,-\mu_2,\mu_0). \tag{15}$$

We designate the n-th approximation for $S(\tau,\mu_1,\mu_0)$ and $T(\tau,\mu_2,\mu_0)$ as $S_n(\tau,\mu_1,\mu_0)$ and $T_n(\tau,\mu_2,\mu_0)$ given below.

$$\begin{aligned} S_n(\tau,\mu_1,\mu_0) &= \sum_{m=1}^{n} \Delta S_m(\tau,\mu_1,\mu_0) \\ T_n(\tau,\mu_2,\mu_0) &= \sum_{m=1}^{n} \Delta T_m(\tau,\mu_2,\mu_0). \end{aligned} \tag{16}$$

2.2 The First Approximation

Substituting the first iteration $\exp(-\frac{\tau}{\mu_0})$, we obtain $\Delta S_1(\tau,\mu_1,\mu_0)$

$$\begin{aligned} \Delta S_1(\tau,\mu_1,\mu_0) &= \int_0^{\tau} \exp(-\frac{y}{\mu_1}-\frac{y}{\mu_0})dy = (\frac{1}{\mu_1}+\frac{1}{\mu_0})^{-1}[1-\exp(-\frac{\tau}{\mu_1}-\frac{\tau}{\mu_0})] \\ &= \tau\sum_{n=0}^{\infty}\frac{1}{(n+1)!}(\frac{1}{\mu_1}+\frac{1}{\mu_0})^n(-\tau)^n. \end{aligned} \tag{17}$$

$\Delta T_1(\tau,\mu_2,\mu_0)$ is obtained from $\Delta S_1(\tau,\mu_1,\mu_0)$ shown below.

$$\begin{aligned} \Delta T_1(\tau,\mu_2,\mu_0) &= (\frac{1}{\mu_2}-\frac{1}{\mu_0})^{-1}[\exp(-\frac{\tau}{\mu_0})-\exp(-\frac{\tau}{\mu_2})] \\ &= \tau\sum_{n=0}^{\infty}\frac{1}{(n+1)!}[\sum_{r=0}^{n}(\frac{1}{\mu_2})^{n-r}(\frac{1}{\mu_0})^r](-\tau)^n. \end{aligned} \tag{18}$$

It is noted that both ΔS_1 and ΔT_1 are symmetrical with respect to μs.

$$\begin{aligned}
&\Delta S_1(\tau,\mu_1,\mu_0)=\Delta S_1(\tau,\mu_0,\mu_1)\\
&\Delta T_1(\tau,\mu_2,\mu_0)=\Delta T_1(\tau,\mu_0,\mu_2)
\end{aligned} \tag{19}$$

The truncated polynomial of S_1 in τ is given below,

$$\Delta S_1(\tau,\mu_1,\mu_0) = \tau - \frac{1}{2}(\frac{1}{\mu_1}+\frac{1}{\mu_0})\tau^2 + \frac{1}{6}(\frac{1}{\mu_1}+\frac{1}{\mu_0})^2\tau^3 + \cdots . \tag{20}$$

2.3 The second Approximation

$J_2(\tau,y,\mu_0)$ is given below.

$$J_2(\tau,y,\mu_0) = \int_0^1\int_0^\tau \exp(-\frac{|y-x|}{\mu})\exp(-\frac{x}{\mu_0})dx\frac{d\mu}{2\mu}. \tag{21}$$

$\Delta S_2(\tau,\mu_1,\mu_0)$ is evaluated by equation (13),

$$\begin{aligned}
&\Delta S_2(\tau,\mu_1,\mu_0) = \int_0^\tau\int_0^1\int_0^\tau \exp(-\frac{y}{\mu_1})\exp(-\frac{|y-x|}{\mu})\exp(-\frac{x}{\mu_0})dx\frac{d\mu}{2\mu}dy\\
&= \int_0^\tau\int_0^1\int_0^y \exp(-\frac{y}{\mu_1})\exp(-\frac{y-x}{\mu})\exp(-\frac{x}{\mu_0})dx\frac{d\mu}{2\mu}dy\\
&+ \int_0^\tau\int_0^1\int_y^\tau \exp(-\frac{y}{\mu_1})\exp(-\frac{x-y}{\mu})\exp(-\frac{x}{\mu_0})dx\frac{d\mu}{2\mu}dy
\end{aligned} \tag{22}$$

The sequence of integration in the above equation is $dx, d\mu, dy$. We change it to $dx, dy, d\mu$. It is proven in the next section that the sequence of integration does not affect the integration. We can easily integrate with respect to dx, dy. The integration with respect to μ requires a special derivation that is described in the next section. We obtain the first integration in the above equation.

$$\begin{aligned}
&\int_0^1\int_0^\tau\int_0^y \exp(-\frac{y}{\mu_1})\exp(-\frac{y-x}{\mu})\exp(-\frac{x}{\mu_0})dxdy\frac{d\mu}{2\mu}\\
&= (\frac{1}{\mu_0}+\frac{1}{\mu_1})^{-1}\int_0^1[-\exp(-\frac{\tau}{\mu_1})\Delta T_1(\mu_0,\mu)+\Delta S_1(\mu_1,\mu)]\frac{d\mu}{2\mu}\\
&= (\frac{1}{\mu_0}+\frac{1}{\mu_1})^{-1}[U_2(\mu_1)-\exp(-\frac{\tau}{\mu_1})V_2(\mu_0)]
\end{aligned} \tag{23}$$

Since the equation (22) is symmetric with respect to μ_1 and μ_0, the second integration is obtained from the first integration by exchanging μ_0 with μ_1. Summing up the both integrations in equation (22), we obtain $\Delta S_2(\tau, \mu_1, \mu_0)$ below.

$$\begin{aligned}&\Delta S_2(\tau, \mu_1, \mu_0) \\ &= (\frac{1}{\mu_0} + \frac{1}{\mu_1})^{-1}[U_2(\mu_0) + U_2(\mu_1) - \exp(-\frac{\tau}{\mu_0})V_2(\mu_1) - \exp(-\frac{\tau}{\mu_1})V_2(\mu_0)]\end{aligned} \tag{24}$$

where $U_2(\tau, \mu^*)$ and $V_2(\tau, \mu^*)$ are given as

$$U_2(\tau, \mu^*) = \int_0^1 \Delta S_1(\mu, \mu^*)\frac{d\mu}{2\mu} \tag{25}$$

$$V_2(\tau, \mu^*) = \int_0^1 \Delta T_1(\mu, \mu^*)\frac{d\mu}{2\mu}. \tag{26}$$

We obtain the relation between $U_2(\tau, \mu^*)$ and $V_2(\tau, \mu^*)$ by equation (15) as below,

$$V_2(\tau, \mu^*) = \exp(-\frac{\tau}{\mu^*})U_2(\tau, -\mu^*). \tag{27}$$

By the above equation and the equation (15), we obtain $\Delta T_2(\tau, \mu_2, \mu_0)$

$$\begin{aligned}&\Delta T_2(\tau, \mu_2, \mu_0) \\ &= (\frac{1}{\mu_0} - \frac{1}{\mu_2})^{-1}[\exp(-\frac{\tau}{\mu_2})U_2(\mu_0) + V_2(\mu_2) - \exp(-\frac{\tau}{\mu_0})U_2(\mu_2) - V_2(\mu_0)].\end{aligned} \tag{28}$$

$\Delta T_2(\tau, \mu_2, \mu_0)$ is also symmetric with respect to μ_2 and μ_0.

$U_2(\tau, \mu^*)$ and $V_2(\tau, \mu^*)$ are evaluated by as the polynomial in τ [6] as below.

$$U_2(\tau, p^*) = \frac{\tau}{2}\sum_{n=0}^{\infty}\{-C - \sum_{m=1}^{\infty}\frac{(-\tau)^m}{mm!} + \sum_{m=0}^{\infty}\frac{(-\tau)^m}{(n+1+m)m!}\}\frac{(-\tau p^*)^n}{(n+1)!} \tag{29}$$

$$V_2(\tau, p^*) = \frac{\tau}{2}\sum_{n=0}^{\infty}[-C + \sum_{m=1}^{n+1}\frac{1}{m} - \sum_{m=1}^{\infty}\frac{(-\tau)^m(n+1)!}{m(n+1+m)!})\,]\frac{(-\tau p^*)^n}{(n+1)!}. \tag{30}$$

where $C = \log\tau + \gamma$ and $\gamma = 0.577216$ is Euler's constant.

We obtain the second iterations as polynomials in τ by substituting $U_2(\tau, p^*)$ and $V_2(\tau, p^*)$ into $\Delta S_2(\tau, \mu_1, \mu_0)$ and $\Delta T_2(\tau, \mu_2, \mu_0)$.

The third iteration $\Delta S_(\tau, \mu_1, \mu_0)$ is evaluated by the recurrent form in equation (13),

$$\Delta S_3(\tau, \mu_1, \mu_0) = \int_0^\tau \exp(-\frac{z}{\mu_1}) J_3(\tau, z, \mu_0) dz.$$

Substituting $J_3(\tau, z, \mu_0)$ into the above equation, we obtain the the quintuplet integration of $\Delta S_(\tau, \mu_1, \mu_0)$,

$$\begin{aligned}
\Delta S_3(\tau, \mu_1, \mu_0) &= \int_0^\tau \exp(-\frac{z}{\mu_1}) J_3(\tau, z, \mu_0) dz = \int_0^1 \int_0^1 \{\cdot\cdot\} \frac{d\mu_a}{2\mu_a} \frac{d\mu_b}{2\mu_b} \\
&= \int_0^1 \int_0^\tau \int_0^\tau \exp(-\frac{z}{\mu_1}) \exp(-\frac{|z-y|}{\mu_b}) J_2(\tau, y, \mu_0) dz dy \frac{d\mu_b}{2\mu_b} \qquad (31) \\
&= \int_0^1 \int_0^1 \int_0^\tau \int_0^\tau \int_0^\tau \exp(-\frac{z}{\mu_1}) \exp(-\frac{|z-y|}{\mu_b}) \exp(-\frac{|x-y|}{\mu_a}) \exp(-\frac{x}{\mu_0}) dx dy dz \frac{d\mu_a}{2\mu_a} \frac{d\mu_b}{2\mu_b}.
\end{aligned}$$

The sequence of integration due to the iteration scheme is x, μ_a, y, μ_b and z. Because of the interchangeability of integration (refer to the next section), we integrate in the sequence of x, y, z, μ_a and μ_b. We can easily integrate with respect to x, y, z. Then we repeat the special integral derivations twice (refer to the next section).
It is noted that $\Delta S_3(\tau, \mu_1, \mu_0)$ is symmetrical with respect to μ_1, μ_0 as

$$\Delta S_3(\tau, \mu_1, \mu_0) = \Delta S_3(\tau, \mu_0, \mu_1).$$

We divide the integration into four separate domains;
$(y > x, y > z,)$, $(y < x, y < z,)$, $(y > x, y < z,)$ and $(y < x, y > z,)$.

We integrate the first domain or $(y > x, y > z,)$ with respect to x, y, and z and obtain the following,

$$
\begin{aligned}
&\int_0^1\int_0^1\int_0^\tau\int_0^y\int_0^y \exp(-\frac{z}{\mu_1})\exp(-\frac{y-z}{\mu_b})\exp(-\frac{y-x}{\mu_a})\exp(-\frac{x}{\mu_0})dzdydx\frac{d\mu_a}{2\mu_a}\frac{d\mu_b}{2\mu_b}\\
&=\int_0^1\int_0^1(-\frac{1}{\mu_1}+\frac{1}{\mu_b})^{-1}\{(-\frac{1}{\mu_0}+\frac{1}{\mu_a})^{-1}(-\frac{1}{\mu_0}-\frac{1}{\mu_1})^{-1}[\exp(-\frac{\tau}{\mu_0}-\frac{\tau}{\mu_1})-1]\\
&\qquad -(-\frac{1}{\mu_0}+\frac{1}{\mu_a})^{-1}(-\frac{1}{\mu_a}-\frac{1}{\mu_1})^{-1}[\exp(-\frac{\tau}{\mu_a}-\frac{\tau}{\mu_1})-1]\\
&\qquad -(-\frac{1}{\mu_0}+\frac{1}{\mu_a})^{-1}(-\frac{1}{\mu_0}-\frac{1}{\mu_b})^{-1}[\exp(-\frac{\tau}{\mu_0}-\frac{\tau}{\mu_b})-1] \qquad (32)\\
&\qquad +(-\frac{1}{\mu_0}+\frac{1}{\mu_a})^{-1}(-\frac{1}{\mu_a}-\frac{1}{\mu_b})^{-1}[\exp(-\frac{\tau}{\mu_a}-\frac{\tau}{\mu_b})-1]\}\frac{d\mu_a}{2\mu_a}\frac{d\mu_b}{2\mu_b}.
\end{aligned}
$$

For the integration with respect to μ_a, and μ_b, the evaluation procedure is described in the next section and the detailed derivation is given in the Appendix 1. We obtain the following,

$$
\begin{aligned}
&\int_0^1\int_0^1\int_0^\tau\int_0^y\int_0^y \exp(-\frac{z}{\mu_1})\exp(-\frac{y-z}{\mu_b})\exp(-\frac{y-x}{\mu_a})\exp(-\frac{x}{\mu_0})dzdydx\frac{d\mu_a}{2\mu_a}\frac{d\mu_b}{2\mu_b}\\
&=(-\frac{1}{\mu_0}-\frac{1}{\mu_1})^{-1}\{V_2(\mu_1)V_2(\mu_0)-U_3^-(\mu_1)+U_3^-(-\mu_0)-V_2(-\mu_0)V_2(\mu_0)]\}\\
&=(\frac{1}{\mu_0}+\frac{1}{\mu_1})^{-1}\{-V_2(\mu_1)V_2(\mu_0)+U_3^-(\mu_1)+U_3^-(\mu_0)\} \qquad (33)
\end{aligned}
$$

where a new function $U_3^-(\tau,\mu^*)$, is given as below.

$$
U_3^-(\tau,\mu*)=\int_0^1(\frac{1}{\mu}-\frac{1}{\mu^*})^{-1}[U_2(\mu^*)-U_2(\mu)]\frac{d\mu}{2\mu}. \qquad (34)
$$

For $\mu* < 0$, $U_3^-(\tau,\mu^*)$ is further derived as below, (refer to Appendix 4)

$$
U_3^-(\tau,-\mu*)=V_2(\mu^*)V_2(-\mu^*)-U_3^-(\mu^*) \qquad (35)
$$

We integrate the second domain or $(y < x, y < z,)$ with respect to x, y, z, μ_a, and μ_b and obtain the following (refer to Appendix 2).

$$
\begin{aligned}
&\int_0^1\int_0^1\int_0^\tau\int_y^\tau\int_y^\tau \exp(-\frac{z}{\mu_1})\exp(\frac{y-z}{\mu_b})\exp(\frac{y-x}{\mu_a})\exp(-\frac{x}{\mu_0})dzdydx\frac{d\mu_a}{2\mu_a}\frac{d\mu_b}{2\mu_b}\\
&=(\frac{1}{\mu_0}+\frac{1}{\mu_1})^{-1}[U_2(\mu_0)U_2(\mu_1)+\exp(-\frac{\tau}{\mu_1}-\frac{\tau}{\mu_0})[U_3^-(\mu_0)+U_3^-(\mu_1)]\\
&\quad -(\frac{1}{\mu_0}+\frac{1}{\mu_1})^{-1}\exp(-\frac{\tau}{\mu_0}-\frac{\tau}{\mu_1})[V_2(\mu_1)V_2(-\mu_1)+\exp(\frac{\tau}{\mu_0})V_2(\mu_0)U_2(\mu_0)]. \qquad (36)
\end{aligned}
$$

We integrate the third domain or $(y > x, y < z,)$ with respect to x, y, z, μ_a, and μ_b and obtain the following (refer to Appendix 3).

$$\int_0^\tau \int_y^\tau \int_0^y \int_0^\tau \int_y^\tau \int_0^y \exp(-\frac{z}{\mu_1}) \exp(\frac{y-z}{\mu_b}) \exp(\frac{x-y}{\mu_a}) \exp(-\frac{x}{\mu_0}) dzdydx \frac{d\mu_a}{2\mu_a}\frac{d\mu_b}{2\mu_b}$$
$$= (\frac{1}{\mu_1} + \frac{1}{\mu_0})^{-1} \exp(-\frac{\tau}{\mu_1})[-V_3^-(\mu_0) + V_3^-(-\mu_1)] \quad (37)$$

where another a new function $V_3^-(\tau, -\mu*)$ is given as below,

$$V_3^-(\tau, \mu*) = \int_0^1 (\frac{1}{\mu} - \frac{1}{\mu^*})^{-1}[V_2(\mu^*) - V_2(\mu)]\frac{d\mu}{2\mu}. \quad (38)$$

For the fourth domain, the integration is made the same as in the third domain by exchanging μ_0 and μ_1.

Summing up the integrations in the four domains, we obtain $\Delta S_3(\tau, \mu_1, \mu_0)$

$$\begin{aligned}
&\Delta S_3(\tau, \mu_1, \mu_0) \\
&= (\frac{1}{\mu_1} + \frac{1}{\mu_0})^{-1}\{-V_2(\mu_1)V_2(\mu_0) + U_3^-(\mu_1) + U_3^-(\mu_0) \\
&\qquad + U_2(\mu_0)U_2(\mu_1) + \exp(-\frac{\tau}{\mu_1} - \frac{\tau}{\mu_0})[U_3^-(\mu_0) + U_3^-(\mu_1)] \\
&\qquad - \exp(-\frac{\tau}{\mu_0})V_2(\mu_1)U_2(\mu_1) - \exp(-\frac{\tau}{\mu_1})V_2(\mu_0)U_2(\mu_0) \qquad (39) \\
&\quad + \exp(-\frac{\tau}{\mu_1})[-V_3^-(\mu_0) + V_3^-(-\mu_1)] + \exp(-\frac{\tau}{\mu_0})[-V_3^-(\mu_1) + V_3^-(-\mu_0)]\}
\end{aligned}$$

The problem or evaluation of the the third iterations is reduced to obtain the two integrations $U_3^-(\tau, \mu*)$ and $V_3^-(\tau, \mu^*)$, that are solved by the author [6] and are given as below,

$$\begin{aligned}
U_3^-(\tau, p^*) &= \frac{(-\tau)^2}{4}(-C - \sum_{m=1}^{\infty} \frac{(-\tau)^m}{mm!}) \sum_{n=0}^{\infty}(-C + \sum_{r=1}^{n+2}\frac{1}{r})\frac{(-\tau p^*)^n}{(n+2)!} \\
&+ \frac{(-\tau)^2}{4}\sum_{n=0}^{\infty}[\sum_{m=0}^{\infty}[-C + \sum_{r=1}^{n+2}\frac{1}{r} + \frac{1}{n+2+m}]\frac{(-\tau)^m}{(n+2+m)m!}]\frac{(-\tau p^*)^n}{(n+2)!} \\
&- \frac{(-\tau)^2}{4}\sum_{n=0}^{\infty}(-C - \sum_{m=1}^{\infty}\frac{(-\tau)^m}{mm!})[\sum_{q=1}^{\infty}\frac{(n+2)!(-\tau)^q}{q(n+2+q)!})]\frac{(-\tau p^*)^n}{(n+2)!} \qquad (40) \\
&- \frac{(-\tau)^2}{4}\sum_{n=0}^{\infty}[\sum_{q=1}^{\infty}\{\sum_{m=0}^{\infty}\frac{(-\tau)^m}{(n+2+q+m)m!}\}\frac{(n+2)!(-\tau)^q}{q(n+2+q)!}]\frac{(-\tau p^*)^n}{(n+2)!}
\end{aligned}$$

$$
\begin{aligned}
V_3^-(\tau,p^*) = &\frac{(-\tau)^2}{4}[-C\sum_{n=0}^{\infty}(-C+\sum_{r=1}^{n+2}\frac{1}{r})]\frac{(-\tau p^*)^n}{(n+2)!} \\
&+\frac{(-\tau)^2}{4}\sum_{n=0}^{\infty}[(\sum_{r=1}^{n+2}\frac{1}{r})(-C+\sum_{q=1}^{n+2}\frac{1}{q})+\sum_{r=1}^{n+2}\frac{1}{r^2}-\frac{\pi^2}{6}]\frac{(-\tau p^*)^n}{(n+2)!} \\
&-\frac{(-\tau)^2}{4}\sum_{n=0}^{\infty}[\sum_{m=1}^{\infty}(-C+\sum_{q=1}^{n+2}\frac{1}{q})\frac{(-\tau)^m(n+2)!}{m(n+2+m)!}]\frac{(-\tau p^*)^n}{(n+2)!} \\
&-\frac{(-\tau)^2}{4}\sum_{n=0}^{\infty}\sum_{q=1}^{\infty}(-C+\sum_{r=1}^{n+2+q}\frac{1}{r})\frac{(n+2)!(-\tau)^q}{q(n+2+q)!}\}\frac{(-\tau p^*)^n}{(n+2)!} \\
&+\frac{(-\tau)^2}{4}\sum_{n=0}^{\infty}[\sum_{q=1}^{\infty}\{\sum_{m=1}^{\infty}\frac{(n+2+q)!(-\tau)^m}{m(n+2+q+m)!}\}\frac{(n+2)!(-\tau)^q}{q(n+2+q)!})]\frac{(-\tau p^*)^n}{(n+2)!}.
\end{aligned} \tag{41}
$$

Substituting $U_3^-(\tau,p^*)$, $V_3^-(\tau,p^*)$, $U_2(\tau,p^*)$, and $V_2(\tau,p^*)$ into the equation (39), we obtain the $\Delta S_3(\tau,\mu_1,\mu_0)$ that is given in the section 4.

We can further derive $\Delta S_3(\tau,\mu_1,\mu_0)$ as below,

$$
\begin{aligned}
\Delta S_3(\tau,\mu_1,\mu_0) = &(\frac{1}{\mu_1}+\frac{1}{\mu_0})^{-1}[U_2(\mu_0)U_2(\mu_1)-V_2(\mu_1)V_2(\mu_0)] \\
+&(\frac{1}{\mu_1}+\frac{1}{\mu_0})^{-1}[U_3(\mu_0)+U_3(\mu_1)-\exp(-\frac{\tau}{\mu_1})V_3(\mu_0)-\exp(-\frac{\tau}{\mu_0})V_3(\mu_1)]
\end{aligned} \tag{42}
$$

where $U_3(\mu*)$ and $V_3(\mu*)$ are given below,

$$
U_3(\mu*) = \int_0^1 \Delta S_2(\tau,\mu*,\mu)\frac{d\mu}{2\mu} = U_3^-(\mu*)+\exp(-\frac{\tau}{\mu*})V_3^-(-\mu*) \tag{43}
$$

$$
V_3(\mu*) = \int_0^1 \Delta T_2(\tau,\mu*,\mu)\frac{d\mu}{2\mu} = V_3^-(\mu*)+V_2(\mu*)U_2(\mu*)-\exp(-\frac{\tau}{\mu*})U_3^-(\mu*). \tag{44}
$$

3 Consideration on the integration of the Milne equation

3.1 Sequence of integration

In the previous section, we integrate the iterations with respect to x, y, and z first and then μ_a and μ_b. We prove that the change of sequence of integration is possible in the second approximation for the equation (22). The integration is given as below,

$$\int_0^1 \int_0^\tau \int_0^y \exp(-\frac{y}{\mu_1}) \exp(-\frac{y-x}{\mu}) \exp(-\frac{x}{\mu_0}) dx dy \frac{d\mu}{2\mu}. \tag{45}$$

The integrand is regular except for $\mu = 0$.
For $(y-x) > 0$ the integrand tends to 0 as $\mu \to 0$.

$$\lim_{\mu \to 0} \exp(-\frac{y-x}{\mu}) \frac{1}{\mu} = 0 \tag{46}$$

However at $(\mu = 0, y - x = 0)$ the integrand tends to a different value by different sequence of 'lim' ,

$$\lim_{\mu \to 0} \lim_{(y-x) \to 0} \exp(-\frac{y-x}{\mu}) \frac{1}{\mu} = \infty \tag{47}$$

$$\lim_{(y-x) \to 0} \lim_{\mu \to 0} \exp(-\frac{y-x}{\mu}) \frac{1}{\mu} = 0 \tag{48}$$

The integration should be regarded as the limit value of the integration with two infinitesimal parameters δ_μ and δ_x approaching to 0.

$$\int_{\delta_\mu}^1 \int_0^\tau \int_0^{y-\delta_x} \exp(-\frac{y}{\mu_1}) \exp(-\frac{y-x}{\mu}) \exp(-\frac{x}{\mu_0}) dx dy \frac{d\mu}{2\mu} \tag{49}$$

We can integrate the equation above with any sequence of integration by x, y, and μ, because the integrand is regular within the integral domain. The problem is, then, whether the following residual integration converges to 0 in any sequence of integration.

$$\int_0^{\delta_\mu} \int_0^\tau \int_{y-\delta_x}^y \exp(-\frac{y}{\mu_1}) \exp(-\frac{y-x}{\mu}) \exp(-\frac{x}{\mu_0}) dx dy \frac{d\mu}{2\mu}. \tag{50}$$

Ignoring the non-essential terms, the integral is reduced to the following,

$$\int_0^{\delta_\mu} \int_0^{\delta_x} \exp(-\frac{x'}{\mu}) \frac{1}{\mu} dx' d\mu. \tag{51}$$

We change the variables from (x', μ) to $(z = x'/\mu, \mu)$. The associated Jacobian is given as below.

$$\frac{\partial(x', \mu)}{\partial(z, \mu)} = \begin{vmatrix} \mu & z \\ 0 & 1 \end{vmatrix} = \mu. \tag{52}$$

Thus, the integration is given in the new variables z and μ below.

$$\int_0^{\delta_\mu}\int_0^\infty \exp(-z)\frac{1}{\mu}\frac{\partial(x',\mu)}{\partial(z,\mu)}dzd\mu = \int_0^{\delta_\mu}\int_0^\infty \exp(-z)dzd\mu. \tag{53}$$

The above integration tends to 0 either as δ_μ approaches 0 and then δ_x approaches 0 or as δ_x approaches 0 and then δ_μ approaches 0. The residual integration in equation (50), accordingly, tends to 0 as δ_μ and δ_x approach to 0. We can thus change the integral sequence with respect to x, y and μ in equation (45).
We can prove the third approximation in the same manner.

3.2 Integration with respect to μ

After integrating equation (22) with respect to x and y, we obtain the following integration with respect to μ,

$$\begin{aligned}
&\int_0^1\int_0^\tau\int_0^y \exp(-\frac{y}{\mu_1})\exp(-\frac{y-x}{\mu})\exp(-\frac{x}{\mu_0})dxdy\frac{d\mu}{2\mu} \\
&= \int_0^1(\frac{1}{\mu}-\frac{1}{\mu_0})^{-1}(-\frac{1}{\mu_0}-\frac{1}{\mu_1})^{-1}[\exp(-\frac{\tau}{\mu_0}-\frac{\tau}{\mu_1})-1]\,\frac{d\mu}{2\mu} \\
&- \int_0^1(\frac{1}{\mu}-\frac{1}{\mu_0})^{-1}(-\frac{1}{\mu}-\frac{1}{\mu_1})^{-1}[\exp(-\frac{\tau}{\mu}-\frac{\tau}{\mu_1})-1]\,\frac{d\mu}{2\mu}.
\end{aligned} \tag{54}$$

We cannot integrate the two integrations separately in the above equation because each integrand has a singularity at $\mu=\mu_0$ in common. This singularity disappears by subtracting the second integrand from the first. Ignoring the non-essential terms, the subtracted integrand is given below.

$$(-\frac{1}{\mu_0}-\frac{1}{\mu_1})^{-1}[\exp(-\frac{\tau}{\mu_0}-\frac{\tau}{\mu_1})-1]-(-\frac{1}{\mu}-\frac{1}{\mu_1})^{-1}[\exp(-\frac{\tau}{\mu}-\frac{\tau}{\mu_1})-1]. \tag{55}$$

Substituting $1/\mu_0$ in the place of $1/\mu$ in the above equation, it becomes 0. The subtracted term includes the factor $(1/\mu-1/\mu_0)$. Therefore $(1/\mu-1/\mu_0)^{-1}$ is canceled from the equation (54).

We introduce a dummy integrand and subtract it from the first integration and add it to the second integration. The dummy integrand has also has a singularity at $\mu=\mu_0$and has the form as below.

$$(\frac{1}{\mu}-\frac{1}{\mu_0})^{-1}(-\frac{1}{\mu_0}-\frac{1}{\mu_1})^{-1}[\exp(-\frac{\tau}{\mu}-\frac{\tau}{\mu_1})-1]. \tag{56}$$

Subtracting the dummy integrand from the first integration and adding it to the second, we obtain the following,

$$
\begin{aligned}
&= \int_0^1 (\frac{1}{\mu} - \frac{1}{\mu_0})^{-1}(-\frac{1}{\mu_0} - \frac{1}{\mu_1})^{-1}[\exp(-\frac{\tau}{\mu_0} - \frac{\tau}{\mu_1}) - \exp(-\frac{\tau}{\mu} - \frac{\tau}{\mu_1})]\,\frac{d\mu}{2\mu} \\
&- \int_0^1 (\frac{1}{\mu} - \frac{1}{\mu_0})^{-1}[(-\frac{1}{\mu} - \frac{1}{\mu_1})^{-1} - (-\frac{1}{\mu_0} - \frac{1}{\mu_1})^{-1}][\exp(-\frac{\tau}{\mu} - \frac{\tau}{\mu_1}) - 1]\,\frac{d\mu}{2\mu} \\
&= (-\frac{1}{\mu_0} - \frac{1}{\mu_1})^{-1} \int_0^1 (\frac{1}{\mu} - \frac{1}{\mu_0})^{-1}[\exp(-\frac{\tau}{\mu_0} - \frac{\tau}{\mu_1}) - \exp(-\frac{\tau}{\mu} - \frac{\tau}{\mu_1})]\,\frac{d\mu}{2\mu} \\
&+(-\frac{1}{\mu_0} - \frac{1}{\mu_1})^{-1} \int_0^1 (\frac{1}{\mu} + \frac{1}{\mu_1})^{-1}][\exp(-\frac{\tau}{\mu} - \frac{\tau}{\mu_1}) - 1]\,\frac{d\mu}{2\mu} \\
&= (\frac{1}{\mu_0} + \frac{1}{\mu_1})^{-1}[-\exp(-\frac{\tau}{\mu_1}) \int_0^1 \Delta T_1(\mu_0, \mu)\,\frac{d\mu}{2\mu} + \int_0^1 \Delta S_1(\mu_1, \mu)\,\frac{d\mu}{2\mu}] \quad (57)
\end{aligned}
$$

The two integrations in the above equation have been evaluated for the integration of Chandrasekhar's integral equation as $V_2(\mu_0)$ and $U_2(\mu_1)$, [5] [6]. Thus we obtain the integrated form of equation (22).

$$
\begin{aligned}
&\int_0^1 \int_0^\tau \int_0^y \exp(-\frac{y}{\mu_1}) \exp(-\frac{y-x}{\mu}) \exp(-\frac{x}{\mu_0}) dx dy \frac{d\mu}{2\mu} \\
&= (\frac{1}{\mu_0} + \frac{1}{\mu_1})^{-1}[U_2(\mu_1) - \exp(-\frac{\tau}{\mu_1}) V_2(\mu_0)] \qquad (58)
\end{aligned}
$$

In the above derivation, the following relation is used,

$$
(\frac{1}{\mu_a} - \frac{1}{\mu_b})^{-1}[(\frac{1}{\mu_c} - \frac{1}{\mu_a})^{-1} - (\frac{1}{\mu_c} - \frac{1}{\mu_b})^{-1}] = (\frac{1}{\mu_c} - \frac{1}{\mu_a})^{-1}(\frac{1}{\mu_c} - \frac{1}{\mu_a})^{-1} (59)
$$

For the third iteration, we repeat this procedure for μ_a and μ_b and obtain $\Delta S(\tau, \mu_1, \mu_0)$.

4 Comparison with the Solution by Chandrasekhar's integral equation

The author obtained the solution of Chandrasekhar's integral equation by the iterative integration [5] [6]. The first approximation $S_1(\tau, \mu_1, \mu_0)$ for Chandrasekhar is identical to the Milne equation.

$$S_1(\tau,\mu_1,\mu_0) = \Delta S_1(\tau,\mu_1,\mu_0) = \tau\sum_{n=0}^{\infty}\frac{1}{(n+1)!}(\frac{1}{\mu_1}+\frac{1}{\mu_0})^n(-\tau)^n \tag{60}$$

The second approximation $\Delta S_2(\tau,\mu_1,\mu_0)$ is given below [6],

$$\begin{aligned}\Delta S_2(\tau,\mu_1,\mu_0) &= (\frac{1}{\mu_0}+\frac{1}{\mu_1})^{-1}\{U_2(\mu_0)+U_2(\mu_1) \\ &- \exp(-\frac{\tau}{\mu_0})V_2(\mu_1) - \exp(-\frac{\tau}{\mu_1})V_2(\mu_0) + U_2(\mu_0)U_2(\mu_1) - V_2(\mu_1)V_2(\mu_0)\}\end{aligned} \tag{61}$$

where $U_2(\mu_0)$ and $V_2(\mu_0)$ are identical to those in the Milne equation. The non-linear terms in the above equation, $U_2(\mu_0)U_2(\mu_1) - V_2(\mu_1)V_2(\mu_0)$ are omitted in the Milne equation. However, the truncated polynomial of $\Delta S_2(\tau,\mu_1,\mu_0)$ up to the second power in τ are identical because the omitted terms do not include the second order power in τ.

The third iterations is given below [6].

$$\begin{aligned}\Delta S_3(\tau,\mu_1,\mu_0) &= (\frac{1}{\mu_0}+\frac{1}{\mu_1})^{-1}\{U_3^c(\mu_0)+U_3^c(\mu_1) \\ &- \exp(-\frac{\tau}{\mu_0})V_3^c(\mu_1) - \exp(-\frac{\tau}{\mu_1})V_3^c(\mu_0) + U_3^c(\mu_0)U_3^c(\mu_1) - V_3^c(\mu_1)V_3^c(\mu_0)\}\end{aligned} \tag{62}$$

where $U_3^c(\mu*)$ and $V_3^c(\mu*)$ are different from $U_3(\mu*)$ and $V_3(\mu*)$ and are given below,

$$\begin{aligned}U_3^c(\tau,p^*) &= U_3^-(\tau,p^*) + \exp(-\tau p^*)V_3^-(\tau,-p^*) \\ &+V_2(p^*)V_3^-(\tau,-p^*) - U_2(p^*)U_3^-(\tau,-p^*) \qquad (63)\\ V_3^c(\tau,p^*) &= U_2(p^*)V_2(p^*) + V_3^-(\tau,p^*) - \exp(-\tau p^*)U_3^-(\tau,p^*) \\ &+U_2(p^*)V_3^-(\tau,p^*) - V_2(p^*)U_3^-(\tau,p^*). \qquad (64)\end{aligned}$$

ΔS_3 of Milne includes the product terms $[U_2(\mu_0)U_2(\mu_1) - V_2(\mu_1)V_2(\mu_0)]$ that are included in ΔS_2 of Chandrasekhar. All the other terms of U_3 (V_3) of Milne are included in U_3^c (V_3^c) of Chandrasekhar. The terms which are included in U_3^c (V_3^c) of Chandrasekhar but not in that of Milne do not have the third or lower power in τ.

Thus the truncated polynomials of the third approximation S_3 of Milne and Chandrasekhar are identical up to the third power in τ, though the apparent forms of the third approximations S_3 are different.
The truncated form of $S_3(\tau,\mu_1,\mu_0)$ up to third power in τ is given below,

$$
\begin{aligned}
&S_3(\tau,\mu_1,\mu_0)\\
&=\tau-\frac{1}{2}(\frac{1}{\mu_1}+\frac{1}{\mu_0})\tau^2+\frac{1}{6}(\frac{1}{\mu_1}+\frac{1}{\mu_0})^2\tau^3\\
&+\frac{1}{2}(-\log\tau)\tau^2+(\frac{3}{4}-\frac{\gamma}{2})\tau^2\\
&+(\frac{1}{8}-\frac{1}{4}(\frac{1}{\mu_1}+\frac{1}{\mu_0}))(-\log\tau)\tau^3+[\frac{-\gamma}{8}+\frac{7}{24}-(-\frac{\gamma}{4}+\frac{3}{4})(\frac{1}{\mu_1}+\frac{1}{\mu_0})]\tau^3\\
&+\frac{1}{4}(-\log\tau)^2\tau^3+(-\frac{\gamma}{2}+\frac{5}{8})(-\log\tau)\tau^3+(\frac{\gamma^2}{4}-\frac{5}{8}\gamma+\frac{7}{12}-\frac{\pi^2}{72})\tau^3 \quad (65)
\end{aligned}
$$

where γ is the Euler constant (0.5772).

5 Discussion on Uniqueness of Solution

The solution by iterative integration of Chandrasekhar's equation does not guarantee the uniqueness of solution [1]. Chandrasekhar argues that X and Y functions have multiple solutions. Using X and Y, $S(\tau,\mu_1,\mu_0)$ and $T(\tau,\mu_1,\mu_0)$ are given below.

$$
\begin{aligned}
S(\tau,\mu_1,\mu_0) &= (\frac{1}{\mu_1}+\frac{1}{\mu_0})^{-1}[X(\mu_1)X(\mu_0)-Y(\mu_1)Y(\mu_0)]\\
T(\tau,\mu_1,\mu_0) &= (\frac{1}{\mu_1}-\frac{1}{\mu_0})^{-1}[Y(\mu_0)X(\mu_1)-X(\mu_0)Y(\mu_1)] \quad (66)
\end{aligned}
$$

The author shows that $X(\mu*)$ and $Y(\mu*)$ are identical to the infinite series series of $U_n^c(\tau,\mu*)$ and $V_n^c(\tau,\mu*)$ [6],

$$
\begin{aligned}
X(\tau,\mu*) &= 1+U_2(\tau,\mu*)+U_3^c(\tau,\mu*)+\cdot+\cdot\\
Y(\tau,\mu*) &= \exp(-\tau/\mu*)+V_2(\tau,\mu*)+V_3^c(\tau,\mu*)+\cdot+\cdot \quad (67)
\end{aligned}
$$

where $U_n^c(\tau,\mu*)$ and $V_n^c(\tau,\mu*)$ are different with those of the Milne equation, and $V_n^c(\tau,\mu*)$ is given by the following iterative integration with $U_2(\tau,\mu*)$ and $V_2(\tau,\mu*)$ as the first iteration [6].

$$
\begin{aligned}
&V_{n+1}^c(\tau,\mu*) = U_2(\tau,\mu*)V_n^c(\tau,\mu*)\\
&+(1+U_n^c(p^*))\int_0^1(\frac{1}{\mu}-\frac{1}{\mu*})^{-1}[V_n^c(\tau,\mu*)-V_n^c(\tau,\mu)]\frac{d\mu}{2\mu}\\
&-(1+\exp(-\frac{\tau}{\mu*})\int_0^1)(\frac{1}{\mu}-\frac{1}{\mu*})^{-1}[U_n^c(\tau,\mu^*)-U_n^c(\tau,\mu)]\frac{d\mu}{2\mu} \quad (68)
\end{aligned}
$$

And $U_n^c(\tau, \mu*)$ is obtained from $\exp(-\tau/\mu*)V_n^c(\tau, -\mu*)$.

By iterative integration, $X(\tau, \mu*)$ is expressed below [6].

$$
\begin{aligned}
X(\tau, \mu*) = & [a_{01}\tau + a_{02}\tau^2 + \cdot + \cdot + \cdot + \cdot + \cdot] \\
& + \log(\tau)[a_{11}\tau + a_{12}\tau^2 + \cdot + \cdot + \cdot + \cdot + \cdot] \\
& +(\log(\tau))^2[a_{22}\tau^2 + a_{23}\tau^3 + \cdot + \cdot + \cdot + \cdot + \cdot] \\
& + \cdot + \cdot + \cdot
\end{aligned} \tag{69}
$$

The function $X(\tau, \mu*)$ is not regular at $\tau = 0$, though it is continuous at $\tau = 0$. The singularity of $X(\tau, \mu*)$ at $\tau = 0$ is due to $\log(\tau)$, $(\log(\tau))^2$ etc. The function $\log(\tau)$ is a multiple function around $\tau = 0$.

$$
\log(\tau) = \log(|\tau|) + 2n\pi i \tag{70}
$$

where n is $0, \pm 1, \pm 2, \pm 3, \cdots$.
Substituting the above equation in equation (69), we obtain other solutions on the complex domain or branches $n \neq 0$. And if all the imaginary parts vanish, the solution might be another solution with real number coefficients on the branch $n \neq 0$. The new solution is a sum of the basic solution (BS) that is obtained on the branch $n = 0$ and the add-on (AO) terms that comes from the imaginary parts of the solution on the branch $n \neq 0$. This additional characteristics of BS and AO for the multiple solutions agrees with the works that concerns the uniqueness of the X and Y functions [1] [3]. The possibility of multiple solution is not only limited to $X(\tau, \mu*)$ but also for other functions like $S(\tau, \mu_1, \mu_0)$ and $T(\tau, \mu_2, \mu_0)$ because of their inclusion of $\log(\tau)$.

If we choose the branch $n = 0$ and choose $\tau_0 \neq 0$ as the point around which we expand the solution, the resulting function is regular on the domain $(0, 2\tau_0)$. $\tau = 0$ is excluded in the convergent domain. The solution thus obtained might be unique. The solution has a true singularity at $\tau = 0$. This is clear if you differentiate the solution, the derivatives at $\tau = 0$ is ∞. Nevertheless it is noted that $\lim_{\tau \to 0} X(\tau, \mu*) = 0$ holds.

6 Conclusion

We obtain a solution of radiative transfer in the anisotropic plane-parallel atmosphere by iterating integration of the Milne integral equation. The truncated polynomials of the scattering function, $S_3(\tau, \mu_1, \mu_0)$, up to the third power in τ is identical with that by Chandrasekhar's integral equation.

$$
\begin{aligned}
&S_3(\tau,\mu_1,\mu_0)\\
=&\tau-\frac{1}{2}(\frac{1}{\mu_1}+\frac{1}{\mu_0})\tau^2+\frac{1}{6}(\frac{1}{\mu_1}+\frac{1}{\mu_0})^2\tau^3\\
&+\frac{1}{2}(-\log\tau)\tau^2+(\frac{3}{4}-\frac{\gamma}{2})\tau^2\\
&+(\frac{1}{8}-\frac{1}{4}(\frac{1}{\mu_1}+\frac{1}{\mu_0}))(-\log\tau)\tau^3+[\frac{-\gamma}{8}+\frac{7}{24}-(-\frac{\gamma}{4}+\frac{3}{4})(\frac{1}{\mu_1}+\frac{1}{\mu_0})]\tau^3\\
&+\frac{1}{4}(-\log\tau)^2\tau^3+(-\frac{\gamma}{2}+\frac{5}{8})(-\log\tau)\tau^3+(\frac{\gamma^2}{4}-\frac{5}{8}\gamma+\frac{7}{12}-\frac{\pi^2}{72})\tau^3 \quad (71)
\end{aligned}
$$

where γ is the Euler constant (0.5772).

The solution has a true singularity at $\tau = 0$ since it has $\log(\tau)$. And $\log(\tau)$ terms imply the possibility of multiple solution around $\tau = 0$.

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Appendix 1. Derivation of Equation (33)

$$
\begin{aligned}
&\int_0^1\int_0^1\int_0^\tau\int_0^y\int_0^y \exp(-\frac{z}{\mu_1})\exp(-\frac{y-z}{\mu_b})\exp(-\frac{y-x}{\mu_a})\exp(-\frac{x}{\mu_0})dzdydx\frac{d\mu_a}{2\mu_a}\frac{d\mu_b}{2\mu_b} \\
&=\int_0^1\int_0^1\int_0^\tau \exp(-\frac{y}{\mu_b}-\frac{y}{\mu_a})\{\int_0^y \exp(-\frac{z}{\mu_1}+\frac{z}{\mu_b})dz\}\int_0^y \exp(-\frac{x}{\mu_0}+\frac{x}{\mu_a})dx\}dy\frac{d\mu_a}{2\mu_a}\frac{d\mu_b}{2\mu_b} \\
&=\int_0^1\int_0^1(-\frac{1}{\mu_1}+\frac{1}{\mu_b})^{-1}\{(-\frac{1}{\mu_0}+\frac{1}{\mu_a})^{-1}(-\frac{1}{\mu_0}-\frac{1}{\mu_1})^{-1}[\exp(-\frac{\tau}{\mu_0}-\frac{\tau}{\mu_1})-1] \\
&\qquad -(-\frac{1}{\mu_0}+\frac{1}{\mu_a})^{-1}(-\frac{1}{\mu_a}-\frac{1}{\mu_1})^{-1}[\exp(-\frac{\tau}{\mu_a}-\frac{\tau}{\mu_1})-1] \\
&\qquad -(-\frac{1}{\mu_0}+\frac{1}{\mu_a})^{-1}(-\frac{1}{\mu_0}-\frac{1}{\mu_b})^{-1}[\exp(-\frac{\tau}{\mu_0}-\frac{\tau}{\mu_b})-1] \\
&\qquad +(-\frac{1}{\mu_0}+\frac{1}{\mu_a})^{-1}(-\frac{1}{\mu_a}-\frac{1}{\mu_b})^{-1}[\exp(-\frac{\tau}{\mu_a}-\frac{\tau}{\mu_b})-1]\}\frac{d\mu_a}{2\mu_a}\,\}\,\frac{d\mu_b}{2\mu_b} \\
&=\int_0^1\{(-\frac{1}{\mu_1}+\frac{1}{\mu_b})^{-1}(-\frac{1}{\mu_0}-\frac{1}{\mu_1})^{-1}\exp(-\frac{\tau}{\mu_1})V_2(\mu_0) \\
&\quad -(-\frac{1}{\mu_1}+\frac{1}{\mu_b})^{-1}(-\frac{1}{\mu_0}-\frac{1}{\mu_1})^{-1}U_2(\mu_1) \\
&\quad -(-\frac{1}{\mu_1}+\frac{1}{\mu_b})^{-1}(-\frac{1}{\mu_0}-\frac{1}{\mu_b})^{-1}\exp(-\frac{\tau}{\mu_b})V_2(\mu_0) \\
&\quad +(-\frac{1}{\mu_1}+\frac{1}{\mu_b})^{-1}(-\frac{1}{\mu_0}-\frac{1}{\mu_b})^{-1}U_2(\mu_b)\,\}\,\frac{d\mu_b}{2\mu_b} \\
&=\int_0^1\{(-\frac{1}{\mu_1}+\frac{1}{\mu_b})^{-1}(-\frac{1}{\mu_0}-\frac{1}{\mu_1})^{-1}[\exp(-\frac{\tau}{\mu_1})-\exp(-\frac{\tau}{\mu_b})]V_2(\mu_0) \\
&\quad -(-\frac{1}{\mu_1}+\frac{1}{\mu_b})^{-1}(-\frac{1}{\mu_0}-\frac{1}{\mu_1})^{-1}[U_2(\mu_1)-U_2(\mu_b)] \\
&\quad +(-\frac{1}{\mu_0}-\frac{1}{\mu_1})^{-1}(-\frac{1}{\mu_0}-\frac{1}{\mu_b})^{-1}[U_2(\mu_b)-U_2(-\mu_0)] \\
&\quad -(-\frac{1}{\mu_0}-\frac{1}{\mu_1})^{-1}(-\frac{1}{\mu_0}-\frac{1}{\mu_b})^{-1}[\exp(-\frac{\tau}{\mu_b})V_2(\mu_0)-U_2(-\mu_0)]\,\}\,\frac{d\mu_b}{2\mu_b} \\
&=(-\frac{1}{\mu_0}-\frac{1}{\mu_1})^{-1}\{V_2(\mu_1)V_2(\mu_0)-U_3^-(\mu_1)+U_3^-(-\mu_0)-V_2(-\mu_0)V_2(\mu_0)]\} \\
&=(\frac{1}{\mu_0}+\frac{1}{\mu_1})^{-1}\{-V_2(\mu_1)V_2(\mu_0)+U_3^-(\mu_1)+U_3^-(\mu_0)\}
\end{aligned}
$$

Appendix 2. Derivation of Equation 36)

$$
\begin{aligned}
&\int_0^1\int_0^1\int_0^\tau\int_y^\tau\int_y^\tau \exp(-\frac{z}{\mu_1})\exp(\frac{y-z}{\mu_b})\exp(\frac{y-x}{\mu_a})\exp(-\frac{x}{\mu_0})dzdydx\frac{d\mu_a}{2\mu_a}\frac{d\mu_b}{2\mu_b}\\
&=\int_0^1\int_0^1\int_0^\tau \exp(\frac{y}{\mu_a}+\frac{y}{\mu_a})\{\int_y^\tau \exp(-\frac{z}{\mu_1}-\frac{z}{\mu_b})dz\}\int_y^\tau \exp(-\frac{x}{\mu_0}-\frac{x}{\mu_a})dx\}dy\frac{d\mu_a}{2\mu_a}\frac{d\mu_b}{2\mu_b}=\int_0^1\int_0^1\\
&\{\ (-\frac{1}{\mu_1}-\frac{1}{\mu_b})^{-1}(-\frac{1}{\mu_0}-\frac{1}{\mu_a})^{-1}\exp(-\frac{\tau}{\mu_1}-\frac{\tau}{\mu_b})(\frac{1}{\mu_a}+\frac{1}{\mu_b})^{-1}\exp(-\frac{\tau}{\mu_0}-\frac{\tau}{\mu_a})[\exp(\frac{\tau}{\mu_a}+\frac{\tau}{\mu_b})-1]\\
&\quad-(-\frac{1}{\mu_1}-\frac{1}{\mu_b})^{-1}(-\frac{1}{\mu_0}-\frac{1}{\mu_a})^{-1}(-\frac{1}{\mu_0}+\frac{1}{\mu_b})^{-1}\exp(-\frac{\tau}{\mu_1}-\frac{\tau}{\mu_b})[\exp(-\frac{\tau}{\mu_0}+\frac{\tau}{\mu_b})-1]\\
&\quad-(-\frac{1}{\mu_1}-\frac{1}{\mu_b})^{-1}(-\frac{1}{\mu_0}-\frac{1}{\mu_a})^{-1}(-\frac{1}{\mu_1}+\frac{1}{\mu_a})^{-1}\exp(-\frac{\tau}{\mu_0}-\frac{\tau}{\mu_a})[\exp(-\frac{\tau}{\mu_1}+\frac{\tau}{\mu_a})-1]\\
&\quad+(-\frac{1}{\mu_1}-\frac{1}{\mu_b})^{-1}(-\frac{1}{\mu_0}-\frac{1}{\mu_a})^{-1}(-\frac{1}{\mu_0}-\frac{1}{\mu_1})^{-1}[\exp(-\frac{\tau}{\mu_0}-\frac{\tau}{\mu_1})-1]\,\}\,\frac{d\mu_a}{2\mu_a}\frac{d\mu_b}{2\mu_b}\\
&=\int_0^1\ \{+(-\frac{1}{\mu_1}-\frac{1}{\mu_b})^{-1}\exp(-\frac{\tau}{\mu_1}-\frac{\tau}{\mu_b})(-\frac{1}{\mu_0}+\frac{1}{\mu_b})^{-1}\exp(-\frac{\tau}{\mu_0})V_2(-\mu_b)\\
&\qquad-(-\frac{1}{\mu_1}-\frac{1}{\mu_b})^{-1}\exp(-\frac{\tau}{\mu_1}-\frac{\tau}{\mu_b})(-\frac{1}{\mu_0}+\frac{1}{\mu_b})^{-1}U_2(\mu_0)\\
&\qquad-(-\frac{1}{\mu_1}-\frac{1}{\mu_b})^{-1}(-\frac{1}{\mu_0}-\frac{1}{\mu_1})^{-1}\exp(-\frac{\tau}{\mu_0})V_2(\mu_1)\\
&\qquad+(-\frac{1}{\mu_1}-\frac{1}{\mu_b})^{-1}(-\frac{1}{\mu_0}-\frac{1}{\mu_1})^{-1}U_2(\mu_0)\,\}\,\frac{d\mu_b}{2\mu_b}\\
&=(\frac{1}{\mu_0}+\frac{1}{\mu_1})^{-1}[U_2(\mu_0)U_2(\mu_1)+\exp(-\frac{\tau}{\mu_1}-\frac{\tau}{\mu_0})U_3^-(\mu_0)]\\
&\quad-(\frac{1}{\mu_0}+\frac{1}{\mu_1})^{-1}\exp(-\frac{\tau}{\mu_0}-\frac{\tau}{\mu_1})[U_3^-(-\mu_1)+\exp(\frac{\tau}{\mu_0})V_2(\mu_0)U_2(\mu_0)]\\
&=(\frac{1}{\mu_0}+\frac{1}{\mu_1})^{-1}[U_2(\mu_0)U_2(\mu_1)+\exp(-\frac{\tau}{\mu_1}-\frac{\tau}{\mu_0})U_3^-(\mu_0)]\\
&\quad-(\frac{1}{\mu_0}+\frac{1}{\mu_1})^{-1}\exp(-\frac{\tau}{\mu_0}-\frac{\tau}{\mu_1})[V_2(\mu_1)V_2(-\mu_1)-U_3^-(\mu_1)+\exp(\frac{\tau}{\mu_0})V_2(\mu_0)U_2(\mu_0)]\\
&=(\frac{1}{\mu_0}+\frac{1}{\mu_1})^{-1}[U_2(\mu_0)U_2(\mu_1)+\exp(-\frac{\tau}{\mu_1}-\frac{\tau}{\mu_0})[U_3^-(\mu_0)+U_3^-(\mu_1)]\\
&\quad-(\frac{1}{\mu_0}+\frac{1}{\mu_1})^{-1}\exp(-\frac{\tau}{\mu_0}-\frac{\tau}{\mu_1})[V_2(\mu_1)V_2(-\mu_1)+\exp(\frac{\tau}{\mu_0})V_2(\mu_0)U_2(\mu_0)]
\end{aligned}
$$

Appendix 3. Derivation of Equation (37)

$$
\begin{aligned}
&\int_0^1\int_0^1\int_0^\tau\int_y^\tau\int_0^y \exp(-\frac{z}{\mu_1})\exp(\frac{y-z}{\mu_b})\exp(\frac{x-y}{\mu_a})\exp(-\frac{x}{\mu_0})dzdydx\frac{d\mu_a}{2\mu_a}\frac{d\mu_b}{2\mu_b}\\
&=\int_0^1\int_0^1\int_0^\tau \exp(\frac{y}{\mu_b}-\frac{y}{\mu_a})\{\int_y^\tau \exp(-\frac{z}{\mu_1}-\frac{z}{\mu_b})dz\}\{\int_0^y \exp(-\frac{x}{\mu_0}+\frac{x}{\mu_a})dx\}dy\,\}\,\frac{d\mu_a}{2\mu_a}\frac{d\mu_b}{2\mu_b}\\
&=\int_0^1\int_0^1\{(-\frac{1}{\mu_1}-\frac{1}{\mu_b})^{-1}\exp(-\frac{\tau}{\mu_1}-\frac{\tau}{\mu_b})(-\frac{1}{\mu_0}+\frac{1}{\mu_a})^{-1}(-\frac{1}{\mu_0}+\frac{1}{\mu_b})^{-1}[\exp(-\frac{\tau}{\mu_0}+\frac{\tau}{\mu_b})-1]\\
&\qquad -(-\frac{1}{\mu_1}-\frac{1}{\mu_b})^{-1}\exp(-\frac{\tau}{\mu_1}-\frac{\tau}{\mu_b})(-\frac{1}{\mu_0}+\frac{1}{\mu_a})^{-1}(-\frac{1}{\mu_a}+\frac{1}{\mu_b})^{-1}[\exp(-\frac{\tau}{\mu_a}+\frac{\tau}{\mu_b})-1]\\
&\qquad -(-\frac{1}{\mu_1}-\frac{1}{\mu_b})^{-1}(-\frac{1}{\mu_0}+\frac{1}{\mu_a})^{-1}(-\frac{1}{\mu_1}-\frac{1}{\mu_0})^{-1}[\exp(-\frac{\tau}{\mu_1}-\frac{\tau}{\mu_0})-1]\\
&\qquad +(-\frac{1}{\mu_1}-\frac{1}{\mu_b})^{-1}(-\frac{1}{\mu_0}+\frac{1}{\mu_a})^{-1}(-\frac{1}{\mu_a}-\frac{1}{\mu_1})^{-1}[\exp(-\frac{\tau}{\mu_a}-\frac{\tau}{\mu_1})-1]\}\,\frac{d\mu_a}{2\mu_a}\frac{d\mu_b}{2\mu_b}\\
&=\int_0^1\{(-\frac{1}{\mu_1}-\frac{1}{\mu_b})^{-1}(-\frac{1}{\mu_0}+\frac{1}{\mu_b})^{-1}\exp(-\frac{\tau}{\mu_1})V_2(\mu_0)\\
&\qquad -(-\frac{1}{\mu_1}-\frac{1}{\mu_b})^{-1}(-\frac{1}{\mu_0}+\frac{1}{\mu_b})^{-1}\exp(-\frac{\tau}{\mu_1})V_2(\mu_b)\\
&\qquad -(-\frac{1}{\mu_1}-\frac{1}{\mu_b})^{-1}(-\frac{1}{\mu_1}-\frac{1}{\mu_0})^{-1}\exp(-\frac{\tau}{\mu_1})V_2(\mu_0)\\
&\qquad +(-\frac{1}{\mu_1}-\frac{1}{\mu_b})^{-1}(-\frac{1}{\mu_1}-\frac{1}{\mu_0})^{-1}U_2(\mu_1)\}\,\frac{d\mu_b}{2\mu_b}\\
&=\int_0^1\{(-\frac{1}{\mu_0}+\frac{1}{\mu_b})^{-1}(-\frac{1}{\mu_1}-\frac{1}{\mu_0})^{-1}\exp(-\frac{\tau}{\mu_1})[V_2(\mu_0)-V_2(\mu_b)]\\
&\quad +(-\frac{1}{\mu_1}-\frac{1}{\mu_b})^{-1}(-\frac{1}{\mu_1}-\frac{1}{\mu_0})^{-1}\exp(-\frac{\tau}{\mu_1})[V_2(-\mu_1)-V_2(\mu_b)]\}\,\frac{d\mu_b}{2\mu_b}\\
&=(\frac{1}{\mu_1}+\frac{1}{\mu_0})^{-1}\exp(-\frac{\tau}{\mu_1})[-V_3^-(\mu_0)+V_3^-(-\mu_1)]
\end{aligned}
$$

Appendix 4. Derivation of Equation (35)

$$
\begin{aligned}
&U_3^-(\tau,-\mu*) = \int_0^1 (\frac{1}{\mu}+\frac{1}{\mu^*})^{-1}[U_2(-\mu^*)-U_2(\mu)]\frac{d\mu}{2\mu}\\
&= \int_0^1\int_0^1 (\frac{1}{\mu}+\frac{1}{\mu^*})^{-1}(\frac{1}{\mu'}-\frac{1}{\mu^*})^{-1}[1-\exp(-\frac{\tau}{\mu'}+\frac{\tau}{\mu^*})]\\
&- \int_0^1\int_0^1 (\frac{1}{\mu}+\frac{1}{\mu^*})^{-1}(\frac{1}{\mu'}+\frac{1}{\mu})^{-1}[1-\exp(-\frac{\tau}{\mu'}-\frac{\tau}{\mu})] \quad \frac{d\mu'}{2\mu'}\frac{d\mu}{2\mu}\\
&= \int_0^1\int_0^1 \{(\frac{1}{\mu'}-\frac{1}{\mu^*})^{-1}(\frac{1}{\mu}+\frac{1}{\mu^*})^{-1}[\exp(-\frac{\tau}{\mu'}-\frac{\tau}{\mu})-\exp(-\frac{\tau}{\mu'}+\frac{\tau}{\mu^*})]\}\\
&\qquad +(\frac{1}{\mu'}-\frac{1}{\mu^*})^{-1}(\frac{1}{\mu'}+\frac{1}{\mu})^{-1}[1-\exp(-\frac{\tau}{\mu'}-\frac{\tau}{\mu})]\} \quad \frac{d\mu}{2\mu}\frac{d\mu'}{2\mu'}\\
&= \int_0^1 (\frac{1}{\mu'}-\frac{1}{\mu^*})^{-1}[-\exp(-\frac{\tau}{\mu'})V_2(-\mu^*)+U_2(\mu')]\} \quad \frac{d\mu'}{2\mu'}\\
&= \int_0^1 (\frac{1}{\mu'}-\frac{1}{\mu^*})^{-1}\{[\exp(-\frac{\tau}{\mu^*})-\exp(-\frac{\tau}{\mu'})]V_2(-\mu^*)+[U_2(\mu')-U_2(\mu^*)]\} \quad \frac{d\mu'}{2\mu'}\\
&= V_2(\mu^*)V_2(-\mu^*)-U_3^-(\mu^*)
\end{aligned}
$$

Figure Captions

Figure 1: Geometry of Radiative Transfer