International Symposium on Information Theory and its Applications, ISITA2008
Auckland, New Zealand, 7-10, December, 2008

Generalized Wigner-Yanase Skew Information And Generalized Fisher Information

Kenjiro YANAGI[‡], Shigeru FURUICHI[†] and Ken KURIYAMA[‡]

[‡] Applied Mathematical Science, Graduate School of Science and Engineering, Yamaguchi University, 2-16-1, Tokiwadai, Ube, 755-8611, Japan
E-mail: yanagi@yamaguchi-u.ac.jp
E-mail: kuriyama@yamaguchi-u.ac.jp

[†] Department of Computer Science and System Analysis, College of Humanities and Sciences, Nihon University, 3-25-40, Sakurajyousui, Setagaya-ku, Tokyo, 156-8550, Japan
E-mail: furuichi@cssa.chs.nihon-u.ac.jp

Abstract

We introduce a generalized Wigner-Yanase skew information and then derive the trace inequality related to the uncertainty relation. This inequality is a non-trivial generalization of the uncertainty relation derived by S.Luo for the quantum uncertainty quantity excluding the classical mixture. And we introduce a generalized Fisher information and then derive a generalized Cramér-Rao inequality. We also give an example for our generalized Fisher information and then derive the uncertainty relation for two observables.

1. INTRODUCTION

As a degree for non-commutativity between a quantum state ρ and an observable H, Wigner-Yanase skew information

$$I_\rho(H) \equiv \frac{1}{2} Tr\left[\left(i\left[\rho^{1/2}, H\right]\right)^2\right]$$

was defined in [10]. Here we denote the commutator by $[X, Y] = XY - YX$. This quantity was generalized by Dyson

$$I_{\rho,\alpha}(H) = \frac{1}{2} Tr\left[(i[\rho^\alpha, H])(i[\rho^{1-\alpha}, H])\right]$$

which is known as the Wigner-Yanase-Dyson skew information. It is famous that the convexity of $I\rho, \alpha(H)$ with respect to ρ was successfully proven by E.Lieb in [7]. From the physical point of view, an observable H is generally considered to be an unbounded operator, however in the present paper, unless otherwise stated, we consider $H \in B(\mathcal{H})$, where $B(\mathcal{H})$ represents the set of all bounded linear operators on the Hilbert space $\mathcal{H}$, as a mathematical interest. We also denote the set of all self-adjoint operators (observables) by $\mathcal{L}_h(\mathcal{H})$ and the set of all density operators (quantum states) by $\mathcal{S}(\mathcal{H})$ on the Hilbert space $\mathcal{H}$. The relation between the Wigner-Yanase skew information and the uncertainty relation was studied in [9]. Moreover the relation between the Wigner-Yanase-Dyson skew information and the uncertainty relation was studied in [6, 11]. In our previous paper [11], we defined a generalized skew information and then derived a kind of an uncertainty relation. In the section 2, we introduce a new generalized Wigner-Yanase-Dyson skew information. On a generalization of the original Wigner-Yanase-Dyson skew information, our generalization is different from the Wigner-Yanase-Dyson skew information and a generalized skew information defined in our previous paper [11].

On the other hand, we have some definitions for the Fisher information in quantum mechanical system. In the section 3, we consider the standard definition and its one-parameter extended one. For a parameterized density operator $\rho_\theta \in \mathcal{S}_\theta(\mathcal{H})$, we define the Fisher information by

$$I(\rho_\theta, L_\theta) \equiv Tr[\rho_\theta L_\theta L_\theta^*],$$

where the logarithmic derivative L_θ is defined by

$$\frac{\partial \rho_\theta}{\partial \theta} \equiv \frac{1}{2}(\rho_\theta L_\theta + L_\theta^* \rho_\theta)$$

and $\mathcal{S}_\theta(\mathcal{H})$ represents the set of all quantum states with one-parameter $\theta \in \mathbb{R}$. In the section 3 of the present paper, we define a one-parameter extended Fisher information and study some trace inequalities between this quantity and the variance (a generalized Cramér-Rao type inequality). See the literatures [2, 3] on recent advances of the skew information, the Fisher information and the uncertainty relation.

This work was partially supported by Grant-in-Aids for Scientific Research (C), 20540175, 20540176 and (B), 18300003.

2. TRACE INEQUALITIES ON A GENERALIZED WIGNER-YANASE SKEW INFORMATION

We review the relation between the Wigner-Yanase skew information and the uncertainty relation. In quantum mechanical system, the expectation value of an observable H in a quantum state ρ is expressed by $Tr[\rho H]$. It is natural that the variance for a quantum state ρ and an observable H is defined by $V_\rho(H) \equiv Tr[\rho(H - Tr[\rho H]I)^2] = Tr[\rho H^2] - Tr[\rho H]^2$. It is famous that we have the Heisenberg's uncertainty relation:

$$V_\rho(A)V_\rho(B) \geq \frac{1}{4}|Tr[\rho[A,B]]|^2 \tag{1}$$

for a quantum state ρ and two observables A and B. The further strong result was given by Schrödinger

$$V_\rho(A)V_\rho(B) - |Cov_\rho(A,B)|^2 \geq \frac{1}{4}|Tr[\rho[A,B]]|^2,$$

where the covariance is defined by $Cov_\rho(A,B) \equiv Tr[\rho(A - Tr[\rho A]I)(B - Tr[\rho B]I)]$. However, the uncertainty relation for the skew information failed. (See [9, 6, 11].)

$$I_\rho(A)I_\rho(B) \geq \frac{1}{4}|Tr[\rho[A,B]]|^2.$$

Recently S.Luo introduced the quantity $U_\rho(H)$ representing a quantum uncertainty excluding the classical mixture:

$$U_\rho(H) \equiv \sqrt{V_\rho(H)^2 - (V_\rho(H) - I_\rho(H))^2}.$$

Note that we have the relation among quantities as

$$0 \leq I_\rho(H) \leq U_\rho(H) \leq V_\rho(H). \tag{2}$$

For a quantum state ρ and observables X, Y, he derived the following uncertainty relation in [8]:

$$U_\rho(X)U_\rho(Y) \geq \frac{1}{4}|Tr[\rho[X,Y]]|^2. \tag{3}$$

The inequality (3) is a refinement of the inequality (1) in the sense of (2). In this section, we study two types of one-parameter extended inequalities for the inequality (3).

Definition 2.1 *For $0 \leq \alpha \leq 1$, a quantum state ρ and an observable H, we define the Wigner-Yanase-Dyson skew information*

$$I_{\rho,\alpha}(H) \equiv \frac{1}{2}Tr\left[(i[\rho^\alpha, H_0])(i[\rho^{1-\alpha}, H_0])\right]$$

and we also define

$$J_{\rho,\alpha}(H) \equiv \frac{1}{2}Tr\left[\{\rho^\alpha, H_0\}\{\rho^{1-\alpha}, H_0\}\right],$$

where $H_0 \equiv H - Tr[\rho H]I$ and we denote the anti-commutator by $\{X,Y\} = XY - YX$.

Note that we have

$$\begin{aligned} & \frac{1}{2}Tr\left[(i[\rho^\alpha, H_0])(i[\rho^{1-\alpha}, H_0])\right] \\ = & \frac{1}{2}Tr\left[(i[\rho^\alpha, H])(i[\rho^{1-\alpha}, H])\right], \end{aligned}$$

but we have

$$\frac{1}{2}Tr[\{\rho^\alpha, H_0\}\{\rho^{1-\alpha}, H_0\}] \neq \frac{1}{2}Tr[\{\rho^\alpha, H\}\{\rho^{1-\alpha}, H\}].$$

Then we have the following inequalities:

$$I_{\rho,\alpha}(H) \leq I_\rho(H) \leq J_\rho(H) \leq J_{\rho,\alpha}(H), \tag{4}$$

since we have $Tr[\rho^{1/2}H\rho^{1/2}H] \leq Tr[\rho^\alpha H \rho^{1-\alpha}H]$. If we define

$$U_{\rho,\alpha}(H) \equiv \sqrt{V_\rho(H)^2 - (V_\rho(H) - I_{\rho,\alpha}(H))^2},$$

as a direct generalization of Eq.(1), then we have

$$0 \leq I_{\rho,\alpha}(H) \leq U_{\rho,\alpha}(H) \leq U_\rho(H)$$

due to the first inequality of (9). We also have

$$U_{\rho,\alpha}(H) = \sqrt{I_{\rho,\alpha}(H)J_{\rho,\alpha}(H)}.$$

In this paper, we introduce a generalized Wigner-Yanase skew information which is a generalized Wigner-Yanase skew information by

$$K_{\rho,\alpha}(H) \equiv \frac{1}{2}Tr\left[\left(i\left[\frac{\rho^\alpha + \rho^{1-\alpha}}{2}, H_0\right]\right)^2\right]$$

and we also define

$$L_{\rho,\alpha}(H) \equiv \frac{1}{2}Tr\left[\left(\left\{\frac{\rho^\alpha + \rho^{1-\alpha}}{2}, H_0\right\}\right)^2\right].$$

Throughout this section, we put $X_0 \equiv X - Tr[\rho X]I$ and $Y_0 \equiv Y - Tr[\rho Y]I$. Then we show the following trace inequality.

Theorem 2.2 *For a quantum state ρ and observable X, Y and $\alpha \in [0,1]$, we have*

$$W_{\rho,\alpha}(X)W_{\rho,\alpha}(Y) \geq \frac{1}{4}\left|Tr\left[\left(\frac{\rho^\alpha + \rho^{1-\alpha}}{2}\right)^2 [X,Y]\right]\right|^2,$$

where

$$W_{\rho,\alpha}(X) \equiv \sqrt{K_{\rho,\alpha}(X)L_{\rho,\alpha}(X)}.$$

Remark 2.3 *Theorem 2.2 is not trivial by the following two reasons.*

(1) *There is no relation between*

$$\left|Tr\left[(\frac{\rho^{\alpha}+\rho^{1-\alpha}}{2})^2[X,Y]\right]\right|^2$$

and $|Tr\,[\rho[X,Y]]|^2$.

(2) *Though* $U_{\rho,\alpha}(H) \leq U_{\rho}(H)$ *and* $U_{\rho,\alpha}(H) \leq \tilde{U}_{\rho,\alpha}(H)$ *hold, there is no relation between* $U_{\rho}(H)$ *and* $\tilde{U}_{\rho,\alpha}(H)$.

Proof of Theorem 2.2. We put

$$A_{\alpha}(H) \equiv i[\rho^{\alpha}, H_0], B_{\alpha}(H) \equiv \{\rho^{\alpha}, H_0\},$$

$$K = \frac{1}{2}(A_{\alpha}(X)+A_{1-\alpha}(X))x + \frac{1}{2}(B_{\alpha}(Y)+B_{1-\alpha}(Y)).$$

It follows from $K^* = K$ that

$$\begin{aligned}
0 &\leq Tr[KK^*] \\
&= \frac{1}{4}Tr[(A_{\alpha}(X)+A_{1-\alpha}(X))^2]x^2 \\
&\quad +\frac{1}{2}Tr[(A_{\alpha}(X)+A_{1-\alpha}(X))(B_{\alpha}(Y)+B_{1-\alpha}(Y))]x \\
&\quad +\frac{1}{4}Tr[(B_{\alpha}(Y)+B_{1-\alpha}(Y))^2] \\
&= \left(\frac{1}{4}Tr[A_{\alpha}(X)^2+A_{1-\alpha}(X)^2]+I_{\rho,\alpha}(X)\right)x^2 \\
&\quad +\frac{1}{2}Tr[(A_{\alpha}(X)+A_{1-\alpha}(X))(B_{\alpha}(Y)+B_{1-\alpha}(Y))]x \\
&\quad +\left(\frac{1}{4}Tr[B_{\alpha}(Y)^2+B_{1-\alpha}(Y)^2]+J_{\rho,\alpha}(Y)\right).
\end{aligned}$$

Then

$$\begin{aligned}
&\frac{1}{4}\left(Tr[(A_{\alpha}(X)+A_{1-\alpha}(X))(B_{\alpha}(Y)+B_{1-\alpha}(Y))]\right)^2 \\
\leq\ & 4\left(\frac{1}{4}Tr[A_{\alpha}(X)^2+A_{1-\alpha}(X)^2+I_{\rho,\alpha}(X)\right) \qquad (5) \\
&\left(\frac{1}{4}Tr[B_{\alpha}(Y)^2+B_{1-\alpha}(Y)^2]+J_{\rho,\alpha}(Y)\right).
\end{aligned}$$

Now we have

$$\begin{aligned}
&Tr[(A_{\alpha}(X)+A_{1-\alpha}(X))(B_{\alpha}(Y)+B_{1-\alpha}(Y))] \\
=\ & Tr[(i[\rho^{\alpha},X_0]+i[\rho^{1-\alpha},X_0])(\{\rho^{\alpha},Y_0\}+\{\rho^{1-\alpha},Y_0\})] \\
=\ & iTr[(\rho^{\alpha}+\rho^{1-\alpha})^2X_0Y_0 - Y_0X_0(\rho^{\alpha}+\rho^{1-\alpha})^2] \\
=\ & Tr[(\rho^{\alpha}+\rho^{1-\alpha})^2(i[X_0,Y_0])] \\
=\ & Tr[(\rho^{\alpha}+\rho^{1-\alpha})^2(i[X,Y])].
\end{aligned}$$

Then (5) is equivalent to the following;

$$\begin{aligned}
&\frac{1}{4}\left(Tr[(\rho^{\alpha}+\rho^{1-\alpha})^2(i[X,Y])]\right)^2 \qquad (6) \\
\leq\ & 4\left(\frac{1}{4}Tr[A_{\alpha}(X)^2+A_{1-\alpha}(X)^2]+I_{\rho,\alpha}(X)\right) \\
&\left(\frac{1}{4}Tr[B_{\alpha}(Y)^2+B_{1-\alpha}(Y)^2]+J_{\rho,\alpha}(Y)\right).
\end{aligned}$$

And we also have

$$\begin{aligned}
&\frac{1}{4}\left|Tr[(\rho^{\alpha}+\rho^{1-\alpha})^2(i[X,Y])]\right|^2 \qquad (7) \\
\leq\ & 4\left(\frac{1}{4}Tr[A_{\alpha}(Y)^2+A_{1-\alpha}(Y)^2]+I_{\rho,\alpha}(Y)\right) \\
&\left(\frac{1}{4}Tr[B_{\alpha}(X)^2+B_{1-\alpha}(X)^2]+J_{\rho,\alpha}(X)\right).
\end{aligned}$$

By taking a square root of (6) × (7), we have

$$\begin{aligned}
&\left\{\frac{1}{4}\left(Tr[(\rho^{\alpha}+\rho^{1-\alpha})^2(i[X,Y])]\right)^2\right\}^2 \\
\leq\ & 4\left(\frac{1}{4}Tr[A_{\alpha}(X)^2+A_{1-\alpha}(X)^2]+I_{\rho,\alpha}(X)\right) \\
&\left(\frac{1}{4}Tr[B_{\alpha}(Y)^2+B_{1-\alpha}(Y)^2]+J_{\rho,\alpha}(Y)\right) \\
&4\left(\frac{1}{4}Tr[A_{\alpha}(Y)^2+A_{1-\alpha}(Y)^2]+I_{\rho,\alpha}(Y)\right) \\
&\left(\frac{1}{4}Tr[B_{\alpha}(X)^2+B_{1-\alpha}(X)^2]+J_{\rho,\alpha}(X)\right).
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{1}{4}\left(Tr[(\rho^{\alpha}+\rho^{1-\alpha})^2(i[X,Y])]\right)^2 \\
\leq\ & 2\sqrt{\left(\frac{1}{4}Tr[A_{\alpha}(X)^2+A_{1-\alpha}(X)^2]+I_{\rho,\alpha}(X)\right)} \\
&\sqrt{\left(\frac{1}{4}Tr[B_{\alpha}(Y)^2+B_{1-\alpha}(Y)^2]+J_{\rho,\alpha}(Y)\right)} \\
&2\sqrt{\left(\frac{1}{4}Tr[A_{\alpha}(Y)^2+A_{1-\alpha}(Y)^2]+I_{\rho,\alpha}(Y)\right)} \\
&\sqrt{\left(\frac{1}{4}Tr[B_{\alpha}(X)^2+B_{1-\alpha}(X)^2]+J_{\rho,\alpha}(X)\right)}.
\end{aligned}$$

Therefore

$$\frac{1}{4}\left(Tr\left[\left(\frac{\rho^{\alpha}+\rho^{1-\alpha}}{2}\right)^2(i[X,Y])\right]\right)^2$$

$$\leq W_{\alpha}(\rho,X)W_{\alpha}(\rho,Y).$$

Since

$$\overline{Tr\left[\left(\frac{\rho^{\alpha}+\rho^{1-\alpha}}{2}\right)^2[X,Y]\right]}$$

$$= -Tr\left[\left(\frac{\rho^{\alpha}+\rho^{1-\alpha}}{2}\right)^2[X,Y]\right],$$

we have

$$Re\left[\left(\frac{\rho^{\alpha}+\rho^{1-\alpha}}{2}\right)^2[X,Y]\right]=0.$$

And then

$$Tr\left[\left(\frac{\rho^{\alpha}+\rho^{1-\alpha}}{2}\right)^2[X,Y]\right]$$

$$= i\mathrm{Im}Tr\left[\left(\frac{\rho^{\alpha}+\rho^{1-\alpha}}{2}\right)^2[X,Y]\right].$$

Hence

$$\begin{aligned}
&\left(Tr\left[\left(\frac{\rho^{\alpha}+\rho^{1-\alpha}}{2}\right)^2(i[X,Y])\right]\right)^2\\
=&-\left(Tr\left[\left(\frac{\rho^{\alpha}+\rho^{1-\alpha}}{2}\right)^2[X,Y]\right]\right)^2\\
=&-\left(i\mathrm{Im}Tr\left[\left(\frac{\rho^{\alpha}+\rho^{1-\alpha}}{2}\right)^2[X,Y]\right]\right)^2\\
=&\left(\mathrm{Im}Tr\left[\left(\frac{\rho^{\alpha}+\rho^{1-\alpha}}{2}\right)^2[X,Y]\right]\right)^2\\
=&\left|Tr\left[\left(\frac{\rho^{\alpha}+\rho^{1-\alpha}}{2}\right)^2[X,Y]\right]\right|^2.
\end{aligned}$$

q.e.d.

We also define the followings to obtain another uncertainty relation.

Definition 2.4 *For a quantum state ρ and observable H and $\alpha\in[0,1]$, we define*

$$\begin{aligned}
&\tilde{W}_{\rho,\alpha}(H)\\
\equiv&\frac{1}{4}\sqrt{Tr\left[(i[\rho^{\alpha},H_0])^2\right]Tr\left[(i[\rho^{1-\alpha},H_0])^2\right]}\\
&\sqrt{Tr\left[\{\rho^{\alpha},H_0\}^2\right]Tr\left[\{\rho^{1-\alpha},H_0\}^2\right]}.
\end{aligned}$$

The we have the following theorem.

Theorem 2.5 *For a quantum state ρ and observable X,Y and $\alpha\in[0,1]$, we have*

$$\begin{aligned}
&\sqrt{\tilde{W}_{\rho,\alpha}(X)\tilde{W}_{\rho,\alpha}(Y)}\\
\geq&\frac{1}{4}\left|Tr\left[\rho^{2\alpha}[X,Y]\right]Tr\left[\rho^{2(1-\alpha)}[X,Y]\right]\right|.
\end{aligned}$$

Remark 2.6 *There is no relation between Theorem 2.2 and Theorem 2.5 by the following (1), (2).*

(1) *There is no relation between $4\tilde{W}_{\rho,\alpha}(X)$ and*

$$\left(Tr\left[\frac{(i[\rho^{\alpha},X_0])^2+(i[\rho^{1-\alpha},X_0])^2}{4}\right]+I_{\rho,\alpha}(X)\right)$$

$$\left(Tr\left[\frac{(\{\rho^{\alpha},X_0\})^2+(\{\rho^{1-\alpha},X_0\})^2}{4}\right]+J_{\rho,\alpha}(X)\right).$$

That is, there are no relation between

$$\sqrt{Tr\left[(i[\rho^{\alpha},X_0])^2\right]Tr\left[(i[\rho^{1-\alpha},X_0])^2\right]}$$

and

$$Tr\left[\frac{(i[\rho^{\alpha},X_0])^2+(i[\rho^{1-\alpha},X_0])^2}{4}\right]$$

$$+\frac{1}{2}Tr\left[(i[\rho^{\alpha},X_0])(i[\rho^{1-\alpha},X_0])\right].$$

and there is no relation between

$$\sqrt{Tr\left[\{\rho^{\alpha},X_0\}^2\right]Tr\left[\{\rho^{1-\alpha},X_0\}^2\right]}$$

and

$$Tr\left[\frac{\{\rho^{\alpha},X_0\}^2+\{\rho^{1-\alpha},X_0\}^2}{4}\right]$$

$$+\frac{1}{2}Tr\left[\{\rho^{\alpha},X_0\}\{\rho^{1-\alpha},X_0\}\right].$$

(2) *There is no relation between*

$$\left|Tr\left[\rho^{2\alpha}[X,Y]\right]Tr\left[\rho^{2(1-\alpha)}[X,Y]\right]\right|$$

and

$$\left|Tr\left[(\frac{\rho^{\alpha}+\rho^{1-\alpha}}{2})^2[X,Y]\right]\right|^2.$$

That is, there is no relation between

$$\left|Tr\left[\rho^{2\alpha}[X,Y]\right]\right|$$

and

$$\left|Tr\left[\left(\frac{\rho^{\alpha}+\rho^{1-\alpha}}{2}\right)^2[X,Y]\right]\right|.$$

and there in no relation between

$$\left|Tr\left[\rho^{2(1-\alpha)}[X,Y]\right]\right|$$

and

$$\left|Tr\left[\left(\frac{\rho^{\alpha}+\rho^{1-\alpha}}{2}\right)^{2}[X,Y]\right]\right|.$$

(3) *When $\alpha = 1/2$, both Theorem 2.2 and Theorem 2.5 reduce the result of Luo.*

Proof of Theorem 2.5. We put

$$K = i[\rho^{\alpha}, X_0]x + \{\rho^{\alpha}, Y_0\}.$$

It follows from $K^* = K$ that

$$\begin{aligned} 0 &\leq Tr\left[KK^*\right] \\ &= Tr\left[(i[\rho^{\alpha},X_0]x+\{\rho^{\alpha},Y_0\})^2\right] \\ &= Tr\left[(i[\rho^{\alpha},X_0])^2\right]x^2 + 2iTr\left[[\rho^{\alpha},X_0]\{\rho^{\alpha},Y_0\}\right]x \\ &\quad + Tr\left[\{\rho^{\alpha},Y_0\}^2\right] \\ &= Tr\left[(i[\rho^{\alpha},X_0])^2\right]x^2 + 2ii\text{Im}Tr\left[\rho^{2\alpha}[X,Y]\right]x \\ &\quad + Tr\left[\{\rho^{\alpha},Y_0\}^2\right]. \end{aligned}$$

Then

$$|Tr\left[\rho^{2\alpha}[X,Y]\right]|^2 = (\text{Im}Tr\left[\rho^{2\alpha}[X,Y]\right])^2$$
$$\leq Tr\left[(i[\rho^{\alpha},X_0])^2\right]Tr\left[\{\rho^{\alpha},Y_0\}^2\right].$$

By exchanging X and Y we have

$$|Tr\left[\rho^{2\alpha}[X,Y]\right]|^2$$
$$\leq Tr\left[(i[\rho^{\alpha},Y_0])^2\right]Tr\left[\{\rho^{\alpha},X_0\}^2\right].$$

And we also have

$$|Tr\left[\rho^{2(1-\alpha)}[X,Y]\right]|^2$$
$$\leq Tr\left[(i[\rho^{1-\alpha},X_0])^2\right]Tr\left[\{\rho^{1-\alpha},Y_0\}^2\right].$$

By exchanging X and Y we have

$$|Tr\left[\rho^{2(1-\alpha)}[X,Y]\right]|^2$$
$$\leq Tr\left[(i[\rho^{1-\alpha},Y_0])^2\right]Tr\left[\{\rho^{1-\alpha},X_0\}\right].$$

We put as follows;

$$S_{\rho,\alpha}(X) \equiv \frac{1}{2}Tr\left[(i[\rho^{\alpha},X_0])^2\right],$$

$$S_{\rho,1-\alpha}(X) \equiv \frac{1}{2}Tr\left[(i[\rho^{1-\alpha},X_0])^2\right],$$

$$S_{\rho,\alpha}(Y) \equiv \frac{1}{2}Tr\left[(i[\rho^{\alpha},Y_0])^2\right],$$

$$S_{\rho,1-\alpha}(Y) \equiv \frac{1}{2}Tr\left[(i[\rho^{1-\alpha},Y_0])^2\right],$$

$$T_{\rho,\alpha}(X) \equiv \frac{1}{2}Tr\left[\{\rho^{\alpha},X_0\}^2\right],$$

$$T_{\rho,1-\alpha}(X) \equiv \frac{1}{2}Tr\left[\{\rho^{1-\alpha},X_0\}^2\right],$$

$$T_{\rho,\alpha}(Y) \equiv \frac{1}{2}Tr\left[\{\rho^{\alpha},Y_0\}^2\right],$$

$$T_{\rho,1-\alpha}(Y) \equiv \frac{1}{2}Tr\left[\{\rho^{1-\alpha},Y_0\}^2\right].$$

Then we have

$$|Tr\left[\rho^{2\alpha}[X,Y]\right]|^2$$
$$\leq 4\sqrt{S_{\rho,\alpha}(X)T_{\rho,\alpha}(X)S_{\rho,\alpha}(Y)T_{\rho,\alpha}(Y)}.$$

$$|Tr\left[\rho^{2(1-\alpha)}[X,Y]\right]|^2$$
$$\leq 4\sqrt{S_{\rho,1-\alpha}(X)T_{\rho,1-\alpha}(X)S_{\rho,1-\alpha}(Y)T_{\rho,1-\alpha}(Y)}.$$

By putting

$$\tilde{W}_{\rho,\alpha}(X) \equiv \sqrt{S_{\rho,\alpha}(X)S_{\rho,1-\alpha}(X)T_{\rho,\alpha}(X)T_{\rho,1-\alpha}(X)},$$

$$\tilde{W}_{\rho,\alpha}(Y) \equiv \sqrt{S_{\rho,\alpha}(Y)S_{\rho,1-\alpha}(Y)T_{\rho,\alpha}(Y)T_{\rho,1-\alpha}(Y)},$$

we have

$$\sqrt{\tilde{W}_{\rho,\alpha}(X)\tilde{W}_{\rho,\alpha}(Y)}$$
$$\geq \frac{1}{4}|Tr\left[\rho^{2\alpha}[X,Y]\right]Tr\left[\rho^{2(1-\alpha)}[X,Y]\right]|.$$

q.e.d.

3. A GENERALIZED FISHER INFORMATION AND A GENERALIZED CRAMÉR-RAO INEQUALITY

We review the Fisher information and the Cramér-Rao inequality in quantum mechanical system. We consider the set of all quantum states:

$$\mathcal{S}_{\theta}(\mathcal{H}) \equiv \{\rho_{\theta} \in B(\mathcal{H}) | \rho_{\theta} \geq 0, Tr[\rho_{\theta}] = 1\},$$

with one parameter $\theta \in \mathbb{R}$. Let $H \in \mathcal{L}_h(\mathcal{H}) \equiv \{H \in B(\mathcal{H}) | H = H^*\}$ be an estimater of the parameter θ. In the sequel, we consider the case which an estimater is unbiased, that is, $E_{\theta}[H] \equiv Tr[\rho_{\theta}H] = \theta$. The variance $V_{\theta}[H]$ of the estimater H is defined by $V_{\theta}[H] \equiv Tr[\rho_{\theta}(H - Tr[\rho_{\theta}H]I)^2]$. Then the famous Cramér-Rao inequality, which is a relation between the Fisher information and the variance, $V_{\theta}[H] \geq \frac{1}{I(\rho_{\theta},L_{\theta})}$ holds. We should note that the logarithmic derivative

$L_\theta \in B(\mathcal{H})$ is not uniquely determined. Thus we define the symmetric logarithmic derivative $L_\theta^S \in \mathcal{L}_h(\mathcal{H})$ by

$$\frac{\partial \rho_\theta}{\partial \theta} \equiv \frac{1}{2}(\rho_\theta L_\theta^S + L_\theta^S \rho_\theta).$$

Then the symmetric logarithmic derivative L_θ^S is uniquely determined [1, 4, 5] and we have

$$I(\rho_\theta, L_\theta) \geq I(\rho_\theta, L_\theta^S). \quad (8)$$

In addition, for the symmetric logarithmic derivative L_θ^S, we have the Cramér-Rao inequality [1, 4, 5]:

$$V_\theta[H] \geq \frac{1}{I(\rho_\theta, L_\theta^S)}. \quad (9)$$

Due to the inequality (8), we have the following theorem known as Cramér-Rao inequality.

Theorem 3.1

$$V_\theta[H] \geq \frac{1}{I(\rho_\theta, L_\theta)}.$$

That is, the symmetric logarithmic derivative L_θ^S gives the best estimation of the lower bound for the variance $V_\theta[H]$.

We here introduce a generalized Fisher information with one-parameter $\alpha \in [0, 1]$.

Definition 3.2 *We define a generalized Fisher information by*

$$I_\alpha(\rho_\theta, L_{\theta,\alpha}) \equiv Tr[\rho_\theta^\alpha L_{\theta,\alpha} \rho_\theta^{1-\alpha} L_{\theta,\alpha}^*], \ \alpha \in [0, 1],$$

where a generalized logarithmic derivative $L_{\theta,\alpha}$ is defined by

$$\frac{\partial \rho_\theta}{\partial \theta} \equiv \frac{1}{2}\left(\rho_\theta^{\frac{1+\alpha}{2}} L_{\theta,\alpha} \rho_\theta^{\frac{1-\alpha}{2}} + \rho_\theta^{\frac{1-\alpha}{2}} L_{\theta,\alpha}^* \rho_\theta^{\frac{1+\alpha}{2}}\right). \quad (10)$$

Note that $\alpha = 1$ or $[\rho_\theta, L_{\theta,\alpha}] = 0$ recovers $I_\alpha(\rho_\theta, L_{\theta,\alpha}) = I(\rho_\theta, L_\theta)$. We also have $I_\alpha(\rho_\theta, L_{\theta,\alpha}) \geq 0$ and the following trace inequality.

Theorem 3.3 *For a self-adjoint operator H, a density operator ρ_θ with the parameter θ and $\alpha \in [0, 1]$, if we have $E_\theta[H] = \theta$, then we have the inequality*

$$V_\theta[H] \geq \frac{1}{I_\alpha(\rho_\theta, L_{\theta,\alpha})}. \quad (11)$$

It is clear that (11) is obtained by putting

$$L_{\rho,\theta} = \rho_\theta^{\frac{1-\alpha}{2}} L_\theta \rho_\theta^{\frac{\alpha-1}{2}}.$$

References

[1] S.Amari and H.Nagaoka, Methods of information geometry, AMS Oxford University Press, 2000.

[2] P.Gibilisco, D.Imparato and T.Isola, A Robertson-type uncertainty principle and quantum Fisher information, Linear Alg. Appl., vol.428(2008), pp.1706-1724.

[3] F.Hansen, Metric adjusted skew information, arXiv:math-ph/0607049v4.

[4] C.W.Helstrom, Quantum detection and estimation theory, Academic Press, 1976.

[5] A.S.Holevo, Probabilistic and statistical aspects of quantum theory, North-Holland, 1982.

[6] H.Kosaki, Matrix trace inequality related to uncertainty principle, International J. Math., vol.16(2005), pp.629-646.

[7] E.H.Lieb, Convex trace functions and the Wigner-Yanase-Dyson conjecture, Adv. Math., vol.11(1973), pp.267-288.

[8] S.Luo, Heisenberg uncertainty relation for mixed states, Phys. Rev.A, vol.72(2005), 042110.

[9] S.Luo and Q.Zhang, On skew information, IEEE Trans. Information Theory, vol.50(2004), pp.1778-1782, and Correction to "On skew information", IEEE Trans. Information Theory, vol.51(2005), p.4432.

[10] E.P.Wigner and M.M.Yanase, Information content of distribution, Proc. Nat. Acad. Sci, U.S.A., vol.49(1963), pp.910-918.

[11] K.Yanagi, S.Furuichi and K.Kuriyama, A generalization skew information and uncertainty relation, IEEE Trans. Information Theory, vol.51(2005), pp.4401-4404.