Necessary Conditions for Optimal Solution of Control Problems with State-variable Inequality Constraints

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(Received November 6, 2001)

Abstract. In this paper, necessary optimality conditions for optimal control problems subject to state-variable inequality constraints are derived under regularity assumptions for the problems. It is shown that so-called adjoint variables are continuous on whole interval on which the optimal state variable are defined. The regularity assumptions are sufficient condition for continuity of the adjoint variables.

Key Words. Optimal control, State constraint, Necessary condition, Regularity condition

1. Introduction

In Necessary conditions for optimal control problems with restricted coordinates were first presented by Gamkrelidze in 1960 [1]. Many works have been done on the problem since then. The necessary conditions have been shown by various means, for example, the calculus of variations [2-4], the abstract theory of optimality [5-7], etc. Further, the necessary conditions for the problem with higher-order state inequality constraints have been derived [3,8,9]. In this paper, the adjoint variables in the necessary conditions may be discontinuous, that is, may have jumps, when the optimal state on the boundary of the restricted region. But we can guess that the adjoint variables are continuous everywhere the optimal solution is defined, depending on the problem. In

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this paper, we shall consider optimal control problems with state-variable inequality constraints. Under regularity assumptions for our problems, it will be proved that the state variables, the optimal controls and the adjoint variables of the optimal control problems satisfy the necessary conditions which are similar to those of ordinary optimal control problem. In particular, the adjoint variables are absolutely continuous on whole interval on which the optimal state variables are defined.

In section 2, we will give a property with respect to linear operations and cones in normed spaces. In section 3, we will formulate an optimal control problems with state-variable inequality constraints and give notations, terminologies and assumptions for our problems. In section 4, we will introduce the conditions necessary for continuity of adjoint variables and give their properties. In section 5, we will prove necessary conditions for the optimal solutions of our problems. And in section 6, we will give a simple example with respect to time optimal control problem with state-variable inequality constraints.

2. Preliminary Results

In this section, we will show a mathematical concept on linear operators and cones in normed spaces, which will be used in the sequel.

Let $\mathscr{X}$, $\mathscr{Y}$ and $\mathscr{Z}$ be real normed spaces and let $\| \bullet \|_{\mathscr{X}}$, $\| \bullet \|_{\mathscr{Y}}$ and $\| \bullet \|_{\mathscr{Z}}$ be norms on the spaces $\mathscr{X}$, $\mathscr{Y}$ and $\mathscr{Z}$, respectively. Let Q be a non-empty convex subset of $\mathscr{X}$ containing the origin, let Y and P be a convex cone in $\mathscr{Y}$ with non-empty interior and a subset of $\mathscr{Z}$, and let Z be a convex cone in $\mathscr{Z}$. Throughout this paper, a set S of the underlying linear space will be called a cone, if it is not empty and if $\mu S \subset S$ whenever $\mu > 0$. Let $\hat{y}$ and $\hat{z}$ be an elements of $\mathscr{Y}$ and $\mathscr{Z}$ such that $\hat{y} \in Y$ and $\hat{z} \in Z$, respectively, and let G and H be a continuous linear operator from $\mathscr{X}$ into $\mathscr{Y}$ and a linear operator from $\mathscr{X}$ into $\mathscr{Z}$, respectively. For any subset S and any element a of underlying space, $\operatorname{int} S$, $\operatorname{cl} S$ and $[S|a]$ shall respectively denote the interior of S, the closure of S and the conical hull of $S - a$, that is,

(1) $$[S|a] = \bigcup_{\gamma > 0} \{\gamma(S - a)\}.$$

Further, each origin of the spaces $\mathscr{X}$, $\mathscr{Y}$ and $\mathscr{Z}$ shall be denoted the same notation $\mathit{0}$.

Now let us consider the following condition.

Condition 1. Besides the original norms $\| \bullet \|_{\mathscr{X}}$ on $\mathscr{X}$ and $\| \bullet \|_{\mathscr{Z}}$ on $\mathscr{Z}$, additional norms $] \bullet [_{\mathscr{X}}$ on $\mathscr{X}$ and $] \bullet [_{\mathscr{Z}}$ on $\mathscr{Z}$ can be defined so as to possess the following properties:

(C1) the convex cone Z has non-empty interior under the additional norm $] \bullet [_{\mathscr{Z}}$,

(C2) there exists an $\alpha \geq 1$ such that

$$\|x\|_{\mathscr{X}} \leq \alpha]x[_{\mathscr{X}}, \quad \|z\|_{\mathscr{Z}} \leq \alpha]z[_{\mathscr{Z}} \quad \text{for all} \quad x \in \mathscr{X} \text{ and } z \in \mathscr{Z},$$

(C3) $$\overline{P} = \mathscr{Z},$$

(C4) there exists a positive β such that, for each $p \in P$, we can find an $x \in \overline{[Q|0]}$ satisfying the relations

$$p \in H(x) - \overline{[Z|\hat{z}]} \quad \text{and} \quad \|x\|_{\mathscr{X}} \leq \beta \|p\|_{\mathscr{Z}}.$$

Here $\overline{S}$ denotes the closure of the set S in $\mathscr{X}$ or $\mathscr{Z}$ under the additional norms $] \bullet [_{\mathscr{X}}$ or $] \bullet [_{\mathscr{Z}}$, respectively. Further, all topological properties hereafter mean those under the original norm, unless explicitly stated otherwise.

We can now show the following lemma. Since the assumptions of the lemma is different from those of Lemma 3 in [10], we shall give the proof.

Lemma 1. Suppose that the sets Q, Z, P, the element $\hat{z}$, the norms $\| \bullet \|_{\mathscr{X}}$, $\| \bullet \|_{\mathscr{Z}}$ and the linear operator $H : \mathscr{X} \to \mathscr{Z}$ satisfy the Condition 1. If

(2) $$\left\{G^{-1}\left(\text{int}\,[Y|\hat{y}]\right)\right\} \cap \left\{H^{-1}\left([Z|\hat{z}]^{\circ}\right)\right\} \cap \overline{[Q|0]} = \varnothing,$$

then there exists a non-zero linear continuous functional y^* defined on $\mathscr{Y}$ and a linear continuous functional z^* defined on $\mathscr{Z}$ such that

$$y^*(y) \leq 0, \quad z^*(z) \leq 0 \quad \text{for all} \quad y \in Y \text{ and } z \in Z,$$

$$y^*(\hat{y}) = 0, \quad z^*(z) = 0,$$

$$y^*(G(x)) + z^*(H(x)) \geq 0 \quad \text{for all} \quad x \in Q,$$

where S° denotes the interior of the set S in $\mathscr{X}$ or $\mathscr{Z}$ under the additional norms $] \bullet [_{\mathscr{X}}$ or $] \bullet [_{\mathscr{Z}}$, respectively.

Proof. For any $\rho > 0$, we shall respectively define the set U_ρ and V_ρ as follows:

$$U_\rho = \left\{x \in \mathscr{X} \,\middle|\, \|x\|_{\mathscr{X}} < \rho\right\} \quad \text{and} \quad V_\rho = \left\{z \in \mathscr{Z} \,\middle|\, \|z\|_{\mathscr{Z}} < \rho\right\}.$$

We will first show that the following statement is true:

(3) *for any* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that*
$$V_\delta \subset H\left(\overline{[Q|0]} \cap U_\varepsilon\right) - \overline{[Z|\hat{z}]}.$$

Let $\varepsilon > 0$ be an arbitrary. We set $\delta = \varepsilon/\beta\,(> 0)$. By means of (C2) and (C3), it is easily verified that

(4)
$$V_\delta \subset \overline{P \cap V_\delta}.$$

It follows, from (C4), that $p \cap V_\delta \subset H\left(\overline{[Q|0]} \cap U_\varepsilon\right) - \overline{[Z|\hat{z}]}$, which, by virtue of (4), that implies that

(5)
$$V_\delta \subset \overline{\left\{H\left(\overline{[Q|0]} \cap U_\varepsilon\right) - \overline{[Z|\hat{z}]}\right\}}.$$

It is easily verified that $\overline{[Q|0]}$ and $\overline{[Z|\hat{z}]}$ are respectively convex cones in $\mathscr{X}$ and $\mathscr{Z}$, which, together with (C1) and linearity of H, implies that $H\left(\overline{[Q|0]} \cap U_\varepsilon\right) - \overline{[Z|\hat{z}]}$ is a convex cone in $\mathscr{Z}$ with non-empty interior under the additional norm $] \bullet [_{\mathscr{Z}}$. Using Lemma 11.A in [11], we obtain that

$$\left\{\overline{\left\{H\left(\overline{[Q|0]} \cap U_\varepsilon\right) - \overline{[Z|\hat{z}]}\right\}}\right\}^\circ = \left\{H\left(\overline{[Q|0]} \cap U_\varepsilon\right) - \overline{[Z|\hat{z}]}\right\}^\circ \subset H\left(\overline{[Q|0]} \cap U_\varepsilon\right) - \overline{[Z|\hat{z}]}.$$

This together with (5) implies that

$$V_\delta \subset H\left(\overline{[Q|0]} \cap U_\varepsilon\right) - \overline{[Z|\hat{z}]},$$

and thus proving the statement (3).

We now turn to the proof of the lemma. Let $O = G^{-1}\left(\mathrm{int}\,[Y|\hat{y}]\right)$. We first consider the case where $\overline{[Q|0]} \cap O \neq \varnothing$. Since G is a continuous linear operator from $\mathscr{X}$ into $\mathscr{Y}$ and H is a linear operator from $\mathscr{X}$ into $\mathscr{Z}$, it follows, from Lemma 11.A in [10], that O is a convex cone in $\mathscr{X}$ and

(6)
$$H\left(\overline{[Q|0]} \cap O\right) - [Z|\hat{z}]^\circ \text{ is a convex cone in } \mathscr{Z}.$$

Let $z \in H\left(\overline{[Q|0]} \cap O\right) - [Z|\hat{z}]^\circ$ be arbitrary, then there is an $x \in \overline{[Q|0]} \cap O$ such that $z \in H(x) - [Z|\hat{z}]^\circ$. Since O is open, there is an $\varepsilon > 0$ such that $x + U_\varepsilon \subset O$. It follows from (3) that there is a $\delta > 0$ such that $V_\delta \subset H\left(\overline{[Q|0]} \cap U_\varepsilon\right) - \overline{[Z|\hat{z}]}$. We know that $\overline{[Z|\hat{z}]} + [Z|\hat{z}]^\circ \subset [Z|\hat{z}]^\circ$, which, by virtue of the linearity of H, implies that

$$\begin{aligned} z + V_\delta &\subset H(x) + H\left(\overline{[Q|0]} \cap U_\varepsilon\right) - \left\{\overline{[Z|\hat{z}]} + [Z|\hat{z}]\right\} \subset H\left(\overline{[Q|0]} \cap \{x + U_\varepsilon\}\right) - [Z|\hat{z}]^\circ \\ &\subset H\left(\overline{[Q|0]} \cap O\right) - [Z|\hat{z}]^\circ, \end{aligned}$$

that is,

(7) $$H\left(\overline{[Q|0]} \cap O\right) - [Z|\hat{z}]^{\circ} \text{ is open}$$

(under the original norm $\| \bullet \|_{\mathscr{Z}}$). By means of (2), we can obtain that

(8) $$0 \notin H\left(\overline{[Q|0]} \cap O\right) - [Z|\hat{z}]^{\circ}.$$

We conclude, from (6)-(8), that $H\left(\overline{[Q|0]} \cap O\right) - [Z|\hat{z}]^{\circ}$ is an open (non-empty) convex cone in $\mathscr{Z}$ and does not contain the origin of the space $\mathscr{Z}$. Hence using the standard separation theorem (see e.g., Theorem 2.7.8 in [12]), there is a non-zero continuous linear functional $\bar{z}^*$ defined on $\mathscr{Z}$ such that

(9) $$\bar{z}^*(z) \geq 0 \text{ whenever } z \in H\left(\overline{[Q|0]} \cap O\right) - [Z|\hat{z}]^{\circ}.$$

It follows from (C2) that $\overline{[Z|\hat{z}]} = \overline{\left\{[Z|\hat{z}]^{\circ}\right\}} = \text{cl}\left\{[Z|\hat{z}]^{\circ}\right\}$, which, by virtue of (9), implies that

(10) $$\bar{z}^*(z) \leq 0 \leq \bar{z}^*(H(x)) \text{ whenever } z \in \overline{[Z|\hat{z}]} \text{ and } x \in \overline{[Q|0]} \cap O,$$

because $\overline{[Q|0]} \cap O$ and $\overline{[Z|\hat{z}]}$ are both convex cones. Let us define the norm $\| \bullet \|$ on the product space $\mathscr{Y} \times R^1$ by $\|(y,\gamma)\| = \|y\|_{\mathscr{Y}} + |\gamma|$ and let us define subset S_1 and S_2 of $\mathscr{Y} \times R^1$ as follows:

$$S_1 = \left\{(y,\gamma) \mid y \in \text{int}\,[Y|\hat{y}],\ \gamma < 0\right\}, \qquad S_2 = \left\{\left(G(x), \bar{z}^*(H(x))\right) \mid x \in \overline{[Q|0]}\right\}.$$

Since $\varnothing \neq \text{int}\, Y \subset [Y|\hat{y}]$, it is verified that S_1 is an open convex cone in $\mathscr{Y} \times R^1$. Since $\overline{[Q|0]}$ is a convex cone in $\mathscr{X}$ with $\overline{[Q|0]} \cap O \neq \varnothing$ and $\left(G(\ \bullet\ ),\ \bar{z}^*(H(\ \bullet\ ))\right)$ is a linear map from $\mathscr{X}$ into $\mathscr{Y} \times R^1$, S_2 is (non-empty) convex cone in $\mathscr{Y} \times R^1$. Further it follows from (10) that $S_1 \cap S_2 = \varnothing$. Therefore, once again using the standard separation theorem, there exists a continuous linear functional y^* and an $\bar{\eta} \geq 0$, not both zero, such that

(11) $$y^*(y) \leq 0 \text{ whenever } y \in \text{cl}\,[Y|\hat{y}].$$

(12) $$y^*(G(x)) + \bar{\eta}\bar{z}^*(H(x)) \geq 0 \text{ whenever } x \in \overline{[Q|0]}.$$

It follows from (10) that

(13) $$\bar{\eta}\bar{z}^*(z) \leq 0 \text{ whenever } z \in \overline{[Z|\hat{z}]}.$$

Since $\bar{z}^*$ is continuous with respect to the original norm $\| \bullet \|_{\mathscr{Z}}$, it follows from (C2) that $\bar{z}^*$, which is non-zero, is also continuous with respect to the additional norm $]\bullet[_{\mathscr{Z}}$. Hence we obtain, from (C3), (C4) and (12), (13), that

(14) y^* is a non-zero functional on $\mathscr{Y}$.

It follows from (C2) that $\text{cl}\left\{\overline{[Z|\hat{z}]}\right\} = \text{cl}\,[Z|\hat{z}]$. Hence (11)-(13) imply that, if we set $z^* = \overline{\eta}\overline{z}^*$, then

(15) $$y^*(y) \le 0 \text{ whenever } y \in \text{cl}\,[Y|\hat{y}],$$

(16) $$z^*(y) \le 0 \text{ whenever } z \in \text{cl}\,[Z|\hat{z}],$$

(17) $$y^*(G(x)) + z^*(H(x)) \ge 0 \text{ whenever } x \in [Q|0].$$

We now consider the case where $\overline{[Q|0]} \cap O = \varnothing$. $G(\overline{[Q|0]})$ is a convex cone in $\mathscr{Y}$ and $\text{int}\,[Y|\hat{y}]$ is an open convex cone in $\mathscr{Y}$ such that $\text{int}\,[Y|\hat{y}] \cap G(\overline{[Q|0]}) = \varnothing$. Using the standard separation theorem, there is a non-zero continuous linear functional y^* such that

$$y^*(y) \le 0 \le y^*(G(x)) \text{ whenever } y \in \text{int}\,[Y|\hat{y}],\ x \in \overline{[Q|0]}.$$

If z^* denotes the functional on $\mathscr{Z}$ such that $z^*(z) = 0$ for all $z \in \mathscr{Z}$, then it is easily verified that (14)-17) hold. Therefore, we can obtain that there exist continuous linear functionals y^* and z^* defined on $\mathscr{Y}$ and $\mathscr{Z}$,respectively, such that (14)-(17) hold, whether or not the set $\overline{[Q|0]} \cap O$ is empty. Since $\hat{y} \in Y$ and $\hat{z} \in Z$, it follows from (1) that $\pm\hat{y} \in \text{cl}\,[Y|\hat{y}]$ and $\pm\hat{z} \in \text{cl}\,[Z|\hat{z}]$. Hence, by virtue of (14)-(17), we conclude that the lemma is true. □

3. Formulation of Optimal Control Problems

In this section, we will formulate an optimal control problems with state-variable inequality constraints and give notations, terminologies and assumptions, which will be used throughout this paper.

Let D, U and J be respectively a non-empty open connected set in R^n, a non-empty set in R^m and a bounded open interval containing 0, and let A be an open connected set in R^{n+m+1} such that $D \times U \times J \subset A$, where R^j denotes the Euclidean j-dimensional space. Let $f(\xi, v, \tau)$ be an n-vector valued function defined on A, where $\xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} \in R^n$, $v = \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix} \in R^m$, $\tau \in R^1$. Let $f_\xi(\xi, v, \tau)$ is the matrix with element $\dfrac{\partial f_i(\xi, v, \tau)}{\partial \xi_j}$ in ith row and jth column ($i, j = 1, \cdots, n$). Let $g_0(\xi), g_1(\xi) \cdots, g_{q+r}(\xi)$ and $h_1(\xi), \cdots, h_s(\xi)$ be real valued functions defined on D which are of class C^1 on D. Let Λ and Ω be a set of all absolutely continuous functions from J into D and a set of all measurable essentially bounded functions from J into U, respectively, where a function is measurable if the preimage of every Borel set is a Borel set.

We are interested in the functions $x(\tau)$, $u(\tau)$ and real number T that satisfy the following relations:

(18) $$x \in \Lambda, \quad u \in \Omega, \quad [0, T] \subset J,$$

(19) $$\frac{dx(\tau)}{d\tau} = f(x(\tau), u(\tau), \tau) \quad \text{for almost all } \tau \in [0, T],$$

(20) $$h_j(x(\tau)) \leq 0, \quad \text{for all } \tau \in [0, T] \text{ and each } j = 1, \cdots, s,$$

(21) $$g_k(x(0)) \leq 0 \quad \text{for each } k = 1, \cdots, q,$$

(22) $$g_{q+i}(x(T)) \leq 0 \quad \text{for each } i = 1, \cdots, r.$$

If functions $x(\tau)$, $u(\tau)$ and a real number T satisfy the relations (18)-(22), then we shall call the triple (x, u, T) a feasible solution of the control system (18)-(22).

In this paper, we shall consider the following optimal control problem.

(OCP) *Find a feasible solution* (x, u, T) *of the control system* (18) - (22) *such that* $g_0(x(T))$ *achieve a minimum* (*subject to the relations* (18) - (22)).

Further we shall consider so-called time optimal control problem, which is one of the most important optimal control problems.

(TOCP) *Find a feasible solution* (x, u, T) *of the control system* (18) - (22) *such that* T *achieve a minimum* (*subject to the relations* (18) - (22)).

As regards terminology, a triple (x, u, T) shall be called an admissible solution of the equation (19) if x, u and T are respectively the n-vector valued function, m-vector valued function and real number that satisfy the relations (18) and (19). A triple $(\hat{x}, \hat{u}, \hat{T})$ shall be called an optimal solution of (OCP) or (TOCP) if it is a feasible solution of the control system (18)-(22) and if $g_0(\hat{x}(\hat{T})) \leq g_0(x(T))$ or $\hat{T} \leq T$ for every feasible solution (x, u, T) of the control system (18)-(22), respectively.

As regards notation, $|\bullet|$ shall denote the ordinary Euclidean norm in the underlying finite dimensional linear space, ${}^t\Psi$ and Ψ^{-1} shall denote the transpose and inverse of the matrix Ψ, respectively, and all vectors shall be interpreted column vectors unless explicitly stated otherwise. For each $k=0,1,\cdots,q+r$ and each $j=1,\cdots,s$, $[g_k]_\xi(\xi)$ and $[h_j]_\xi(\xi)$ shall denote n-dimensional row vector valued functions on D, defined by

$$[g_k]_\xi(\xi)=\left(\frac{\partial g_k(\xi)}{\partial \xi_1},\ \cdots,\frac{\partial g_k(\xi)}{\partial \xi_n}\right) \quad \text{and} \quad [h_j]_\xi(\xi)=\left(\frac{\partial h_j(\xi)}{\partial \xi_1},\ \cdots,\frac{\partial h_j(\xi)}{\partial \xi_n}\right),$$

respectively. $g^-(\xi)$, $g^+(\xi)$ and $h(\xi)$ shall denote a q-vector valued function, r-vector function and s-vector valued function on D defined by

$$g^-(\xi)={}^t(g_1(\xi),\cdots,g_q(\xi)),\quad g^+(\xi)={}^t(g_{q+1}(\xi),\cdots,g_{q+r}(\xi))$$

and

$$h(\xi)={}^t(h_1(\xi),\cdots,h_s(\xi)),$$

respectively. $g_\xi^-(\xi)$, $g_\xi^-(\xi)$ and $h_\xi(\xi)$ shall denote $q\times n$, $r\times n$ and $s\times n$ Jacobian matrices derived from the functions $g^-(\xi)$, $g^+(\xi)$ and $h(\xi)$, respectively, that is,

$$g_\xi^-(\xi)=\begin{pmatrix}[g_1]_\xi(\xi)\\ \vdots \\ [g_q]_\xi(\xi)\end{pmatrix},\quad g_\xi^+(\xi)=\begin{pmatrix}[g_{q+1}]_\xi(\xi)\\ \vdots \\ [g_{q+r}]_\xi(\xi)\end{pmatrix} \quad \text{and} \quad h_\xi(\xi)=\begin{pmatrix}[h_1]_\xi(\xi)\\ \vdots \\ [h_s]_\xi(\xi)\end{pmatrix}$$

for all $\xi\in D$. Further $\mathfrak{F}$ shall denote the family of functions from $D\times J$ into R^n defined by

$$\mathfrak{F}=\{F(\xi,\tau)|F(\xi,\tau)=f(\xi,u(\tau),\tau)-f(\hat{x}(\tau),\hat{u}(\tau),\tau),\ \ u\in\Omega\} \tag{23}$$

and co $\mathfrak{F}$ shall denote the convex hull of the family $\mathfrak{F}$.

In closing this section, we shall suppose that, for every $u\in\Omega$ and every compact subset B of D, there exists a positive M, possibly depending on u and B, such that

$$|f(\xi,u(\tau),\tau)|\le M,\quad |f_\xi(\xi,u(\tau),\tau)|\le M \qquad \text{for every} \quad \xi\in B,\ \ \tau\in J. \tag{24}$$

Under this assumption, we can prove that the family $\mathfrak{F}$ is quasiconvex, using arguments identical to those used by Gamkrelize (see section 4 in [13]). The assumption used by Gamkrelidze is weaker than (24). However we will use the assumption (24), because discussion of the assumption itself is not essential theme of this paper.

4. Regularities of an Admissible Solution

In this section, we will define a regularities of the admissible solution of the equation (19) and we will give their properties.

Throughout this paper, let $(\hat{x}, \hat{u}, \hat{T})$ be the admissible solution of the equation (19) and let $I = [0, \hat{T}]$. Let $\Phi(\tau)$ be the absolutely continuous $n \times n$ matrix valued function on I that satisfies the following relations:

(25)
$$\frac{d\Phi(\tau)}{d\tau} = f_\xi(\hat{x}(\tau), \hat{u}(\tau), \tau) \quad \text{for almost all } \tau \in I,$$
$$\Phi(0) = \text{the identity matrix.}$$

For each $\xi \in R^n$ and each $F \in \text{co}\,\mathfrak{F}$, let $\tilde{x}(\xi, F; \tau)$ be the absolutely continuous n-vector valued function on I defined by

(26)
$$\tilde{x}(\xi, F; \tau) = \Phi(\tau)\left[\xi + \int_0^\tau \Phi(\sigma)^{-1} F(\hat{x}(\sigma), \sigma) d\sigma\right] \quad \text{for all } \tau \in I,$$

where the integral is to be interpreted in the sense of Lebesgue, and let

(27)
$$Q(\hat{x}, \hat{u}, \hat{T}) = \{\tilde{x}(\xi, F; \bullet) \mid \xi \in R^n, \ F \in \text{co}\,\mathfrak{F}\}.$$

Let X be the subset of Λ defined by

$$X = \{x \in \Lambda \mid \text{there exists } u \in \Omega \text{ such that } (x, u, \hat{T}) \text{ is the admissible solution of the equation (19)}\}.$$

By the theory of ordinary differential equations, we have known that the following is true (see e.g., section 3 in [13]).

Lemma 2. If $\tilde{x}(\xi, F; \bullet) \in Q(\hat{x}, \hat{u}, \hat{T})$, then, for each sufficiently small $\varepsilon > 0$, there exist an $x_\varepsilon \in \Lambda$ and $u_\varepsilon \in \Omega$ such that

$$\frac{dx_\varepsilon(\tau)}{d\tau} = f(x_\varepsilon(\tau), u_\varepsilon(\tau), \tau) \quad \text{for almost all } \tau \in I,$$
$$x_\varepsilon(0) = \hat{x}(0) + \varepsilon\xi.$$

that is, $x_\varepsilon \in X$. Moreover,

$$\frac{x_\varepsilon(\tau) - \hat{x}(\tau)}{\varepsilon} \xrightarrow[\varepsilon \to 0]{} \tilde{x}(\xi, F; \tau) \quad \text{uniformly in } \tau \in I.$$

We define the regularities of admissible solution of the equation (19).

Definition 1. (i) An admissible solution $\left(\hat{x},\hat{u},\hat{T}\right)$ of the equation (19) is called regular to the constraints (20) if we can find a positive β satisfying that, for each s-vector valued polynomial $p(\tau)={}^t\left(p_1(\tau),\cdots,p_s(\tau)\right)$, there exist positive numbers ρ, γ and an $\tilde{x}\in Q\left(\hat{x},\hat{u},\hat{T}\right)$, possibly depending on the polynomial $p(\tau)$, such that

(28) $$p_j(\tau)\geq\rho\left[h_j\right]_\xi(\hat{x}(\tau))\tilde{x}(\tau)+\gamma h_j(\hat{x}(\tau)) \quad \text{for all } \tau\in I \text{ and each } j=1,\cdots,s,$$

(29) $$\rho\left[|\tilde{x}(0)|^2+\left|\tilde{x}\left(\hat{T}\right)\right|^2+\int_0^{\hat{T}}|\tilde{x}(\sigma)|^2d\sigma\right]^{1/2}\leq\beta\left[|p(0)|^2+\left|p\left(\hat{T}\right)\right|^2+\int_0^{\hat{T}}|p(\sigma)|^2d\sigma\right]^{1/2}.$$

(ii) An admissible solution $\left(\hat{x},\hat{u},\hat{T}\right)$ of the equation (19) is called strongly regular to the constraints (20) if it satisfies the statement (i), but with (29) replaced by the relation

(30) $$\rho\left[|\tilde{x}(0)|^2+\left|\tilde{x}\left(\hat{T}\right)\right|^2+\int_0^{\hat{T}}|\tilde{x}(\sigma)|^2d\sigma\right]^{1/2}\leq\beta\left[\int_0^{\hat{T}}|p(\sigma)|^2d\sigma\right]^{1/2}.$$

Here vector valued polynomial means that the vector valued function whose components are all polynomials.

Let $\mathscr{X}$ and $\mathscr{Z}$ be the linear spaces of all n-vector valued and s-vector valued continuous functions defined on I, respectively. We define the norm $\|\bullet\|_{\mathscr{X}}$ on $\mathscr{X}$ as follows:

(31) $$\|x\|_{\mathscr{X}}=\left[|x(0)|^2+\left|x\left(\hat{T}\right)\right|^2+\int_0^{\hat{T}}|x(\sigma)|^2d\sigma\right]^{1/2} \quad \text{for each } x\in\mathscr{X}.$$

Besides the norm above, we define an additional norm $]\bullet[_{\mathscr{X}}$ on $\mathscr{X}$ as follows:

$$]x[_{\mathscr{X}}=\left[|x(0)|^2+\left|x\left(\hat{T}\right)\right|^2+\sup_{\tau\in I}|x(\tau)|^2\right]^{1/2} \quad \text{for each } x\in\mathscr{X}.$$

Let $T(X,\hat{x})$ be the sequential tangent cone to X at $\hat{x}$ (cf. [14,15]), that is,

(32) $$T(X,\hat{x})=\left\{\tilde{x}\in\mathscr{X}\;\middle|\; \begin{array}{l} x^k\in X,\ \gamma^k>0 \text{ for each } k=1,2,\cdots \\ \text{and } \lim_{k\to\infty}\gamma^k=0,\ \lim_{k\to\infty}\left]\tilde{x}-\frac{x^k-\hat{x}}{\gamma^k}\right[_{\mathscr{X}}=0 \end{array}\right\}.$$

Further let

(33) $$\alpha=\max\left\{1,\sqrt{\hat{T}}\right\}$$

We can verify that $Q\left(\hat{x},\hat{u},\hat{T}\right)$ is a convex subset of $\mathscr{X}$ containing the origin $\mathit{0}$ of $\mathscr{X}$ and $T(X,\hat{x})$ is a cone which is closed in the sense of the additional norm $]\bullet[_{\mathscr{X}}$ (see e.g., [14]). Therefore, by virtue of Lemma 2, the following lemma is straightforward to prove.

Lemma 3. (i) $Q(\hat{x},\hat{u},\hat{T})$ is a convex subset of $\mathscr{X}$ satisfying that

$$0 \in Q(\hat{x},\hat{u},\hat{T}) \subset \overline{\left[Q(\hat{x},\hat{u},\hat{T})\middle|0\right]} \subset T(X,\hat{x}).$$

(ii) $\|x\|_{\mathfrak{X}} \le \alpha\ ]x[_{\mathscr{X}}$ for all $x \in \mathscr{X}$,

where $\overline{\left[Q(\hat{x},\hat{u},\hat{T})\middle|0\right]}$ denotes the closure of the set $\left[Q(\hat{x},\hat{u},\hat{T})\middle|0\right]$ under the additional norm $]\bullet[_{\mathscr{X}}$.

Let Z be the convex cone in $\mathscr{Z}$ defined by

(34) $$Z = \left\{ z(\bullet) = {}^t(z_1(\bullet),\cdots,z_s(\bullet)) \in \mathscr{Z} \;\middle|\; z_j(\tau) \le 0 \text{ for all } \tau \in I \text{ and each } j = 1,\cdots,s \right\},$$

and let P be the set of all s-vector valued polynomials. Let $\bar{h}$ and H be the functions from $\mathscr{X}$ into $\mathscr{Z}$ such that, for each $x \in \mathscr{X}$,

(35) $$\bar{h}(x) = h(x(\bullet)), \quad H(x) = h_\xi(\hat{x}(\bullet))x(\bullet).$$

We now consider the case where $(\hat{x},\hat{u},\hat{T})$ is regular to the constraints (20), In this case we define the norm $\|\bullet\|_{\mathscr{Z}}$ on $\mathscr{Z}$ as follows:

(36) $$\|z\|_{\mathscr{Z}} = \left[|z(0)|^2 + \left|z(\hat{T})\right|^2 + \int_0^{\hat{T}} |z(\sigma)|^2 d\sigma\right]^{1/2} \quad \text{for each } z \in \mathscr{Z}.$$

Besides the norm above, we define an additional norm $]\bullet[_{\mathscr{Z}}$ on $\mathscr{Z}$ as follows:

(37) $$]\,z\,[_{\mathscr{Z}} = \left[|z(0)|^2 + \left|z(\hat{T})\right|^2 + \sup_{\tau \in I} |z(\tau)|^2\right]^{1/2} \quad \text{for each } z \in \mathscr{Z}.$$

As was defined in section 2, for each subset S of $\mathscr{X}$ (or $\mathscr{Z}$), $\operatorname{int} S$ $\operatorname{cl} S$ shall denote the interior and the closure of the set S under the original norm $\|\bullet\|_{\mathscr{X}}$ (or $\|\bullet\|_{\mathscr{Z}}$), respectively. S° and $\bar{S}$ shall denote the interior and the closure of the set S under the additional norm $]\bullet[_{\mathscr{X}}$ (or $]\bullet[_{\mathscr{Z}}$), respectively. We will show the following lemma.

Lemma 4. (i) H is a linear operator from $\mathscr{X}$ into $\mathscr{Z}$ such that, for each $\tilde{x} \in \mathscr{X}$,

$$\lim_{\substack{\gamma \downarrow 0 \\ ]x-\tilde{x}[_{\mathscr{X}} \to 0}} \left] \frac{\bar{h}(\hat{x}+\gamma x)}{\gamma} - H(\tilde{x}) \right[_{\mathscr{Z}} = 0.$$

(ii) The statements (C1)-(C3) in Condition 1 hold, in addition if we set $\hat{z} = \bar{h}(\hat{x})$ and $Q = Q(\hat{x},\hat{u},\hat{T})$, then the statement (C4) holds.

Proof. Since $h_1,\cdots,h_s$ are of class C^1, (i) follows from (35). Now we will show that (ii) is true. By means of (33), (34), (36), (37) together with (ii) in Lemma 3, it is easily verified that (C1) and (C2) hold It follows, from Weierstarass' theorem, that (C3) holds.

To show (C4), let $p={}^t(p_1,\cdots,p_s)\in P$. It follows, from (i) in Definition 1, that there are positive numbers ρ, γ and an $\tilde{x}\in Q(\hat{x},\hat{u},\hat{T}))$ which satisfy the relations (28) and (29). Let $z_1(\tau),\cdots,z_s(\tau)$ be the function on I defined by

$$z_j(\tau)=\frac{\rho[h_j]_\xi(\hat{x}(\tau))\tilde{x}(\tau)-p_j(\tau)}{\gamma}+h_j(\hat{x}(\tau)) \quad \text{for all } \tau\in I \text{ and each } j=1,\cdots,s$$

and let $z(\tau)={}^t(z_1(\tau),\cdots,z_s(\tau))$ for all $\tau\in I$. Since $h_1,\cdots,h_s$ are of class C^1 on D, it follows, from (28), (34) and (35), that $z\in Z$ and $p=H(\rho\tilde{x})-\gamma[z-\bar{h}(\hat{x})]$. Let $x=\rho\tilde{x}$, $\hat{z}=\bar{h}(\hat{x})$ and $Q=Q(\hat{x},\hat{u},\hat{T})$, then $x\in[Q|0]\subset\overline{[Q|0]}$ and $p\in H(x)-\overline{[Z|\hat{z}]}$, because $\gamma[z-\bar{h}(\hat{x})]\subset[Z|\hat{z}]\subset\overline{[Z|\hat{z}]}$. Further, (29) together with (31) and (36) imply that $\|x\|_{\mathscr{X}}\le\beta\|p\|_{\mathscr{Z}}$. Hence (C4) holds, and thus proving the lemma. □

Let us consider a continuous linear functional defined on the normed space $\mathscr{Z}$.

Lemma 5. Let z^* be a continuous linear functional on the normed space $\mathscr{Z}$ such that

(38) $$z^*(z)\le 0 \quad \text{for all } z\in Z,$$

then there is a square Lebesgue integrable s-dimensional row vector valued function $\lambda(\tau)=(\lambda_1(\tau),\cdots,\lambda_s(\tau))$ defined on I such that

(39) $$z^*(z)=\lambda(0)z(0)+\lambda(\hat{T})z(\hat{T})+\int_0^{\hat{T}}\lambda(\sigma)z(\sigma)d\sigma \quad \text{for each } z\in\mathscr{Z}$$

and, for each $j=1,\cdots,s$,

(40) $$\lambda_j(0)\ge 0,\ \lambda_j(\hat{T})\ge 0 \ \text{ and } \ \lambda_j(\tau)\ge 0 \ \text{ for almost all } \tau\in(0,\hat{T}).$$

Proof. Let $\mathscr{L}^2$ be a linear space of all square Lebesgue integrable functions from I into R^s and let $\|\bullet\|_2$ be the norm on $\mathscr{L}^2$ defined by

$$\|\tilde{z}\|_2=\left[\int_0^{\hat{T}}|\tilde{z}(\sigma)|^2d\sigma\right]^{1/2} \quad \text{for each } \tilde{z}\in\mathscr{L}^2.$$

Further let $\mathscr{W}=R^s\times R^s\times\mathscr{L}^2$. We define the norm $\|\bullet\|_{\mathscr{W}}$ on $\mathscr{W}$ as follows:

$$\|w\|_{\mathscr{W}}=\left[|\xi|^2+|\zeta|^2+\|\tilde{z}\|_2{}^2\right]^{1/2} \quad \text{for each } w=(\xi,\zeta,\tilde{z}).$$

Let z^* be an arbitrary continuous linear functional on the normed space $\mathscr{Z}$ which satisfies the relation (38). Since the linear space $\mathscr{Z}$ is regarded as a subspace of the normed space $\mathscr{W}$ and is everywhere dense in $\mathscr{W}$ (under the norm $\|\bullet\|_{\mathscr{W}}$), the functional z^* can be uniquely extended onto $\mathscr{W}$. Now let w^* be the extended continuous linear functional on $\mathscr{W}$. Then there exists a square Lebesgue integrable s-dimensional row vector valued function $\lambda(\tau)=(\lambda_1(\tau),\cdots,\lambda_s(\tau))$ defined on I such that

$$w^*(w)=\lambda(0)\xi+\lambda(\hat{T})\zeta+\int_0^{\hat{T}}\lambda(\sigma)\tilde{z}(\sigma)d\sigma \quad \text{for each } w=(\xi,\zeta,\tilde{z})\in\mathscr{W},$$

which implies that the relation (39) holds. It follows, from (34), (38) and (39), that $\lambda_j(0)\geq 0$, $\lambda_j(\hat{T})\geq 0$ for each $j=1,\cdots,s$ and $\int_0^{\hat{T}}\lambda(\sigma)z(\sigma)d\sigma\leq 0$ for all $z\in\mathscr{Z}$, which implies that $\lambda_j(\tau)\geq 0$ for almost all $\tau\in I$ and each $j=1,\cdots,s$, and thus proving the lemma. □

At the end of this section, we consider the case where $(\hat{x},\hat{u},\hat{T})$ is strongly regular to the constraints (20). Besides the norm $\|\bullet\|_2$, we define an additional norm $]\bullet[_\infty$ as follows:

$$]z[_\infty=\operatorname*{ess\,sup}_{\tau\in I}|z(\tau)|\left(=\sup_{\tau\in I}|z(\tau)|\right) \quad \text{for all } z\in Z.$$

Using the identical procedures in the proof of Lemma 4, but with the norms $\|\bullet\|_{\mathscr{Z}}$, $]\bullet[_{\mathscr{Z}}$ on $\mathscr{Z}$ and the relation (29) replaced by the norms $\|\bullet\|_2$, $]\bullet[_\infty$ on $\mathscr{Z}$ and the relation (30), respectively, we can show the following lemma.

Lemma 4'. (i) H is a linear operator from $\mathscr{X}$ into $\mathscr{Z}$ such that, for each $\tilde{x}\in\mathscr{X}$,

$$\lim_{\substack{\gamma\downarrow 0\\ ]x-\tilde{x}[_{\mathscr{X}}\to 0}}\left]\frac{\bar{h}(\hat{x}+\gamma x)}{\gamma}-H(\tilde{x})\right[_\infty=0.$$

(ii) If the element $\hat{z}$, the set Q, the norms $\|\bullet\|_{\mathscr{Z}}$ and $]\bullet[_{\mathscr{Z}}$ are replaced by $\bar{h}(\hat{x})$, $Q(\hat{x},\hat{u},\hat{T})$, $\|\bullet\|_2$ and $]\bullet[_\infty$, respectively all the statements (C1)-(C4) hold.

Since the linear space $\mathscr{Z}$ is regarded as a subspace of the normed space $\mathscr{L}^2$ (presented in the proof of Lemma 5) and is everywhere dense in $\mathscr{L}^2$ (under the norm $\|\bullet\|_2$), each continuous linear functional on the normed space $\mathscr{Z}$ can be uniquely extended onto $\mathscr{L}^2$. Therefore we can prove the following lemma, employing arguments virtually identical to those used in the proof of Lemma 5.

Lemma 5'. Let z^* be a linear functional on $\mathscr{Z}$ which is continuous with respect to the norm $\|\bullet\|_2$ and satisfies the relation (38), then there is a square Lebesgue integrable s-dimensional row vector valued function $\lambda(\tau)=(\lambda_1(\tau),\cdots,\lambda_s(\tau))$ defined on I such that

(41) $$z^*(z)=\int_0^{\hat{T}}\lambda(\sigma)z(\sigma)d\sigma \quad \text{for each } z\in\mathscr{Z}$$

and, for each $j=1,\cdots,s$,

(42) $$\lambda_j(\tau)\geq 0 \quad \text{for almost all } \tau\in(0,\hat{T}).$$

5. Necessary Conditions for Optimal Solutions

In this section, we will introduce necessary conditions for optimal solutions of the problems (OCP) and (TOCP), and give their proofs. We will use all notations in section 4 without further explanation.

we first consider the problem (0CP).

Theorem 1. Let $(\hat{x},\hat{u},\hat{T})$ be an optimal solution of the optimal control problem (OCP) and regular to the constraints (20) as well. Then there exist non-negative numbers $\eta_0,\eta_1,\cdots,\eta_{q+r}$, not all being zero, square Lebesgue integrable functions $\lambda_1(\tau),\cdots,\lambda_s(\tau)$ defined on $[0,\hat{T}]$ and absolutely continuous functions $\psi_1(\tau),\cdots,\psi_n(\tau)$, defined on $[0,\hat{T}]$, which satisfy the following conditions:

(N1) $$\eta_k g_k(\hat{x}(0))=0 \quad \text{for all } k=1,\cdots,q,$$
$$\eta_{q+i}g_{q+i}\left(\hat{x}(\hat{T})\right)=0 \quad \text{for all } i=1,\cdots,r,$$

(N2) $$\lambda_j(0)\geq 0,\ \lambda_j(0)h_j(\hat{x}(0))=0,\ \lambda_j(\hat{T})\geq 0,\ \lambda_j(\hat{T})h_j\left(\hat{x}(\hat{T})\right)=0 \text{ for all } j=1,\cdots,s,$$
$$\lambda_j(\tau)\geq 0,\ \lambda_j(\tau)h_j(\hat{x}(\tau))=0 \text{ for almost all } \tau\in(0,\hat{T}) \text{ and all } j=1,\cdots,s,$$

$$\frac{d\psi(\tau)}{d\tau}=-\psi(\tau)f_\xi(\hat{x}(\tau),\hat{u}(\tau),\tau)-\lambda(\tau)h_\xi(\hat{x}(\tau)) \text{ for almost all } \tau\in[0,\hat{T}]$$

(N3) $$\psi(0)=-\eta^- g_\xi^-(\hat{x}(0))-\lambda(0)h_\xi(\hat{x}(0)),$$
$$\psi(\hat{T})=\eta_0[g_0]_\xi\left(\hat{x}(\hat{T})\right)+\eta^+g_\xi^+\left(\hat{x}(\hat{T})\right)+\lambda(\hat{T})h_\xi\left(\hat{x}(\hat{T})\right),$$

(N4) $$\psi(\tau)f(\hat{x}(\tau),\hat{u}(\tau),\tau)=\min_{v\in U}\ \psi(\tau)f(\hat{x}(\tau),v,\tau) \text{ for almost all } \tau\in[0,\hat{T}],$$

where $\psi(\tau)=(\psi_1(\tau),\cdots\psi_n(\tau))$, $\lambda(\tau)=(\lambda_1(\tau),\cdots,\lambda_s(\tau))$, $\eta^-=(\eta_1,\cdots,\eta_q)$, $\eta^+=(\eta_{q+1},\cdots,\eta_{q+r})$, which are all row vectors.

Proof. Let $(\hat{x},\hat{u},\hat{T})$ be an optimal solution of (0CP) as well as regular to the constraints (20). We define a convex cone $Y\subset R^{q+r+1}$ as follows:

(43) $$Y=\left\{{}^t(y_0,y_1,\cdots,y_{q+r})\in R^{q+r+1}\ |\ y_i\leq 0 \text{ for all } i=0,1,\cdots,q+r\right\}.$$

Let $\bar{g}:\mathscr{X}\to R^{q+r+1}$ and $G:\mathscr{X}\to R^{q+r+1}$ be defined by

(44) $$\bar{g}(x)=\begin{pmatrix} g_0\left(x(\hat{T})\right)-g_0\left(\hat{x}(\hat{T})\right) \\ g^-(x(0)) \\ g^+\left(x(\hat{T})\right)\end{pmatrix},\quad G(x)=\begin{pmatrix}[g_0]_\xi\left(\hat{x}(\hat{T})\right)x(\hat{T}) \\ g_\xi^-(\hat{x}(0))x(0) \\ g_\xi^+\left(\hat{x}(\hat{T})\right)x(\hat{T})\end{pmatrix},\quad \text{for each } x\in\mathscr{X}.$$

It is easily verified that Y is a convex cone in R^{q+r+1} with non-empty interior. Since $g_0(\xi), g_1(\xi), \cdots, g_{q+r}(\xi)$ are of class C^1 on D, it can be verified that $G: \mathscr{X} \to R^{q+r+1}$ is a continuous linear operator such that, for each $\tilde{x} \in \mathscr{X}$,

(45) $$\lim_{\substack{\gamma \downarrow 0 \\ ]x-\tilde{x}[_{\mathscr{X}} \to 0}} \left| \frac{\bar{g}(\hat{x}+\gamma x)-\bar{g}(\hat{x})}{\gamma} - G(\tilde{x}) \right| = 0.$$

Since $(\hat{x}, \hat{u}, \hat{T})$ be an optimal solution of (OCP), we can verify that

$$\bar{g}(\hat{x}) \in Y, \qquad \bar{h}(\hat{x}) \in Z.$$

In order to prove that

(46) $$\left\{G^{-1}\left(\text{int}\left[Y|\bar{g}(\hat{x})\right]\right)\right\} \cap \left\{H^{-1}\left(\left[Z|\bar{h}(\hat{x})\right]^{\circ}\right)\right\} \cap \overline{\left[Q(\hat{x}, \hat{u}, \hat{T})|0\right]} = \varnothing,$$

suppose the contrary, that is, there is an

(47) $$\tilde{x} \in \overline{\left[Q(\hat{x}, \hat{u}, \hat{T})|0\right]}$$

such that

(48) $$G(\tilde{x}) \in \text{int}\left[Y|\bar{g}(\hat{x})\right], \quad H(\tilde{x}) \in \left[Z|\bar{h}(\hat{x})\right]^{\circ}.$$

It follows, from (32), (47) and (i) in Lemma 3, that there are sequences $\{x^k\}$ and $\{\gamma^k\}$ such that

(49) $$x^k \in X, \quad \gamma^k > 0 \quad \text{for each } k = 1, 2, \cdots,$$

(50) $$\lim_{k \to \infty} \left] \tilde{x} - \frac{x^k - \hat{x}}{\gamma^k} \right[_{\mathscr{X}} = 0, \qquad \lim_{k \to \infty} \gamma^k = 0.$$

Since Y and Z are respectively convex cones in R^{q+r+1} and $\mathscr{Z}$ with $\text{int}\, Y \neq \varnothing$ and $Z^{\circ} \neq \varnothing$, we can show, by means of Lemma 11.A in [11], that

(51) $$\theta \,\text{int}\, Y + \omega Y \subset \text{int}\, Y, \quad \theta Z + \omega Z \subset Z \qquad \text{whenever} \quad \theta > 0, \ \omega \geq 0,$$

(52) $$\text{int}\left[Y|\bar{g}(\hat{x})\right] = \left[\text{int}\, Y|\bar{g}(\hat{x})\right], \qquad \left[Z|\bar{h}(\hat{x})\right]^{\circ} = \left[Z^{\circ}|\bar{h}(\hat{x})\right].$$

It follows from (48) and (52) that there are elements $y \in \text{int}\, Y$, $z \in Z^{\circ}$ and positive numbers ρ_y, ρ_z such that

(53) $$G(\tilde{x}) = \rho_y\left(y - \bar{g}(\hat{x})\right), \quad H(\tilde{x}) = \rho_z\left(z - \bar{h}(\hat{x})\right),$$

which imply that there is a $\delta > 0$ such that

(54) $$y' \in \text{int}\, Y \quad \text{whenever } |y - y'| < \delta, \qquad z' \in Z^{\circ} \quad \text{whenever } |z - z'| < \delta.$$

If we set positive numbers $\rho_m = \min\{\rho_y, \rho_z\}$ and $\rho_M = \max\{\rho_y, \rho_z\}$, then it follows, from (45), (50) and (i) in Lemma 4, that there is a positive integer ℓ such that

$$\left| \frac{\bar{g}(x^\ell)-\bar{g}(\hat{x})}{\gamma^\ell} - G(\tilde{x}) \right| = \left| \frac{\bar{g}\left(\hat{x}+\gamma^\ell\left[(x^\ell-\hat{x})/\gamma^\ell\right]\right)-\bar{g}(\hat{x})}{\gamma^\ell} - G(\tilde{x}) \right| < \rho_m\delta,$$

$$\left]\frac{\bar{h}(x^\ell)-\bar{h}(\hat{x})}{\gamma^\ell} - H(\tilde{x})\right[_{\mathscr{Z}} = \left]\frac{\bar{h}\left(\hat{x}+\gamma^\ell\left[(x^\ell-\hat{x})/\gamma^\ell\right]\right)-\bar{h}(\hat{x})}{\gamma^\ell} - H(\tilde{x})\right[_{\mathscr{Z}} < \rho_m\delta,$$

and $\gamma^\ell < 1/\rho_M$, which, by virtue of (49), (51), (53) and (54), imply that

(55) $$x^\ell \in X, \quad \bar{g}(x^\ell) \in \operatorname{int} Y, \quad \bar{h}(x^\ell) \in Z^\circ,$$

because $\bar{g}(\hat{x}) \in Y$ and $\bar{h}(\hat{x}) \in Z$. Considering the definitions of the sets X, Y, Z and the functions $\bar{g}$, $\bar{h}$, the relations (55) imply that there is a $u \in \Omega$ such that the triple $(x^\ell, u, \hat{T})$ is a feasible solution of the control system (18)-(22) and $g_0(\hat{x}(\hat{T})) > g_0(x^\ell(\hat{T}))$, which contradicts the optimality of the triple $(\hat{x}, \hat{u}, \hat{T})$. Therefore the relation (46) holds.

Using Lemma 1, it follows, from (ii) in Lemma 4 and (46), that there exists non-zero continuous linear functional y^* on R^{q+r+1} and a continuous linear functional z^* on $\mathscr{Z}$ such that

(56) $$y^*(y) \le 0 \quad \text{for all } y \in Y, \qquad y^*(\bar{g}(\hat{x})) = 0,$$

(57) $$z^*(z) \le 0 \quad \text{for all } z \in Z,$$

(58) $$z^*(\bar{h}(\hat{x})) = 0,$$

(59) $$y^*(G(\tilde{x})) + z^*(H(\tilde{x})) \ge 0 \quad \text{for all } \tilde{x} \in Q(\hat{x}, \hat{u}, \hat{T}).$$

Since y^* is a non-zero continuous linear functional on R^{q+r+1}, there are real numbers $\eta_0, \eta_1, \cdots, \eta_{q+r}$, not all being zero, such that

(60) $$y^*(y) = \sum_{i=0}^{q+r} \eta_i y_i \quad \text{for all } y = {}^t(y_0, y_1, \cdots, y_{q+r}) \in R^{q+r+1},$$

which, by virtue of (56) together with the definitions of the cone Y and the function $\bar{g}$, implies that the numbers $\eta_0, \eta_1, \cdots, \eta_{q+r}$ satisfy (N1), because $\bar{g}(\hat{x}) \in Y$. By virtue of Lemma 5, (57) implies that there exists a square Lebesgue integrable s-dimensional row vector valued function $\lambda(\tau) = (\lambda_1(\tau), \cdots, \lambda_s(\tau))$ defined on I which satisfies (39) and (40). Hence (N2) follows from (39), (40), (58), because $\bar{h}(\hat{x}) \in Z$. Let us define row vectors η^- and η^+ as follows:

$$\eta^- = (\eta_1, \cdots, \eta_q), \quad \eta^+ = (\eta_{q+1}, \cdots, \eta_{q+r}).$$

We know that $\tilde{x}(\xi, F; \bullet) \in Q(\hat{x}, \hat{u}, \hat{T})$ for each $\xi \in R^n$ and $F \in \mathrm{co}\,\mathfrak{F}$, where $\tilde{x}(\xi, F; \bullet)$ is the function from I into R^n defined by (26). Hence, using (35), (39) and (44), (60), the following inequality is obtained from (59):

$$\left[\Gamma + \Delta\Phi(\hat{T})\int_0^{\hat{T}} \lambda(\sigma) h_\xi(\hat{x}(\sigma))\Phi(\sigma)d\sigma\right]\xi + \Delta\Phi(\hat{T})\int_0^{\hat{T}} \Phi(\sigma)^{-1}F(\hat{x}(\sigma),\sigma)d\sigma$$

$$+\int_0^{\hat{T}} \lambda(\sigma)h_\xi(\hat{x}(\sigma))\Phi(\sigma)\left[\int_0^{\sigma}\Phi(\tau)^{-1}F(\hat{x}(\tau),\tau)d\tau\right]d\sigma \geq 0 \tag{61}$$

$$\text{for all } \xi \in R^n \text{ and } F \in \mathrm{co}\,\mathfrak{F},$$

where Γ and Δ denote n-dimensional row vectors defined by

$$\Gamma = \eta^- g_\xi^-(\hat{x}(0)) + \lambda(0)h_\xi(\hat{x}(0)), \tag{62}$$

$$\Delta = \eta_0[g_0]_\xi(\hat{x}(\hat{T})) + \eta^+ g_\xi^+(\hat{x}(\hat{T})) + \lambda(\hat{T})h_\xi(\hat{x}(\hat{T})). \tag{63}$$

Using Fubini's theorem, (61) is rewritten as follows:

$$\Delta\Phi(\hat{T}) + \int_0^{\hat{T}} \lambda(\sigma)h_\xi(\hat{x}(\sigma))\Phi(\sigma)d\sigma = -\Gamma, \tag{64}$$

$$\int_0^{\hat{T}}\left[\Delta\Phi(\hat{T}) + \int_\tau^{\hat{T}} \lambda(\sigma)h_\xi(\hat{x})\Phi(\sigma)d\sigma\right]\Phi(\tau)^{-1}F(\hat{x}(\tau),\tau)d\tau \geq 0 \text{ for all } F \in \mathrm{co}\,\mathfrak{F}. \tag{65}$$

Let $\psi(\tau)$ be an absolutely continuous n-vector valued row vector valued function on I defined by

$$\psi(\tau) = \left[\Delta\Phi(\hat{T}) + \int_\tau^{\hat{T}} \lambda(\sigma)h_\xi(\hat{x}(\sigma))\Phi(\sigma)d\sigma\right]\Phi(\tau)^{-1} \quad \text{for all } \tau \in I, \tag{66}$$

then, we obtain, from (64), (65) and (23), that

$$\psi(0) = -\Gamma, \qquad \psi(\hat{T}) = \Delta \tag{67}$$

and

$$\int_0^{\hat{T}} \psi(\tau)[f(\hat{x}(\tau),u(\tau),\tau) - f(\hat{x}(\tau),\hat{u}(\tau),\tau)]d\tau \geq 0 \quad \text{for all } u \in \Omega, \tag{68}$$

because $\Phi(\tau)$ is a non-singular matrix for all $\tau \in I$ and $\Phi(0)$ is the identity matrix. Hence (N3) follows from (62), (63), (66), (67) and the fact that

$$\frac{d\Phi(\tau)^{-1}}{d\tau} = -\Phi(\tau)^{-1}\frac{d\Phi(\tau)}{d\tau}\Phi(\tau) = -\Phi(\tau)^{-1}f_\xi(\hat{x}(\tau),\hat{u}(\tau),\tau) \quad \text{for almost all } \tau \in I.$$

Using the same argument as those used in the proof of (4.20) in [2], (N4) is obtained from (68), This completes the proof of the theorem. □

In the case where the optimal solution $(\hat{x}, \hat{u}, \hat{T})$ of (OCP) is strongly regular to the constraints (20), we can show the following corollary if we respectively use Lemmas 4', 5', relations (41), (42) and

$$\Gamma = \eta^- g_\xi^-(\hat{x}(0)), \qquad \Delta = \eta_0 [g_0]_\xi \left(\hat{x}(\hat{T})\right) + \eta^+ g_\xi^+ \left(\hat{x}(\hat{T})\right)$$

instead of Lemmas 4, 5, relations (39), (40) and (62), (63) in the proof of Theorem 1.

Corollary 1.1. Let $\left(\hat{x}, \hat{u}, \hat{T}\right)$ be an optimal solution of the optimal control problem (OCP) and strongly regular to the constraints (20) as well. Then there exist non-negative numbers $\eta_0, \eta_1, \cdots, \eta_{q+r}$, not all being zero, square Lebesgue integrable functions $\lambda_1(\tau), \cdots, \lambda_s(\tau)$ and absolutely continuous functions $\psi_1(\tau), \cdots, \psi_n(\tau)$, both defined on $\left[0, \hat{T}\right]$, which satisfy the conditions (N1), (N4) and the following conditions:

(N2') $\quad \lambda_j(\tau) \geq 0,\ \lambda_j(\tau) h_j(\hat{x}(\tau)) = 0$ for almost all $\tau \in \left(0, \hat{T}\right)$ and all $j = 1, \cdots, s$,

$$\frac{d\psi(\tau)}{d\tau} = -\psi(\tau) f_\xi(\hat{x}(\tau), \hat{u}(\tau), \tau) - \lambda(\tau) h_\xi(\hat{x}(\tau)) \text{ for almost all } \tau \in \left[0, \hat{T}\right],$$

(N3') $\quad \psi(0) = -\eta^- g_\xi^-(\hat{x}(0)),$

$$\psi\left(\hat{T}\right) = \eta_0 [g_0]_\xi \left(\hat{x}\left(\hat{T}\right)\right) + \eta^+ g_\xi^+ \left(\hat{x}\left(\hat{T}\right)\right).$$

Next we consider the problem (TOCP).

Theorem 2. Let $\left(\hat{x}, \hat{u}, \hat{T}\right)$ be an optimal solution of the time optimal control problem (TOCP) and regular to the constraints (20) as well. Then there exist non-negative numbers $\eta_1, \cdots, \eta_{q+r}$, not all being zero, square Lebesgue integrable functions $\lambda_1(\tau), \cdots, \lambda_s(\tau)$ and absolutely continuous functions $\psi_1(\tau), \cdots, \psi_n(\tau)$, defined on $\left[0, \hat{T}\right]$, which satisfy the conditions (N1), (N2), (N4) and the following conditions:

$$\frac{d\psi(\tau)}{d\tau} = -\psi(\tau) f_\xi(\hat{x}(\tau), \hat{u}(\tau), \tau) - \lambda(\tau) h_\xi(\hat{x}(\tau)) \quad \text{for almost all } \tau \in \left[0, \hat{T}\right],$$

(N3'') $\quad \psi(0) = -\eta^- g_\xi^-(\hat{x}(0)) - \lambda(0) h_\xi(\hat{x}(0)),$

$$\psi\left(\hat{T}\right) = \eta^+ g_\xi^+ \left(\hat{x}\left(\hat{T}\right)\right) + \lambda\left(\hat{T}\right) h_\xi\left(\hat{x}\left(\hat{T}\right)\right).$$

Proof. Let $\left(\hat{x}, \hat{u}, \hat{T}\right)$ be an optimal solution of (TOCP) as well as regular to the constraints (20). If we use the space R^{q+r} in place of R^{q+r+1} and replace (43) and (44) by

$$Y = \left\{ {}^t\left(y_1, \cdots, y_{q+r}\right) \in R^{q+r} \mid y_i \leq 0 \text{ for all } i = 1, \cdots, q+r \right\}$$

and

$$\bar{g}(x)=\begin{pmatrix} g^-(x(0)) \\ g^+\left(x(\hat{T})\right) \end{pmatrix},\quad G(x)=\begin{pmatrix} g_\xi^-(\hat{x}(0))x(0) \\ g_\xi^+\left(\hat{x}(\hat{T})\right)x(\hat{T}) \end{pmatrix},\quad \text{for each } x\in\mathscr{X}$$

as the definitions of cone Y and functions $\bar{g}$, G, respectively, then we can show the relation (46), because, for every $x\in\Lambda$, $u\in\Omega$ and $T\in\left(0,\hat{T}\right)$, (x,u,T) is not a feasible solution of the control system (18)-(22). By virtue of Lemma 1, (46) implies that there exist a non-zero continuous linear functional y^* on R^{q+r} and a continuous linear functional z^* on $\mathscr{Z}$ which satisfy the relations (56)-(59). Hence we can prove the theorem in the same way as we have proved in Theorem 1, but (60) and (63) replaced by

$$y^*(y)=\sum_{i=1}^{q+r}\eta_i y_i \quad \text{for all } y={}^t\left(y_1,\cdots,y_{q+r}\right)\in R^{q+r},$$

and

$$\Delta=\eta^+ g_\xi^+\left(\hat{x}\left(\hat{T}\right)\right)+\lambda\left(\hat{T}\right)h_\xi\left(\hat{x}\left(\hat{T}\right)\right),$$

respectively. □

In case the optimal solution $\left(\hat{x},\hat{u},\hat{T}\right)$ of (TOCP) is strongly regular to the constraints (20), we can easily verify that the following is true, by virtue of Lemmas 4' and 5'.

Corollary 2.1. Let $\left(\hat{x},\hat{u},\hat{T}\right)$ be an optimal solution of the time optimal control problem (TOCP) and strongly regular to the constraints (20) as well. Then there exist non-negative numbers $\eta_1,\cdots,\eta_{q+r}$, not all being zero, square Lebesgue integrable functions $\lambda_1(\tau),\cdots,\lambda_s(\tau)$ and absolutely continuous functions $\psi_1(\tau),\cdots,\psi_n(\tau)$, both defined on $\left[0,\hat{T}\right]$, which satisfy the conditions (N1), (N2'), (N4) and the following conditions:

$$\frac{d\psi(\tau)}{d\tau}=-\psi(\tau)f_\xi(\hat{x}(\tau),\hat{u}(\tau),\tau)-\lambda(\tau)h_\xi(\hat{x}(\tau)) \quad \text{for almost all } \tau\in\left[0,\hat{T}\right],$$

(N3''')
$$\psi(0)=-\eta^- g_\xi^-(\hat{x}(0)),$$
$$\psi\left(\hat{T}\right)=\eta^+ g_\xi^+\left(\hat{x}\left(\hat{T}\right)\right).$$

6. Example and Concluding Remarks

In this section, we shall verify that the optimal solution of the following simple example about time optimal control problem with restricted phase coordinates is strongly regular to the constraints (20) and satisfies the Corollary 2.1:

Find a time T, absolutely continuous functions x_1, x_2 and a measurable function u, both defined on $[0,T]$ such that

1. $\dfrac{dx_1(\tau)}{d\tau}=x_2(\tau),\quad \dfrac{dx_2(\tau)}{d\tau}=u(\tau)$ *for almost all* $\tau\in[0,T]$,

2. $x_1(0)\geq 4,\ x_2(0)\geq 0,\quad x_1(T)\leq 0,\ x_2(T)\geq 0$,

3. $|x_2(\tau)|\leq 1$ *for all* $\tau\in[0,T]$,

4. $|u(\tau)|\leq 1$ *for almost all* $\tau\in[0,T]$,

and such that T achieves a minimum (subject to the above constraints).

If we define an absolutely continuous function $\hat{x}(\bullet)={}^t(\hat{x}_1(\bullet),\hat{x}_2(\bullet))$ from $[0,5]$ into R^2 and a measurable essentially bounded functions $\hat{u}(\bullet)$ from $[0,5]$ into R^1 as follows:

(69)
$$\hat{x}(\tau)=\begin{cases} {}^t\left(-\dfrac{\tau^2}{2}+4,\ -\tau\right), & 0\leq\tau<1,\\ {}^t\left(-\tau+\dfrac{9}{2},\ -1\right), & 1\leq\tau<4,\\ {}^t\left(\dfrac{\tau^2}{2}-5\tau+\dfrac{25}{2},\ \tau-5\right), & 4\leq\tau\leq 5,\end{cases}$$

(70)
$$\hat{u}(\tau)=\begin{cases}-1, & 0\leq\tau<1,\\ 0, & 1\leq\tau<4,\\ 1, & 4\leq\tau\leq 5,\end{cases}$$

then the functions $\hat{u}(\tau)$ and $\hat{x}(\tau)$, $0\leq\tau\leq 5$, are so-called optimal control and optimal trajectory of the above problem, respectively.

Let $\tilde{T}$ be a sufficiently large positive number and let $J=(-1,\tilde{T})$. Let Λ and Ω be a set of all absolutely continuous functions from J into R^2 and a set of all measurable functions from J into $U=\{v\in R^1 \mid |v|\leq 1\}$, respectively.

We set that, for each $\xi={}^t(\xi_1,\xi_2)\in R^2$, $v\in R^1$, $\tau\in J$,

(71)
$$f(\xi,v,\tau)=\begin{pmatrix}\xi_2\\ v\end{pmatrix}$$

(72)
$$h_1(\xi)=\xi_2-1,\quad h_2(\xi)=-\xi_2-1,$$

(73)
$$g_1(\xi)=4-\xi_1,\quad g_2(\xi)=-\xi_2,\quad g_3(\xi)=\xi_1,\quad g_4(\xi)=-\xi_2.$$

It is easily verified that the example given in this section is the time optimal control problem (TOCP) introduced in section 3, where $n=q=r=s=2$ and $m=1$. Hence the triple $(\hat{x}.\hat{u},5)$ is the optimal solution of (TOCP), where the function $\hat{x}(\tau)$ and $\hat{u}(\tau)$ are the functions presented by (69) and (70), respectively. It is obtained from (71) that $f_\xi(\hat{x}(\tau),\hat{u}(\tau),\tau)=\begin{pmatrix}0&1\\0&0\end{pmatrix}$. Hence we obtain from (25) that $\Phi(\tau)=\begin{pmatrix}1&\tau\\0&1\end{pmatrix}$, $\Phi(\tau)^{-1}=\begin{pmatrix}1&-\tau\\0&1\end{pmatrix}$ for all $\tau\in[0,5]$. By means of (26), (27) and (71), for each $u\in\Omega$, the function

$$\tilde{x}(\tau)=\Phi(\tau)\left[\begin{pmatrix}0\\0\end{pmatrix}+\int_0^\tau \Phi(\sigma)^{-1}\{f(\hat{x}(\sigma),u(\sigma),\sigma)-f(\hat{x}(\sigma),\hat{u}(\sigma),\sigma)\}d\sigma\right]$$

(74)

$$=\int_0^\tau \begin{pmatrix}(\tau-\sigma)(u(\sigma)-\hat{u}(\sigma))\\ u(\sigma)-\hat{u}(\sigma)\end{pmatrix}d\sigma, \qquad 0\le\tau\le5,$$

is an element of $Q(\hat{x},\hat{u},5)$, because $f(\xi,u(\sigma),\sigma)-f(\hat{x}(\sigma),\hat{u}(\sigma),\sigma)\in\mathfrak{F}$.

Let us check that the optimal solution $(\hat{x},\hat{u},5)$ will be strongly regular to the constraint

(75) $$h_1(\hat{x}(\tau))=\hat{x}_2(\tau)-1\le0,\quad h_2(\hat{x}(\tau))=-\hat{x}_2(\tau)-1\le0 \quad \text{for all } \tau\in[0,5].$$

Let $p(\tau)={}^t(p_1(\tau),\ p_2(\tau))$ be an arbitrary 2-vector valued polynomial. Let us define non-negative continuous functions $z^+(\tau)$ and $z^-(\tau)$ on $[0,5]$ as follows:

$$z^+(\tau)=\frac{|p_2(\tau)|+p_2(\tau)}{2},\quad z^-(\tau)=\frac{|p_2(\tau)|-p_2(\tau)}{2} \quad \text{for all } \tau\in[0,5],$$

then it is easily verified that

(76) $$p_2(\tau)=z^+(\tau)-z^-(\tau),\quad |z^-(\tau)|\le|p_2(\tau)| \quad \text{for all } \tau\in[0,5].$$

Let

(77) $$\delta=\left[\int_0^5|z^-(\tau)|^2d\tau\right]^{1/2}.$$

Let us consider the case where $\delta>0$. There is a polynomial $\tilde{p}(\tau)$ such that

(78) $$-2\delta<\tilde{p}(\tau)+z^-(\tau)<0 \quad \text{for all } \tau\in[0,5].$$

Let ε be a number such that

(79) $$0<\varepsilon<\min\left\{\frac{3\delta^2}{\tilde{p}(1)^2},\ \frac{3\delta^2}{\tilde{p}(4)^2},\ \frac{1}{2}\right\}.$$

Since $z^-(\tau)$ is continuous on $[0,5]$, it follows from (78) that there is an $\varepsilon'\in(0,\varepsilon)$ such that

(80) $$\frac{[\tau-(1-\varepsilon)]\tilde{p}(1)}{\varepsilon}\le-z^-(\tau) \quad \text{whenever } 1-\varepsilon'\le\tau\le1,$$

(81) $$-\frac{[\tau-(4+\varepsilon)]\tilde{p}(4)}{\varepsilon}\le-z^-(\tau) \quad \text{whenever } 4\le\tau\le4+\varepsilon'.$$

Let ρ and γ be sufficiently large numbers such that

$$\rho > max\left\{ \max_{0\le\tau\le5} \left|\tilde{p}'(\tau)\right| , \quad -\frac{\tilde{p}(1)}{2\varepsilon} , \quad -\frac{\tilde{p}(4)}{2\varepsilon} \right\},$$

(82) $$\gamma > \frac{1}{\varepsilon'} \max\left\{ \max_{0\le\tau\le5} \left|p_1(\tau)\right| + \max_{0\le\tau\le5} \left|\tilde{p}(\tau)\right| , \quad \max_{0\le\tau\le5} \left|p_2(\tau)\right| \right\},$$

where $\tilde{p}'(\tau)$ denotes the derivative of the polynomial $\tilde{p}(\tau)$. Now we define a measurable essentially bounded functions $u(\bullet)$ from $[0,5]$ into R^1 as follows:

(83) $$u(\tau) = \begin{cases} -1, & 0 \le \tau < 1-\varepsilon, \\ -\dfrac{\tilde{p}(1)}{\rho\varepsilon} - 1, & 1-\varepsilon \le \tau < 1, \\ -\dfrac{\tilde{p}'(\tau)}{\rho} & 1 \le \tau < 4, \\ \dfrac{\tilde{p}(4)}{\rho\varepsilon} + 1, & 4 \le \tau < 4+\varepsilon, \\ 1, & 4+\varepsilon \le \tau \le 5. \end{cases}$$

Substituting (83) for (74), we can obtain the function

(84) $$\tilde{x}(\tau) \in Q(\hat{x}, \hat{u}, 5)$$

as follows:

(85) $$\tilde{x}(\tau) = \begin{cases} {}^t(0,\ 0), & 0 \le \tau < 1-\varepsilon, \\ \dfrac{-\tilde{p}(1)}{\rho\varepsilon}{}^t\left(\dfrac{(\tau-1+\varepsilon)^2}{2},\ \tau-1+\varepsilon\right), & 1-\varepsilon \le \tau < 1, \\ \dfrac{-1}{\rho}{}^t\left(\dfrac{\varepsilon\tilde{p}(1)}{2} + \tilde{P}(\tau),\ \tilde{p}(\tau)\right), & 1 \le \tau < 4, \\ \dfrac{1}{\rho}{}^t\left(\dfrac{\tilde{p}(4)}{2\varepsilon}(\tau-4)(\tau-4-2\varepsilon) - \dfrac{\varepsilon\tilde{p}(1)}{2} - \tilde{P}(4),\ \dfrac{\tilde{p}(4)}{\varepsilon}(\tau-4-\varepsilon)\right), & 4 \le \tau < 4+\varepsilon, \\ \dfrac{-1}{\rho}{}^t\left(\dfrac{\varepsilon(\tilde{p}(1)+\tilde{p}(4))}{2} + \tilde{P}(4),\ 0\right), & 4+\varepsilon \le \tau \le 5, \end{cases}$$

where $\tilde{P}(\tau) = \int_1^\tau \tilde{p}(\sigma)d\sigma$ for all $\tau \in [0,5]$. Further the function $\tilde{x}(\bullet)$ satisfies the following inequalities:

(86) $$\begin{aligned} p_1(\tau) &\ge \rho(0,\ \hat{x}_2(\tau))\tilde{x}(\tau) + \gamma(\hat{x}_2(\tau) - 1), \\ p_2(\tau) &\ge \rho(0,\ -\hat{x}_2(\tau))\tilde{x}(\tau) + \gamma(-\hat{x}_2(\tau) - 1), \end{aligned} \quad \text{for all } \tau \in [0,5],$$

(87) $$\rho\left[\left|\tilde{x}(0)\right|^2 + \left|\tilde{x}(5)\right|^2 + \int_0^5 \left|\tilde{x}(\tau)\right|^2 d\tau\right]^{1/2} \le 21\left[\int_0^5 \left|p(\tau)\right|^2 d\tau\right]^{1/2}.$$

The inequalities (86) and (87) correspond to the inequalities (28) and (30), respectively.

In case $\delta = 0$, we define positive numbers ρ, γ and absolutely continuous 2-vector valued function $\tilde{x}(\bullet)={}^t(x_1(\bullet), x_2(\bullet))$ on $[0,5]$ as follows:

$$\rho = 1, \quad \gamma = \max\left\{\max_{0\le\tau\le5}|p_1(\tau)|, 1\right\}, \quad \tilde{x}(\tau)={}^t(0, 0) \quad \text{for all } \tau \in [0,5].$$

Then it is easily verified that the function $\tilde{x}(\tau)$ also satisfies the relations (84), (86) and (87). Therefore, we see, from (84), (86) and (87), that the optimal solution $(\hat{x}, \hat{u}, 5)$ is strongly regular to the constraints (75).

Now we see that the optimal solution $(\hat{x}, \hat{u}, 5)$ satisfies the Corollary 2.1, that is, there exist non-negative numbers η_1, η_2, η_3, η_4, not all being zero, square Lebesgue integrable functions $\lambda_1(\tau)$, $\lambda_2(\tau)$ and absolutely continuous functions $\psi_1(\tau)$, $\psi_2(\tau)$, both defined on $[0,5]$, which satisfy the following conditions:

(88) $\quad \eta_1(4 - \hat{x}_1(0)) = \eta_2(-\hat{x}_2(0)) = \eta_3(\hat{x}_1(5)) = \eta_4(-\hat{x}_2(5)) = 0$

(89) $\quad \begin{array}{ll} \lambda_1(\tau) \ge 0, & \lambda_2(\tau) \ge 0, \\ \lambda_1(\tau)(\hat{x}_2(\tau) - 1) = 0, & \lambda_2(\tau)(-\hat{x}_2(\tau) - 1) = 0, \end{array}$ for almost all $\tau \in [0,5]$,

(90) $\quad \dfrac{d\psi_1(\tau)}{d\tau} = 0, \quad \dfrac{d\psi_2(\tau)}{d\tau} = -\psi_1(\tau) + \lambda_1(\tau) + \lambda_2(\tau)$ for almost all $\tau \in [0,5]$,

(91) $\quad \psi_1(0) = \eta_1, \quad \psi_2(0) = \eta_2, \quad \psi_1(5) = \eta_3, \quad \psi_2(0) = -\eta_4,$

(92) $\quad \psi_1(\tau)\hat{x}_1(\tau) + \psi_2(\tau)\hat{u}(\tau) = \inf_{-1\le v\le1}\{\psi_1(\tau)\hat{x}_1(\tau) + \psi_2(\tau)v\}$ for almost all $\tau \in [0,5]$.

Indeed, if we set that

$$\eta_1 = \eta_2 = \eta_3 = \eta_4 = 1,$$

$$(\lambda_1(\tau), \lambda_2(\tau)) = \begin{cases} (0, 0), & 0 \le \tau \le 1, \\ (0, 1), & 1 \le \tau \le 4, \\ (0,0), & 4 \le \tau \le 5, \end{cases}$$

$$(\psi_1(\tau), \psi_2(\tau)) = \begin{cases} (1, 1\text{-}\tau 0), & 0 \le \tau \le 1, \\ (1, 0), & 1 \le \tau \le 4, \\ (1, 4 - \tau), & 4 \le \tau \le 5, \end{cases}$$

then it is easily verified, from (69) and (70), that the relations (88)-(92) hold.

Let us here state concluding remarks. We can obtain the Lemma 1 except the continuity of the linear functional z^*, without the Condition 1. In this case, the functional z^* is continuous with respect to the additional norm $]\bullet[_{\mathscr{X}}$, but it is not always continuous with respect to the original norm $\| \bullet \|_{\mathscr{X}}$, which implies the integral of (66) in the proof of Theorem 1 is in the sense of Stieltjes. Therefore we can not give the assurance that so-called adjoint variables $\psi_1(\bullet), \cdots, \psi_n(\bullet)$ are continuous on all the interval where the optimal control $\hat{u}(\bullet)$ is defined. We now end our discussion by emphasizing that it is Condition 1 that give the assurance.

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