山口大学教育学部附属教育実践総合センター研究紀要第13号（2002.3）

Constraint Qualifications for Lagrange Multiplier Rules satisfied by Extremals

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(Received November 6, 2001)

Abstract. There are two purposes in the paper. One is to introduce equivalent conditions for solutions of extremal problems to satisfy Lagrange multiplier rules in real, linear topological spaces, and the other is to show that, by slightly changing the shape of the Neustadt's conditions, we can obtain equivalent conditions for the solutions to satisfy Lagrange multiplier rules.

Key Words. Constraint Qualification, Lagrange multiplier rule, Exremal, Convex function, Affine function.

1. Introduction.

In research on nonlinear programming problems, many kinds of constraint qualifications have been presented for obtaining meaningful results (see, e.g., [1-3]). Neustadt also gave constraint qualifications for solutions of extremal problems, and showed that such solutions satisfy Lagrange multiplier rule (see Th.3.1 in [4]). For solutions of extremal problems defined in real, linear spaces, similar results were given in [5]. In this paper, we shall introduce some relations between Lagrange multiplier rules and constraint qualifications for the solutions of extremal problems, and give necessary and sufficient conditions for the solutions to satisfy Lagrange multiplier rules by means of a few modification of the Neustadt's constraint qualifications.

In section 2, we shall give notations, definitions and their properties which will be

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used in the sequel. In section 3, we shall define two kinds of extremals, which we shall refer to as (φ, ψ, Z, X)-extremal and (ψ, Z, X)-extremal. These concepts were first introduced by Neustadt (see Def.2.1 and Def.2.2 in [4]). For each extremal, we shall give a Lagrange multiplier rule and two kinds of constraint qualifications. In sections 4 and 5, we shall show that the constraint qualifications and Lagrange multiplier rule are equivalent for the (φ, ψ, Z, X)-extremal and (ψ, Z, X)-extremal, respectively. In section 6, we shall consider the abstract maximum principle given by Neustadt. We shall introduce a constraint qualification for a (φ, ψ, Z, X)-extremal by slightly changing the shape of the Condition 3.1 in [4]. We shall show that if an element is the (φ, ψ, Z, X)-extremal, then the Lagrange multiplier rule is equivalent to the reformed constraint qualifications, and we shall give a theorem which is virtually equivalent to the Theorem 3.1 in [4].

2. Preliminary Results.

In this section, we shall give notations, definitions and show fundamental results which will be used in the sequel.

Let $\mathscr{Y}$ be a topological linear space over the real field, R a set of all real numbers and N a set of all positive integers. For every $k \in N$, we define a simplex P_+^{k-1} of R^k (a k-dimensional vector space) as follows:

$$P_+^{k-1} = \left\{ \left(\alpha^1, \cdots, \alpha^k\right) \,\middle|\, \sum_{i=1}^{k} \alpha^i = 1,\ \ \alpha^i \geq 0,\ \text{ for } i = 1, \cdots, k \right\}.$$

For arbitrary subset E and F of $\mathscr{Y}$, we define sets $E-F$, $E+F$ and $E \backslash F$ as follows:

$$E - F = \{u - v \mid u \in E,\ v \in F\}, \quad E + F = \{u + v \mid u \in E,\ v \in F\}, \quad E \backslash F = \{v \mid v \in E,\ v \notin F\}.$$

A non-empty subset C of $\mathscr{Y}$ will be called a cone if $\alpha C \subset C$ whenever $\alpha \geq 0$, where $\alpha C = \{\alpha x \mid x \in C\}$.

The following lemma was shown by Neustadt (Lemma 3.1 in [1]).

Lemma 2.1. If C is a convex cone in $\mathscr{Y}$ with non-empty interior, then

$$\alpha\, \text{int.}C + \beta\, C = \text{int.}C \quad \text{whenever} \ \ \alpha > 0,\ \ \beta \geq 0,$$

in particular,

$$\text{int.}C + C = \text{int.}C,$$

where $\text{int.}C$ is the interior of the set C.

For an element $v \in \mathscr{Y}$ and a set $A \subset \mathscr{Y}$, the following subset $[A|v]$ was defined by Neustadt ((4.3) in [1]):

$$[A|v] = \{a + \gamma v \mid a \in A,\ \gamma \le 0\}.$$

If A is a convex subset of $\mathscr{Y}$ with non-empty interior, then $\text{int.}A$ is an open convex set. Therefore we can easily show the following lemma, and hence we shall omit the proof.

Lemma 2.2. Let C be a convex cone in $\mathscr{Y}$ and let $v \in \mathscr{Y}$. Then $[C|v]$ is a convex cone in $\mathscr{Y}$ such that $C \subset [C|v]$. If the set C has non-empty interior, then $[\text{int.}C|v]$ is an open convex subset of $[C|v]$.

We shall now show the following lemma (cf. Lemma 3.2 in [5]).

Lemma 2.3. Let C be a convex cone in $\mathscr{Y}$ with non-empty interior and let $v \in \mathscr{Y}$, then $[\text{int.}C|v] = \text{int.}[C|v]$.

Proof. From Lemma 2.2, the set $[\text{int.}C|v]$ is an open convex subset of $\mathscr{Y}$ such that $[\text{int.}C|v] \subset [C|v]$, which implies that $[\text{int.}C|v] \subset \text{int.}[C|v]$ and $\text{int.}\{\text{cl.}[\text{int.}C|v]\} = [\text{int.}C|v]$, where $\text{cl.}S$ is the closure of the set S. Therefore, to prove that $[\text{int.}C|v] = \text{int.}[C|v]$, it suffices show that $[C|v] \subset \text{cl.}[\text{int.}C|v]$. Let $w \in [C|v]$ be arbitrary, and let U be an arbitrary neighborhood of the origin $0_{\mathscr{Y}}$ of the space $\mathscr{Y}$. There exist a $c_w \in C$ and $\gamma_w \le 0$ such that $w = c_w + \gamma_w v$. Since the convex cone C has a non-empty interior, we obtain that $C \subset \text{cl.}\{\text{int.}C\}$. Hence there exists an $c \in \text{int.}C$ such that $c \in c_w + U$, which implies that $c + \gamma_w v \in c_w + U + \gamma_w v \subset w + U$. Since $c + \gamma_w v \in \text{int.}C + \gamma_w v \subset [\text{int.}C|v]$, it is satisfied that $\{w + U\} \cap [\text{int.}C|v] \ne \varnothing$, that is, $w \in \text{cl.}[\text{int.}C|v]$. Thus we obtain that $[C|v] \subset \text{cl.}[\text{int.}C|v]$. □

3. Lagrange Multiplier Rules for Extremals and Constraint Qualifications.

In this section, we shall formulate two kinds of extremals which nave been introduced by Neustadt [4], and give equivalent conditions which the extremals satisfy.

Let $\mathscr{X}$ be a real linear space, and let $\mathscr{Y}$ and $\mathscr{Z}$ be real linear topological spaces. Let $\mathscr{Y}^*$ and $\mathscr{Z}^*$ be the topological duals of $\mathscr{Y}$ and $\mathscr{Z}$ respectively. Let X be a subset of $\mathscr{X}$, Z a convex cone in $\mathscr{Z}$ with non-empty interior. Let $\varphi(\bullet)$ and $\psi(\bullet)$ be functions from $\mathscr{X}$ into $\mathscr{Y}$ and $\mathscr{Z}$, respectively. Now let us present extremals, which were first introduced by Neustadt (see Def.2 and Def. 3 in [4]).

Definition 3.1. (Neustadt) An element $\hat{x} \in \mathscr{X}$ be called a (φ, ψ, Z, X)-extremal, if

(i) $\varphi(\hat{x}) = 0_{\mathscr{Y}}$ (the origin of $\mathscr{Y}$),

(ii) $\psi(\hat{x}) \in Z$,

(iii) $\hat{x} \in X$,

(iv) $\left\{x \in X \mid \varphi(x) = 0_{\mathscr{Y}},\ \psi(x) \in \text{int.}Z\right\} = \varnothing$.

Definition 3.2. (Neustadt). An element $\hat{x} \in \mathscr{X}$ be called a (ψ, Z, X)-extremal, if

(i) $\psi(\hat{x}) \in Z$,

(ii) $\hat{x} \in X$,

(iii) $\left\{x \in X \mid \psi(x) \in \text{int.}Z\right\} = \varnothing$.

In the first place, we shall consider the (φ, ψ, Z, X)-extremal. We shall give a Lagrange multiplier rule and constraint qualifications for a (ψ, Z, X)-extremal.

Lagrange Multiplier Rule A (LMR:A). There exist a non-empty convex subset K of $\mathscr{X}$, an affine function $f: K \to \mathscr{Y}$, a Z-convex function $g: K \to \mathscr{Z}$ and elements $z^* \in \mathscr{Z}^*$, $y^* \in \mathscr{Y}^*$ such that

(1) $$(z^*, y^*) \neq (0_{\mathscr{Z}^*}, 0_{\mathscr{Y}^*}),$$

(2) $$z^*(g(x)) + y^*(f(x)) \geq 0 \quad \text{for all } x \in K,$$

(3) $$z^*(z) \leq 0 \quad \text{for all } z \in Z,$$

(4) $$z^*(\psi(\hat{x})) = 0.$$

Here $0_{\mathscr{Z}^*}$ and $0_{\mathscr{Y}^*}$ are the origins of the spaces $\mathscr{Z}^*$ and $\mathscr{Y}^*$, respectively, and the Z-convex function $g: K \to \mathscr{Z}$ is defined as follows: for every positive integer k,

$$g\left(\sum_{i=1}^{k} \alpha_i x_i\right) \in \sum_{i=1}^{k} \alpha_i g(x_i) + Z$$

whenever $v_i \in K$ for each $i = 1, \cdots, k$ and $(\alpha_1, \cdots, \alpha_k) \in P_+^{k-1}$. The Z-convex function was introduced by Neustadt [4].

Basic Constraint Qualification A (BCQ:A). There exist a non-empty convex subset K of $\mathscr{X}$, an affine function $f: K \to \mathscr{Y}$, a Z-convex function $g: K \to \mathscr{Z}$ and a convex subset V of $\mathscr{Y}$ such that

$$0_{\mathscr{Y}} \in V, \quad \text{int.}V \neq \varnothing \quad \text{and} \quad \text{int.}V \cap f\left(g^{-1}\left(\text{int.}\left[Z \middle| \psi(\hat{x})\right]\right)\right) = \varnothing.$$

The following constraint qualification is obtained by changing the shape of the Condition 3.1 in [4].

Generalized Neustadt's Constraint Qualification A (GNCQ:A). There exist a non-empty convex subset K of $\mathscr{X}$, an affine function $f: K \to \mathscr{Y}$ and a Z-convex function $g: K \to \mathscr{Z}$ with the following property: In case $\text{int.} f\left(g^{-1}\left(\text{int}\left[Z|\psi(\hat{x})\right]\right)\right) = \varnothing$, there exists a non-empty convex subset V of $\mathscr{Y}$ such that $0_{\mathscr{Y}} \in V$, $\text{int.}V \neq \varnothing$ and $\text{int.}V \cap f\left(g^{-1}\left(\text{int.}\left[Z|\psi(\hat{x})\right]\right)\right) = \varnothing$, and in case $\text{int.} f\left(g^{-1}\left(\text{int}\left[Z|\psi(\hat{x})\right]\right)\right) \neq \varnothing$, for every triple (S, U, η) — where S is a convex subset of $g^{-1}\left(\text{int.}\left[Z|\psi(\hat{x})\right]\right)$ such that $0_{\mathscr{Y}} \in \text{cr.} f(S) \subset \text{int.} f\left(g^{-1}\left(\text{int.}\left[Z|\psi(\hat{x})\right]\right)\right)$, U is a neighborhood of $0_{\mathscr{Z}}$ and $\eta > 0$ — there are an $\bar{x} \in X$ and an $\bar{\varepsilon} \in (0, \eta)$ (both possibly depending on S, U and η) such that

(5) $$\varphi(\bar{x}) = 0_{\mathscr{Y}},$$

(6) $$\frac{\psi(\bar{x}) - \psi(\hat{x})}{\bar{\varepsilon}} \in \text{co.} g(S) + Z + U,$$

where $0_{\mathscr{Z}}$ is the origin of $\mathscr{Z}$, cr.A and co.A are the core and convex hull of the set A, respectively. (The core of the set was first introduced by Klee (see [6]).

In the second place, we shall consider a (ψ, Z, X)-extremal. We shall give a Lagrange multiplier rule and constraint qualifications for a (ψ, Z, X)-extremal.

Lagrange Multiplier Rule B (LMR:B). There exist a non-empty convex subset K of $\mathscr{X}$, a Z-convex function $g: K \to \mathscr{Z}$ and an element $z^* \in \mathscr{Z}^*$ such that

(7) $$z^* \neq 0_{\mathscr{Z}^*}$$

(8) $$z^*(g(x)) \geq 0 \quad \text{for all } x \in K,$$

(9) $$z^*(z) \leq 0 \quad \text{for all } z \in Z,$$

(10) $$z^*(\psi(\hat{x})) = 0.$$

Basic Constraint Qualification B (BCQ:B). There exist a non-empty convex subset K of $\mathscr{X}$ and a Z-convex function $g: K \to \mathscr{Z}$ such that

$$g^{-1}\left(\text{int.}\left[Z|\psi(\hat{x})\right]\right) = \varnothing.$$

Generalized Neustadt's Constraint Qualification B (GNCQ:B). There exist a non-empty convex subset K of $\mathscr{X}$ and a Z-convex function $g: K \to \mathscr{Z}$ with the following property: For every triple (x, U, η) — where $x \in g^{-1}\left(\text{int.}\left[Z|\psi(\hat{x})\right]\right)$, U is a neighborhood of $0_{\mathscr{Z}}$ and $\eta > 0$ — there are an $\bar{x} \in X$ and an $\bar{\varepsilon} \in (0, \eta)$ (both possibly depending on x, U and η) such that

(11) $$\frac{\psi(\bar{x}) - \psi(\hat{x})}{\bar{\varepsilon}} \in g(x) + Z + U.$$

The BCQ:B and GNCQ:B are obtained slightly changing the shape of the Condition B in [2] and Condition 3.2 in [4], respectively.

4. Properties of (φ, ψ, Z, X)-extremal.

In this section, we shall show that, if an $\hat{x} \in \mathscr{X}$ is a (φ, ψ, Z, X)-extremal, all conditions for the (φ, ψ, Z, X)-extremal are equivalent. We will use all notations and terminologies in sections 2 and 3 without further explanation.

Lemma 4.1. Let $\hat{x} \in \mathscr{X}$ be a (φ, ψ, Z, X)-extremal. The element $\hat{x} \in \mathscr{X}$ satisfies the LMR:A if, and only if, the $\hat{x} \in \mathscr{X}$ satisfies the BCQ:A.

Proof. Let $\hat{x} \in \mathscr{X}$ be a (φ, ψ, Z, X)-extremal which satisfies the LMR:A. Then there exist a non-empty convex set $K \subset \mathscr{X}$, an affine function $f: K \to \mathscr{Y}$, a Z-convex function $g: K \to \mathscr{Z}$, a $z^* \in \mathscr{Z}^*$ and $y^* \in \mathscr{Y}^*$ satisfying the relations (1)-(4), and let $V = \{y \in \mathscr{Y} \mid y^*(y) \leq 0\}$, then the set V is a convex subset of $\mathscr{Y}$ such that $0_{\mathscr{Y}} \in V$ and $\text{int.}V \neq \varnothing$. In order to show that

(12) $$\text{int.}V \cap f\left(g^{-1}\left(\text{int.}\left[Z|\psi(\hat{x})\right]\right)\right) = \varnothing,$$

suppose the contrary. Then there exists an $x \in K$ such that $f(x) \in \text{int.}V$ and $g(x) \in \text{int.}\left[Z|\psi(\hat{x})\right]$. It follows from Lemma 2.3 that there are $z \in \text{int.}Z$ and $\gamma \leq 0$ such that $z = g(x) - \gamma\psi(\hat{x})$. We obtain from (1)-(4) that $y^* \neq 0_{\mathscr{Y}^*}$ Indeed, if $y^* = 0_{\mathscr{Y}^*}$, it follows from (1) that $z^* \neq 0_{\mathscr{Z}^*}$, which, by virtue of (3) together with $z \in \text{int.}Z$, implies that $z^*(z) < 0$. Since $z = g(x) - \gamma\psi(\hat{x})$, we obtain from (4) that $z^*(g(x)) < 0$ which contradicts the inequality (2). We now obtain that $y^*(f(x)) < 0$, because $f(x) \in \text{int.}V$. Hence it follows from (3) and (4) that $z^*(g(x)) + y^*(f(x)) = z^*(g(x)) - \gamma z^*(\psi(\hat{x})) + y^*(f(x)) < 0$ which contradicts the inequality (1). Therefore the relation (12) holds. Since $0_{\mathscr{Y}} \in V$ and $\text{int.}V \neq \varnothing$, the element $\hat{x} \in \mathscr{X}$ satisfies the BCQ:A.

Conversely, let $\hat{x} \in \mathscr{X}$ be a (φ, ψ, Z, X)-extremal which satisfies the BCQ:A. Then there exist a non-empty convex set $K \subset \mathscr{X}$, an affine function $f: K \to \mathscr{Y}$, a Z-convex function $g: K \to \mathscr{Z}$ and a convex subset V of $\mathscr{Y}$ such that $0_{\mathscr{Y}} \in V$, $\text{int.}V \neq \varnothing$ and $\text{int.}V \cap f\left(g^{-1}\left(\text{int.}\left[Z \middle| \psi(\hat{x})\right]\right)\right) = \varnothing$. Let $\mathscr{Z} \times \mathscr{Y}$ be a product topological space, and let Q and T be subset of $\mathscr{Z} \times \mathscr{Y}$ defined by

$$Q = \left[Z \middle| \psi(\hat{x})\right] \times V$$

and

$$T = \bigcup_{x \in K} \{(z, f(x)) \mid g(x) - z \in Z\},$$

respectively. Since Z is a convex cone in $\mathscr{Z}$ with non-empty interior, it follows from Lemma 2.2 that $\left[Z \middle| \psi(\hat{x})\right]$ is a convex cone in $\mathscr{Z}$ with non-empty interior, and hence Q is a convex subset of $\mathscr{Z} \times \mathscr{Y}$ such that $\left(0_{\mathscr{Z}},\ 0_{\mathscr{Y}}\right) \in Q$ and $\text{int.}Q \neq \varnothing$, because $\text{int.}Q = \text{int.}\left[Z \middle| \psi(\hat{x})\right] \times \text{int.}V$. Further it is obvious that T is a non-empty convex subset of $\mathscr{Z} \times \mathscr{Y}$, because the sets K and Z are both convex.

To show that $\text{int.}V \cap T = \varnothing$, suppose the contrary. Then there is a $(z, y) \in \text{int.}V \times T$. Then $z \in \text{int.}\left[Z \middle| \psi(\hat{x})\right]$, $y \in \text{int.}V$ and there is an $x \in K$ such that $g(x) - z \in Z$ and $y = f(x)$. It follows, from Lemmas 2.1 and 2.2, that

$$g(x) \in z + Z \subset \text{int.}\left[Z \middle| \psi(\hat{x})\right] + \left[Z \middle| \psi(\hat{x})\right] = \text{int.}\left[Z \middle| \psi(\hat{x})\right],$$

that is, $x \in g^{-1}\left(\text{int.}\left[Z \middle| \psi(\hat{x})\right]\right)$. This contradicts the relation (12), because $y = f(x) \in \text{int.}V$. We obtain that Q and T are non-empty convex subset of $\mathscr{Z} \times \mathscr{Y}$ such that $\left(0_{\mathscr{Z}},\ 0_{\mathscr{Y}}\right) \in Q$, $\text{int.}Q \neq \varnothing$ and $\text{int.}V \cap T = \varnothing$.

By the separation theorem of convex subsets of a topological linear space (e.g., see Theorem V.2.8 in [7]), there exist $z^* \in \mathscr{Z}^*$, $y^* \in \mathscr{Y}^*$ and real number α such that

(13) $$(z^*,\ y^*) \neq \left(0_{\mathscr{Z}^*},\ 0_{\mathscr{Y}^*}\right)$$

(14) $$z^*(g(x)) - z^*(z) + y^*(f(x)) \geq \alpha \qquad \text{for all } x \in K \text{ and } z \in Z,$$

(15) $$z^*(z) + y^*(y) \leq \alpha \qquad \text{for all } z \in \left[Z \middle| \psi(\hat{x})\right] \text{ and } y \in V.$$

Since $0_{\mathscr{Y}} \in V$ and $0_{\mathscr{Z}} \in Z$, we obtain from (13) and (14) that

(16) $$z^*(g(x)) + y^*(f(x)) \geq \alpha \qquad \text{for all } x \in K,$$

(17) $$z^*(z) \leq \alpha \qquad \text{for all } z \in \left[Z \middle| \psi(\hat{x})\right].$$

Since $\left[Z|\psi(\hat{x})\right]$ is a convex cone in $\mathscr{Z}$ such that $0_{\mathscr{Z}} \in Z \subset \left[Z|\psi(\hat{x})\right]$, the real number α must be 0, and hence the relations (1), (2) and (3) are obtained from (13), (15), (16). Since $-\psi(\hat{x}) \in \left[Z|\psi(\hat{x})\right]$, $\alpha = 0$ and $\psi(\hat{x}) \in Z$, we can obtain the equality (4) from the inequalities (2) and (16). Therefore the element $\hat{x} \in \mathscr{X}$ satisfies LMR:A. □

Lemma 4.2. If $\hat{x} \in \mathscr{X}$ is a (φ, ψ, Z, X)-extremal which satisfies the GNCQ:A, then the element $\hat{x} \in \mathscr{X}$ satisfies the BCQ:A.

Proof. Let $\hat{x} \in \mathscr{X}$ be a (φ, ψ, Z, X)-extremal which satisfies the GNCQ:A. Then there exist a non-empty convex subset K of $\mathscr{X}$, an affine function $f: K \to \mathscr{Y}$ and a Z-convex function $g: K \to \mathscr{Z}$ which have property stated in the definition of the GNCQ:A.

In case $\text{int.} f\left(g^{-1}\left(\text{int}\left[Z|\psi(\hat{x})\right]\right)\right) = \varnothing$, there exists a non-empty convex set $V \subset \mathscr{Y}$ such that

(18) $$0_{\mathscr{Y}} \in V, \quad \text{int.}V \neq \varnothing, \quad \text{int.}V \cap f\left(g^{-1}\left(\text{int.}\left[Z|\psi(\hat{x})\right]\right)\right) = \varnothing.$$

In case $\text{int.} f\left(g^{-1}\left(\text{int}\left[Z|\psi(\hat{x})\right]\right)\right) \neq \varnothing$, it is verified, from Lemmas 2.1-2.3, that $f\left(g^{-1}\left(\text{int.}\left[Z|\psi(\hat{x})\right]\right)\right)$ is a convex subset of $\mathscr{Y}$ with non-empty interior. Suppose that $0_{\mathscr{Y}} \in \text{int.} f\left(g^{-1}\left(\text{int.}\left[Z|\psi(\hat{x})\right]\right)\right)$. Then it follows from Lemma 2.3 that there exist $x \in K$ and $\gamma \leq 0$ such that $g(x) \in \text{int.}Z + \gamma\psi(\hat{x})$ and $f(x) = 0_{\mathscr{Y}}$. Since Z is a convex cone with non-empty interior such that $0_{\mathscr{Z}} \in Z$, it follows from lemma 2.1 that

(19) $$\frac{1}{2} g(x) \in \text{int.}Z + \frac{\gamma}{2}\psi(\hat{x}),$$

which, by virtue of Lemma 2.1, implies that

$$g(x) \in \frac{1}{2}g(x) + \frac{1}{2}g(x) + Z \in Z + \text{int.}Z + \frac{1}{2}\gamma\psi(\hat{x}) + \frac{1}{2}g(x) = \text{int.}Z + \frac{1}{2}\gamma\psi(\hat{x}) + \frac{1}{2}g(x)$$

and that there exists a neighborhood U of $0_{\mathscr{Z}}$ such that

(20) $$\frac{1}{2}g(x) + U \subset \text{int.}Z + \frac{1}{2}\gamma\psi(\hat{x}).$$

Now we can define a non-empty convex subset S' of $g^{-1}\left(\text{int.}\left[Z|\psi(\hat{x})\right]\right)$ as follows:

(21) $$S' = g^{-1}\left(\text{int.}Z + \frac{\gamma}{2}\psi(\hat{x}) + \frac{1}{2}g(x)\right).$$

We obtain that

(22) $$f(S') \subset f\left(g^{-1}\left(\text{int.}\left[Z|\psi(\hat{x})\right]\right)\right).$$

Let y be arbitrary element of $\mathscr{Y}$. Since there is a $\tilde{\lambda} \in (0,1)$ such that

$$\lambda y = (1-\lambda)0_{\mathscr{Y}} + \lambda y \in f\left(g^{-1}\left(\text{int.}\left[Z|\psi(\hat{x})\right]\right)\right) \quad \text{whenever} \quad 0 \leq \lambda \leq \tilde{\lambda}.$$

Further there is an $\tilde{x} \in K$ such that $f(\tilde{x}) = \tilde{\lambda} y$. It follows from (19) that there is a $\bar{\lambda} \in (0,1)$ such that

$$\text{(23)} \qquad \frac{1}{2}g(x) + \lambda[g(\tilde{x}) - g(x)] \in \text{int} Z + \frac{\gamma}{2}\psi(\hat{x}) \quad \text{whenever} \quad 0 \le \lambda \le \bar{\lambda}.$$

Since $g : K \to \mathscr{Z}$ is a Z convex function and K is convex, it follows from Lemma 2.1 and (22) that $(1-\lambda)x + \lambda\tilde{x} \in K$ and

$$\begin{aligned} g((1-\lambda)x + \lambda\tilde{x}) &\in (1-\lambda)g(x) + \lambda g(\tilde{x}) + Z = \frac{1}{2}g(x) + Z + \left\{\frac{1}{2}g(x) + \lambda[g(\tilde{x}) - g(x)]\right\} \\ &\subset \frac{1}{2}g(x) + Z + \text{int} Z + \frac{\gamma}{2}\psi(\hat{x}) \\ &= \text{int} Z + \frac{\gamma}{2}\psi(\hat{x}) + \frac{1}{2}g(x) \end{aligned}$$

whenever $0 \le \lambda \le \bar{\lambda}$, which imply that $(1-\lambda)x + \lambda\tilde{x} \in S'$ whenever $0 \le \lambda \le \bar{\lambda}$, that is,

$$\lambda f(\tilde{x}) = (1-\lambda)f(x) + \lambda f(\tilde{x}) = f((1-\lambda)x + \lambda\tilde{x}) \in f(S') \quad \text{whenever} \quad 0 \le \lambda \le \bar{\lambda}.$$

Since $f(\tilde{x}) = \tilde{\lambda} y$, it is obvious that $\lambda y \in f(S')$ whenever $0 \le \lambda \le (\bar{\lambda}\tilde{\lambda})$, and hence $0_{\mathscr{Y}} \in \text{cr}.f(S')$. Therefore we obtain from (22) that

$$\text{(24)} \qquad 0_{\mathscr{Y}} \in \text{cr}.f(S') \subset f(S') \subset f\left(g^{-1}\left(\text{int}[Z|\psi(\hat{x})]\right)\right).$$

It follows from (21) that $g(S') \subset \text{int} Z + (\gamma/2)\psi(\hat{x}) + (1/2)g(x)$, which implies from (20) that $g(S') + U \subset \text{int} Z + (\gamma/2)\psi(\hat{x}) + (1/2)g(x) + U \subset \text{int} Z + \gamma\psi(\hat{x})$. Since the set $\text{int} Z + \gamma\psi(\hat{x})$ is convex, we obtain that

$$\text{(25)} \qquad \text{co}.g(S') + U \subset \text{int} Z + \gamma\psi(\hat{x}).$$

Let $S = (1/2)S' + (1/2)x$, then S is a convex subset of S', because $x \in S'$, and hence it follows from (25) that

$$\text{(26)} \qquad \text{co}.g(S) + U \subset \text{int} Z + \gamma\psi(\hat{x}).$$

Since $f(S) = (1/2)f(S') \subset \text{cr}.f(S')$ and $f\left(g^{-1}\left(\text{int}[Z|\psi(\hat{x})]\right)\right)$ is a convex set such that $0_{\mathscr{Y}} \in \text{int} f\left(g^{-1}\left(\text{int}[Z|\psi(\hat{x})]\right)\right)$, it follows, from (24) and Theorem V.2.1 in [7], that

$$0_{\mathscr{Y}} \in \text{cr}.f(S) \subset f(S) \subset \text{int}.f\left(g^{-1}\left(\text{int}[Z|\psi(\hat{x})]\right)\right).$$

Let $\eta = 1/(1-\gamma)$, then $1 + \bar{\varepsilon}\gamma > \eta > 0$ whenever $0 < \bar{\varepsilon} < \eta$. By means of the hypothesis of GNCQ:A, for the triple (S, U, η), there exist an $\bar{x} \in X$ and $\bar{\varepsilon} \in (0, \eta)$ such that (5) and (6) hold. Since $1 + \bar{\varepsilon}\gamma > \eta > 0$ and $\psi(\hat{x}) \in Z$, it follows, from (6), (26) and Lemma 2.1, that

$$\begin{aligned} \psi(\bar{x}) &\in \bar{\varepsilon}\, \text{co}.g(S) + \bar{\varepsilon} Z + \bar{\varepsilon} U + \psi(\hat{x}) = \bar{\varepsilon}\{\text{co}.g(S) + U\} + \bar{\varepsilon} Z + \psi(\hat{x}) \\ &\subset \bar{\varepsilon}\{\text{int} Z + \gamma\psi(\hat{x})\} + \bar{\varepsilon} Z + \psi(\hat{x}) = \bar{\varepsilon}\{\text{int} Z + Z\} + \bar{\varepsilon} Z + (1 + \bar{\varepsilon}\gamma)\psi(\hat{x}) \\ &\subset \bar{\varepsilon}\, \text{int} Z + (1 + \bar{\varepsilon}\gamma) Z = \text{int} Z, \end{aligned}$$

that is,

(27) $$\psi(\bar{x}) \in \text{int}.Z.$$

Since $\bar{x} \in X$, the relations (5) and (27) contradict that the element $\hat{x}$ is a $(\varphi, \psi, Z, X)-$extremal. Hence we obtain that

(28) $$0_{\mathscr{Y}} \notin \text{int}.f\left(g^{-1}\left(\text{int}.\left[Z|\psi(\hat{x})\right]\right)\right).$$

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The set $\text{int}.f\left(g^{-1}\left(\text{int}.\left[Z|\psi(\hat{x})\right]\right)\right)$is also a convex subset of $\mathscr{Y}$ with non-empty interior, because $f\left(g^{-1}\left(\text{int}.\left[Z|\psi(\hat{x})\right]\right)\right)$ is a convex subset of $\mathscr{Y}$ with non-empty interior, Hence, by means of (28) together with the separation theorem of convex subsets of a topological linear space again, there exists non-zero $y^* \in \mathscr{Y}^*$ such that

(29) $$y^*(y) \geq 0 \quad \text{for all } y \in f\left(g^{-1}\left(\text{int}.\left[Z|\psi(\hat{x})\right]\right)\right).$$

Let $V = \{y \in Y \mid y^*(y) \leq 0\}$, then it is obvious that all of the relation of (18) hold, and hence the element $\hat{x}$ satisfies the BCQ:A. □

Lemma 4.3. If $\hat{x} \in \mathscr{X}$ is a (φ, ψ, Z, X)-extremal which satisfies the BCQ:A , then the element $\hat{x} \in \mathscr{X}$ satisfies the GNCQ:A.

Proof. Let $\hat{x} \in \mathscr{X}$ be a (φ, ψ, Z, X)-extremal which satisfies the BCQ:A. Then there exist a non-empty convex subset K of $\mathscr{X}$, an affine function $f: K \to \mathscr{Y}$, a Z-convex function $g: K \to \mathscr{Z}$ and a convex subset V of $\mathscr{Y}$ such that $0_{\mathscr{Y}} \in V$, $\text{int}.V \neq \varnothing$ and

(30) $$\text{int}.V \cap f\left(g^{-1}\left(\text{int}.\left[Z|\psi(\hat{x})\right]\right)\right) = \varnothing.$$

In case $\text{int}.f\left(g^{-1}\left(\text{int}\left[Z|\psi(\hat{x})\right]\right)\right) = \varnothing$, It is obvious that the $\hat{x}$ satisfies the GNCQ:A. In case $\text{int}.f\left(g^{-1}\left(\text{int}.\left[Z|\psi(\hat{x})\right]\right)\right) \neq \varnothing$, the relation (30) implies that there is no convex subset S of $g^{-1}\left(\text{int}.\left[Z|\psi(\hat{x})\right]\right)$ such that

$$0_{\mathscr{Y}} \in \text{cr}.f(S) \subset \text{int}.f\left(g^{-1}\left(\text{int}.\left[Z|\psi(\hat{x})\right]\right)\right),$$

and hence the $\hat{x}$ satisfies the GNCQ:A. □

From the Lemmas 4.1-4.3, We obtain the following main result of this section.

Theorem 4.1. Let $\hat{x} \in \mathscr{X}$ be a $(\varphi, \psi, Z, X)-$extremal, then the LMR:A, BCQ:A and GNCQ:A are all equivalent.

5. Properties of (ψ, Z, X)-extremal.

In this section, we shall show that, if an $\hat{x} \in \mathscr{X}$ is a (ψ, Z, X)-extremal, all conditions for the (ψ, Z, X)-extremal are equivalent. We will use all notations and terminologies in sections 2 and 3 without further explanation.

Lemma 5.1. Let $\hat{x} \in \mathscr{X}$ be a (ψ, Z, X)-extremal. The element $\hat{x} \in \mathscr{X}$ satisfies the LMR:B if, and only if, the $\hat{x} \in \mathscr{X}$ satisfies the BCQ:B.

Proof. Let $\hat{x} \in \mathscr{X}$ be a (ψ, Z, X)-extremal which satisfies the LMR:A. Then there are a non-empty convex set $K \subset \mathscr{X}$, a Z-convex function $g : K \to \mathscr{Z}$ and a $z^* \in \mathscr{Z}^*$ and a $y^* \in \mathscr{Y}^*$ satisfying the relations (7)-(10). Suppose that $g^{-1}\left(\text{int.}\left[Z|\psi(\hat{x})\right]\right) \neq \varnothing$, then there is an $x \in K$ such that $g(x) \in \text{int.}\left[Z|\psi(\hat{x})\right]$. It follows from Lemma 2.3 that there are a $z \in \text{int.}Z$ and a $\gamma \leq 0$ such that $z = g(x) - \gamma\psi(\hat{x})$. Hence we obtain from (9) and (10) that

$$z^*(g(x)) = z^*(g(x)) - \gamma z^*(\psi(\hat{x})) = z^*(z) < 0,$$

which contradicts the inequality (8). Therefore $g^{-1}\left(\text{int.}\left[Z|\psi(\hat{x})\right]\right) = \varnothing$, that is, the element $\hat{x} \in \mathscr{X}$ satisfies the BCQ:B.

Conversely, let $\hat{x} \in \mathscr{X}$ be a (ψ, Z, X)-extremal which satisfies the BCQ:B. Then there exist a non-empty convex set $K \subset \mathscr{X}$ and a Z-convex function $g : K \to \mathscr{Z}$ such that $g^{-1}\left(\text{int.}\left[Z|\psi(\hat{x})\right]\right) = \varnothing$. Let T be a subset of $\mathscr{Z}$ defined as follows:

$$T = \bigcup_{x \in K} \left\{ z \in \mathscr{Z} \mid g(x) - z \in Z \right\}.$$

It is obvious that T is a non-empty convex subset of $\mathscr{Z}$. We know that $\left[Z|\psi(\hat{x})\right]$ is a convex cone in $\mathscr{Z}$ with non-empty interior. To show that $\text{int.}\left[Z|\psi(\hat{x})\right] \cap T = \varnothing$, suppose the contrary, that is, there is a $z \in \text{int.}\left[Z|\psi(\hat{x})\right] \cap T$. Hence there is an $x \in K$ such that $g(x) - z \in Z$. It follows from Lemmas 2.1 and 2.2 that

$$g(x) \in z + Z \subset \text{int.}\left[Z|\psi(\hat{x})\right] + \left[Z|\psi(\hat{x})\right] = \text{int.}\left[Z|\psi(\hat{x})\right],$$

that is, $x \in g^{-1}\left(\text{int.}\left[Z|\psi(\hat{x})\right]\right)$, which contradicts our hypothesis. Therefore we obtain that

(31) $$\text{int.}\left[Z|\psi(\hat{x})\right] \cap T = \varnothing.$$

Using the separation theorem of convex subsets of a topological linear space, it follows from (31) that there exist non-zero $z^* \in \mathscr{Z}^*$ and $\alpha \in R$ such that

(32) $$z^*(g(x)) - z^*(z) \geq \alpha \quad \text{for all} \quad x \in K \text{ and } z \in Z,$$

(33) $$z^*(z) \le \alpha \quad \text{for all} \quad z \in [Z|\psi(\hat{x})].$$

Since $[Z|\psi(\hat{x})]$ is a convex cone such that $0_{\mathscr{Z}} \in [Z|\psi(\hat{x})]$, we can easily verified from (33) that

(34) $$\alpha = 0.$$

Since $\psi(\hat{x}) \in [Z|\psi(\hat{x})]$ and $-\psi(\hat{x}) \in [Z|\psi(\hat{x})]$, the relation (33) and (34) implies that

(35) $$z^*(\psi(\hat{x})) = 0.$$

Since $0_{\mathscr{Z}} \in Z \subset [Z|\psi(\hat{x})]$ and $z^* \ne 0_{\mathscr{Z}}$, we obtain from (32)-(35) that the element $\hat{x} \in \mathscr{X}$ satisfies LMR:B. □

Lemma 5.2. Let $\hat{x} \in \mathscr{X}$ be a (ψ, Z, X)-extremal. The element $\hat{x} \in \mathscr{X}$ satisfies the BCQ:B if, and only if, the $\hat{x} \in \mathscr{X}$ satisfies the GNCQ:B.

Proof. Let $\hat{x} \in \mathscr{X}$ be a (ψ, Z, X)-extremal such that the $\hat{x} \in \mathscr{X}$ satisfies the BCQ:B. Then there exist a non-empty convex subset K of $\mathscr{X}$ and a Z-convex function $g: K \to \mathscr{Z}$ such that $g^{-1}(\text{int.}[Z|\psi(\hat{x})]) = \varnothing$, which implies that there is no element $x \in \mathscr{X}$ such that $x \in g^{-1}(\text{int.}[Z|\psi(\hat{x})])$. Therefore the $\hat{x} \in \mathscr{X}$ satisfies the GNCQ:B.

Conversely let $\hat{x} \in \mathscr{X}$ be a (ψ, Z, X)-extremal which satisfies the GNCQ:B. Then there exist a non-empty convex subset K of $\mathscr{X}$ and a Z-convex function $g: K \to \mathscr{Z}$ which have property stated in the definition of the GNCQ:B. To prove that the element $\hat{x}$ satisfies the BCQ:B, it suffices prove that

(36) $$g^{-1}(\text{int.}[Z|\psi(\hat{x})]) = \varnothing.$$

In order to prove (36), suppose the contrary. Then there is an $x \in K$ such that $g(x) \in \text{int.}[Z|\psi(\hat{x})]$, which, by virtue of Lemma 2.3, implies that there is a $\gamma \le 0$ such that $g(x) - \gamma\psi(\hat{x}) \in \text{int.}Z$. Hence there exists a neighborhood U of $0_{\mathscr{Z}}$ such that $g(x) - \gamma\psi(\hat{x}) + U \in \text{int.}Z$, that is,

(37) $$g(x) + U \in \text{int.}Z + \gamma\psi(\hat{x}).$$

Let $\eta = 1/(1-\gamma)$, then $\eta > 0$. It follows, from the hypothesis of GNCQ:B, that, for the triple (x, U, η), there are an $\bar{x} \in X$ and an $\bar{\varepsilon} \in (0, \eta)$ such that (11) hold. Since $1 + \bar{\varepsilon}\gamma > \eta > 0$, it follows, from (37) and Lemma 2.1, that

(38)
$$\begin{aligned}\psi(\bar{x}) &\in \bar{\varepsilon}\{g(x)+U\}+\bar{\varepsilon}Z+\psi(\hat{x}) \subset \bar{\varepsilon}\{\mathrm{int}.Z+\gamma\psi(\hat{x})\}+\bar{\varepsilon}Z+\psi(\hat{x}) \\ &\subset \bar{\varepsilon}\{\mathrm{int}.Z+Z\}+(1+\bar{\varepsilon}\gamma)\psi(\hat{x}) \subset \bar{\varepsilon}\,\mathrm{int}.Z+(1+\bar{\varepsilon}\gamma)Z \\ &= \mathrm{int}.Z,\end{aligned}$$

that is, $\psi(\bar{x}) \in \mathrm{int}.Z$. This contradicts (iii) in the Definition 3.2. Therefore the equality (36) holds, that is the element $\hat{x}$ satisfies the BCQ:B. □

From the Lemmas 5.1 and 5.2, We obtain the following main result of this section.

Theorem 5.1. Let $\hat{x} \in \mathscr{X}$ be a (ψ, Z, X)-extrmal, then the LMR:B, BCQ:B and GNCQ:B are all equivalent.

6. On Neustadt's Abstract Maximum Principle and Concluding Remarks.

In this section, we shall consider the abstract maximum principle introduced by Neustadt [4]. Notations and terminologies in sections 2-5 are used except the space $\mathscr{Y}$ and the cone Z. Let $\mathscr{Y}$ be an m dimensional Euclidean space and Z a closed convex cone in $\mathscr{Z}$ with non-empty interior.

Neustadt presented the following constraint qualification to obtain a Lagrange multiplier rule for the (φ, ψ, Z, X)-extremal (see Condition 3.1 in [4]).

Neustadt's Constraint Qualification A (NCQ:A). There exist a non-empty convex subset K of $\mathscr{X}$, an affine function $f: K \to \mathscr{Y}$ and a Z-convex function $g: K \to \mathscr{Z}$ with the following property: For every triple (Σ, U, η) — where Σ is a k-simplex contained in K such that $k \le m$ and $0_{\mathscr{Y}} \in f(\Sigma)$, U is a neighborhood of $0_{\mathscr{Z}}$ and $\eta > 0$ — there exist real numbers $\varepsilon_0 \ne 0$ and $\varepsilon_1 \in (0, \eta)$ together with a map $\Theta: \Sigma \to X$ (all possibly depending on Σ, U and η) such that

(39) $$\left|\frac{\varphi\circ\Theta(x)-\varphi(\hat{x})}{\varepsilon_0} - f(x)\right| < \eta \qquad \text{for every } x \in \Sigma,$$

(40) $$\frac{\psi\circ\Theta(x)-\psi(\hat{x})}{\varepsilon_1} \in \mathrm{co}.g(\Sigma)+Z+U \qquad \text{for every } x \in \Sigma,$$

(41) the map $\varphi\circ\Theta$ is continuous from Σ (i.e., is continuous as a function of the barycentric coordinates of a point in Σ) into $\mathscr{Y}$.

Here, vertical bars denote the ordinary Euclidean norm.

We can relax the condition in NCQ:A as follows.

Relaxed Neustadt's Constraint Qualification A (RNCQ:A). There are a non-empty convex set $K \subset \mathscr{X}$, an affine function $f: K \to \mathscr{Y}$ and a Z-convex function $g: K \to \mathscr{Z}$ with the following property: For every triple (Σ, U, η) — where Σ is an m-simplex contained in $g^{-1}\left(\text{int.}\left[Z \middle| \psi(\hat{x})\right]\right)$ such that $0_{\mathscr{Y}} \in \text{int.} f(\Sigma) \subset \text{int.} f\left(g^{-1}\left(\text{int.}\left[Z \middle| \psi(\hat{x})\right]\right)\right)$, U is a neighborhood of $0_{\mathscr{Z}}$ and $\eta > 0$ — there exist real numbers $\varepsilon_0 \neq 0$ and $\varepsilon_1 \in (0, \eta)$ together with a map $\Theta: \Sigma \to X$ (all possibly depending on Σ, U and η) which satisfy the conditions (39)-(41) in NCQ:A.

Neustadt also presented the following constraint qualification to obtain a Lagrange multiplier rule for the (ψ, Z, X)-extremal (see Condition 3.2 in [4]).

Neustadt's Constraint Qualification B (NCQ:B). There exist a non-empty convex subset K of $\mathscr{X}$ and a Z-convex function $g: K \to \mathscr{Z}$ with the following property: For every triple (x, U, η) — where $x \in K$, U is a neighborhood of $0_{\mathscr{Z}}$ and $\eta > 0$ — there exist an $x \in X$ and a real number $\varepsilon_1 \in (0, \eta)$ (both possibly depending on x, U and η) such that

$$\frac{\psi(x) - \psi(\hat{x})}{\varepsilon_1} \in g(x) + Z + U. \tag{42}$$

Let us present a property on the (φ, ψ, Z, X)-extremal and give the proof. The only if part of the proof is essentially the same as the proof of Lemma 4.1, together with one of Lemma 4.2 (see section 4 in [4]).

Lemma 6.1. Let $\hat{x} \in \mathscr{X}$ be a (φ, ψ, Z, X)-extremal. The element $\hat{x} \in \mathscr{X}$ satisfies the GNCQ:A if, and only if, the $\hat{x} \in \mathscr{X}$ satisfies the RNCQ:A.

Proof. Let $\hat{x} \in \mathscr{X}$ be a (φ, ψ, Z, X)-extremal which satisfies the GNCQ:A. By virtue of Lemma 4.2, the element $\hat{x}$ satisfies the BCQ:A. Hence there exist a non-empty convex subset K of $\mathscr{X}$, an affine function $f: K \to \mathscr{Y}$, a Z-convex function $g: K \to \mathscr{Z}$ and a convex set $V \subset \mathscr{Y}$ such that

$$0_{\mathscr{Y}} \in V, \quad \text{int.} V \neq \varnothing \quad \text{and} \quad \text{int.} V \cap f\left(g^{-1}\left(\text{int.}\left[Z \middle| \psi(\hat{x})\right]\right)\right) = \varnothing,$$

which imply that there is no m-simplex Σ contained in $g^{-1}\left(\text{int.}\left[Z \middle| \psi(\hat{x})\right]\right)$ such that $0_{\mathscr{Y}} \in \text{int.} f(\Sigma) \subset \text{int.} f\left(g^{-1}\left(\text{int.}\left[Z \middle| \psi(\hat{x})\right]\right)\right)$. Therefore the functions f, g and the set K

satisfy the property stated in RNCQ:A, that is, the element $\hat{x}$ satisfies the RNCQ:A.

Conversely, let $\hat{x} \in \mathscr{X}$ be a (φ, ψ, Z, X)-extremal which satisfies the RNCQ:A. Then there exists a non-empty convex subset K of $\mathscr{X}$, an affine function $f : K \to \mathscr{Y}$ and a Z-convex function $g : K \to \mathscr{Z}$ with the property in RNCQ:A. In case $\text{int}.f\left(g^{-1}\left(\text{int}.\left[Z|\psi(\hat{x})\right]\right)\right) = \varnothing$, there exist s a convex subset V of $\mathscr{Y}$ such that $0_{\mathscr{Y}} \in V$, $\text{int}.V \neq \varnothing$ and $\text{int}.V \cap f\left(g^{-1}\left(\text{int}.\left[Z|\psi(\hat{x})\right]\right)\right) = \varnothing$, because $f\left(g^{-1}\left(\text{int}.\left[Z|\psi(\hat{x})\right]\right)\right)$ is a convex set in $\mathscr{Y} = R^m$. Now let us consider in case where $\text{int}.f\left(g^{-1}\left(\text{int}.\left[Z|\psi(\hat{x})\right]\right)\right) \neq \varnothing$. Let S, U and η be a convex subset of $g^{-1}\left(\text{int}.\left[Z|\psi(\hat{x})\right]\right)$ such that

(43) $$0_{\mathscr{Y}} \in \text{cr}.f(S) \subset \text{int}.f\left(g^{-1}\left(\text{int}.\left[Z|\psi(\hat{x})\right]\right)\right),$$

a neighborhood of $0_{\mathscr{Z}}$ and positive number, respectively. Since $\mathscr{Y} = R^m$ and the function $f : K \to \mathscr{Y}$ is affine, $\text{cr}.f(S) = \text{int}.f(S)$ which, by virtue of (43), imply that there is a $m-$simplex $\overline{\Sigma} \subset f(S)$ such that $0_Y \in \text{int}.\overline{\Sigma} \subset \text{cr}.f(S)$. Hence there is an $\overline{\eta} \in (0, \eta)$ such that

(44) $$y \in \overline{\Sigma} \qquad \text{whenever } y \in \mathscr{Y} \text{ and } |y| < \overline{\eta}.$$

Let $f(\tilde{x}_1), \cdots, f(\tilde{x}_{m+1})$ be the vertices of $\overline{\Sigma}$, where $\tilde{x}_i \in S$ for each $i = 1, \cdots, m+1$. Since the function $f : K \to \mathscr{Y}$ is affine and the points $f(\tilde{x}_1), \cdots, f(\tilde{x}_{m+1})$ are in general position, the points $\tilde{x}_1, \cdots, \tilde{x}_{m+1}$ are in general position, and hence the convex hull of the set $\{\tilde{x}_1, \cdots, \tilde{x}_{m+1}\}$ are an $m-$simplex in S, which is denoted by Σ. It is easily verified that $\Sigma \subset g^{-1}\left(\text{int}.\left[Z|\psi(\hat{x})\right]\right)$ and $0_{\mathscr{Y}} \in \text{int}.f(\Sigma) \subset \text{int}.f\left(g^{-1}\left(\text{int}.\left[Z|\psi(\hat{x})\right]\right)\right)$.

By means of the property of RNCQ:A, for the triple $(\Sigma, U, \overline{\eta})$, there exist real numbers $\varepsilon_0 \neq 0$ and $\overline{\varepsilon} \in (0, \overline{\eta})$ together with a map $\Theta : \Sigma \to X$ such that (39) and (40) hold and such that the map $\varphi \circ \Theta$ is continuous as a function of the barycentric coordinates of a point in Σ. For each $\xi = \sum_{i=1}^{m+1} \xi_i f(\tilde{x}_i) \in \overline{\Sigma}$, let $h(\xi) = \sum_{i=1}^{m+1} \xi_i \tilde{x}_i$. Then $h(\xi) \in \Sigma$ for all $\xi \in \overline{\Sigma}$ and $f(h(\xi)) = \xi$, for all $\xi \in \overline{\Sigma}$, because the function f is affine. Now let us define a function $\chi : \overline{\Sigma} \to \mathscr{Y} \left(= R^m\right)$ as follows:

$$\chi(\xi) = \frac{\varphi \circ \Theta(h(\xi))}{\varepsilon_0} - \xi \qquad \text{for all} \quad \xi \in \overline{\Sigma}.$$

Since $\varphi(\hat{x}) = 0_{\mathscr{Y}}$, it follows from (39) that

$$|\chi(\xi)| = \left| \frac{\varphi \circ \Theta(h(\xi))}{\varepsilon_0} - \xi \right| = \left| \frac{\varphi \circ \Theta(h(\xi)) - \varphi(\hat{x})}{\varepsilon_0} - \xi \right| < \overline{\eta} \qquad \text{for all } \xi \in \overline{\Sigma}.$$

Recall (41) and (44), we conclude that the χ is a continuous function from the compact convex non-empty subset $\bar{\Sigma}$ into itself. Hence, by Brouwer's fixed point theorem, that there exists a $\bar{\xi} \in \bar{\Sigma}$ such that $\chi(\bar{\xi}) = \bar{\xi}$, that is, $\varphi \circ \Theta(h(\bar{\xi})) = 0_{\mathscr{Y}}$(see e.g., Theorem 2.1.11 in [8]). Let $\bar{x} = \Theta(h(\bar{\xi}))$, then $\bar{x} \in X$ and $\varphi(\bar{x}) = 0_{\mathscr{Y}}$. Since $\Sigma \subset S$, it follows from (40) that the relation (6) holds. Since $0 < \bar{\varepsilon} < \bar{\eta} < \eta$, we now conclude that the $\hat{x}$ satisfies the GNCQ:A. □

The following theorem is obtained from Theorem 4.1 and Lemma 6.1.

Theorem 6.1. Let $\hat{x} \in \mathscr{X}$ be a (φ, ψ, Z, X)-extremal. If $\mathscr{Y} = R^m$, then the LMR:A, BCQ:A, GNCQ:A and RNCQ:A are all equivalent.

Let us consider the NCQ:A and NCQ:B. It is obvious that, if an element $\hat{x} \in \mathscr{X}$ is a (φ, ψ, Z, X)-extremal or (ψ, Z, X)-extremal which satisfies the NCQ:A or NCQ:B, then the element $\hat{x}$ satisfies the GNCQ:A or GNCQ:B, respectively. Therefore, we can obtain the following theorem which is virtually equivalent to the Theorem 3.1 in [4].

Theorem 4.2. If $\hat{x} \in \mathscr{X}$ is a (φ, ψ, Z, X)-extremal satisfying the NCQ:A, then the element $\hat{x}$ satisfies the LMR:A. And if $\hat{x} \in \mathscr{X}$ is a (ψ, Z, X)-extremal satisfying the NCQ:B, then the element $\hat{x}$ satisfies the LMR:B.

Finally we shall state concluding remarks. In extremal problem, the weakest constraint qualification to obtain the necessary condition in the form of Lagrange multiplier rule is the basic constraint qualification stated in section 3. Further we can give necessary and sufficient conditions for the solutions to satisfy Lagrange multiplier rules by means of a few relaxation of the Neustadt's constraint qualifications.

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