COLLOQUIUM MATHEMATICUM

Online First version

SINGULARITIES OF PEDAL CURVES OF FRONTAL CURVES IN THE EUCLIDEAN PLANE

BY

KEISUKE TERAMOTO

Abstract. We study pedal curves for planar curves, possibly with singularities, called frontal curves. In particular, we analyze the types of singularities on pedal curves for a given frontal in terms of the geometrical properties.

1. Introduction. The *pedal curve* (with respect to the origin) of a regular plane curve is the locus of orthogonal projection images of the origin to the tangent lines to the curve [2]. If the curve does not pass through the origin, the pedal exhibits singularities at the points which correspond to the *inflection points* of the curve [2]. Thus the geometric properties of the pedal curve are related to inflection points of the original plane curve.

Recent studies on curves with singularities have been conducted from a differential-geometric perspective. Curves belonging to a class called *frontals* have been studied extensively [1, 3–7, 9, 11, 12, 14–16]. A *frontal curve* is a plane curve possibly with singular points which admits a well-defined smooth unit normal vector field, even at singular points. We define the *tangent line* to a frontal curve at a singular point as the line perpendicular to the unit normal vector at the point. Hence the pedal of a frontal curve can be defined as in the case of a regular plane curve (cf. [1, 8, 9, 16]). In [9], Li and Pei studied pedal curves of frontal curves in the Euclidean plane and showed that there is a connection between the singular points of a pedal curve and inflection points of the corresponding frontal curve. Moreover, in [16], Tuncer, Ceyhan, Gök and Ekmekci provide a characterization of singularities on the pedal and contrapedal curves of fronts. However, problems related to characterizations of certain types of singularities on pedal curves for frontals still remain.

In this paper, we provide a characterization of singularities of pedal curves for frontal curves with certain singularities in terms of geometric properties. To investigate pedal curves, we distinguish the following cases: (1) The given

2020 *Mathematics Subject Classification*: Primary 53A04; Secondary 57R45, 58K05.
Key words and phrases: frontal curve, pedal curve, cusp.
Received 30 June 2025; revised 2 December 2025.
Published online 4 March 2026.

DOI: 10.4064/cm9663-12-2025

frontal curve does not pass through the origin. (2) The frontal curve passes through the origin. In the first case, we highlight the following: if the original curve is a front (see Section 2), the pedal becomes regular at the point corresponding to the singular point of the front. Furthermore, we characterize the pedal curves as a 5/2-cusp or a 5/3-cusp by the geometrical properties and singularity types of the initial frontal curves (Theorems 3.4 and 3.6, and Proposition 3.7). In the second case, we examine pedal curves of fronts. In particular, we give conditions for the pedal curves to be a 5/4-cusp based on the type of singularity of the initial fronts (Theorem 3.10).

2. Frontal curves. We consider the fundamental properties of frontal curves (see [3, 10, 17] for the details).

Let $\gamma\colon (\mathbb{R},0) \to \mathbb{R}^2$ be a C^∞ map germ. Then γ is said to be a *frontal curve* (or a *frontal*) if there exists a C^∞ map $\boldsymbol{n}\colon (\mathbb{R},0) \to \mathbb{S}^1$ such that $\gamma'(t)\cdot\boldsymbol{n}(t) = 0$ for any $t \in (\mathbb{R},0)$, where $\mathbb{S}^1$ is the unit circle, $\gamma' = d\gamma/dt$, and the dot '$\cdot$' is the canonical inner product on $\mathbb{R}^2$. The map $\boldsymbol{n}$ is called a *unit normal vector* to γ. Moreover, γ is a *front* if γ is a frontal and the pair $(\gamma,\boldsymbol{n})\colon (\mathbb{R},0) \to \mathbb{R}^2 \times \mathbb{S}^1$ is an immersion.

DEFINITION 2.1. Let $\gamma\colon (\mathbb{R},0) \to \mathbb{R}^2$ be frontal with a unit normal vector $\boldsymbol{n}$. Then $t_0 \in (\mathbb{R},0)$ is called a *singular point* of γ if $\gamma'(t_0) = 0$. Moreover, γ is considered of *A-type* if $\gamma'(0) = 0$ and $\gamma''(0) \neq 0$.

DEFINITION 2.2. Let $\gamma_i\colon (\mathbb{R},0) \to (\mathbb{R}^2,\gamma_i(0))$ $(i = 1,2)$ be C^∞ curve germs. Then γ_1 and γ_2 are *C^r-equivalent* for $r \in \mathbb{N}\cup\{\infty\}$ if there exist C^r-diffeomorphism germs $\varphi\colon (\mathbb{R},0) \to (\mathbb{R},0)$ and $\Phi\colon (\mathbb{R}^2,\gamma_1(0)) \to (\mathbb{R}^2,\gamma_2(0))$ such that $\Phi\circ\gamma_1 = \gamma_2\circ\varphi$. C^∞-equivalence is referred to as *$\mathcal{A}$-equivalence.*

DEFINITION 2.3. Let $\gamma\colon (\mathbb{R},0) \to \mathbb{R}^2$ be a C^∞ map germ. Then

(1) γ is a *3/2-cusp* or an *ordinary cusp* at 0 if γ is $\mathcal{A}$-equivalent to the germ $t \mapsto (t^2,t^3)$ at the origin;
(2) γ is a *4/3-cusp* at 0 if γ is $\mathcal{A}$-equivalent to the germ $t \mapsto (t^3,t^4)$ at the origin;
(3) γ is a *5/4-cusp* at 0 if γ is $\mathcal{A}$-equivalent to the germ $t \mapsto (t^4,t^5)$ at the origin;
(4) γ is a *(5/4; ±7)-cusp* at 0 if γ is $\mathcal{A}$-equivalent to the germ $t \mapsto (t^4,t^5 \pm t^7)$ at the origin;
(5) γ is a *5/2-cusp* or a *rhamphoid cusp* at 0 if γ is $\mathcal{A}$-equivalent to the germ $t \mapsto (t^2,t^5)$ at the origin;
(6) γ is a *5/3-cusp* at 0 if γ is $\mathcal{A}$-equivalent to the germ $t \mapsto (t^3,t^5)$ at the origin.

These singularities appear on frontal curves. In particular, if γ is a 3/2-cusp, 4/3-cusp, or 5/4-cusp at 0, then γ is a front at 0. Moreover, if γ is a

3/2-cusp or 5/2-cusp, then γ is an A-type frontal curve. For n/m-cusps with $(m,n) = (2,3),(3,4),(4,5),(2,5),(3,5)$, the following criteria are useful.

FACT 2.4 ([12, 13, 17]). *Let $\gamma\colon (\mathbb{R},0) \to \mathbb{R}^2$ be a frontal curve. Assume $\gamma'(0) = 0$. Then*

(1) *γ is a 3/2-cusp at 0 if and only if $\det(\gamma''(0), \gamma'''(0)) \neq 0$;*
(2) *γ is a 4/3-cusp at 0 if and only if $\gamma''(0) = 0$ and $\det(\gamma'''(0), \gamma^{(4)}(0)) \neq 0$;*
(3) *γ is C^1-equivalent to a 5/4-cusp at 0 if and only if $\gamma''(0) = \gamma'''(0) = 0$ and $\det(\gamma^{(4)}(0), \gamma^{(5)}(0)) \neq 0$; moreover, γ is a 5/4-cusp at 0 if and only if γ is C^1-equivalent to a 5/4-cusp at 0 and*

$$P = -77Y^2 + 105XW + 60XZ = 0, \tag{2.1}$$

where

$$X = \det(\gamma^{(4)}(0), \gamma^{(5)}(0)), \qquad Y = \det(\gamma^{(4)}, \gamma^{(6)}),$$
$$Z = \det(\gamma^{(4)}(0), \gamma^{(7)}(0)), \qquad W = \det(\gamma^{(5)}(0), \gamma^{(6)}(0));$$

(4) *γ is $\mathcal{A}$-equivalent to a $(5/4;+7)$-cusp (resp. $(5/4;-7)$-cusp) at 0 if and only if $\gamma''(0) = \gamma'''(0) = 0$, $X \neq 0$ and $P > 0$ (resp. $P < 0$), where X and P are given in (3);*
(5) *γ is a 5/2-cusp at 0 if and only if there exists a constant $C \in \mathbb{R}$ such that $\gamma'''(0) = C\gamma''(0)$ and*

$$\det(\gamma''(0), 3\gamma^{(5)}(0) - 10C\gamma^{(4)}(0)) \neq 0;$$

(6) *γ is a 5/3-cusp at 0 if and only if $\gamma''(0) = 0$, $\det(\gamma'''(0), \gamma^{(4)}(0)) = 0$ and $\det(\gamma'''(0), \gamma^{(5)}(0)) \neq 0$.*

Criteria for other singularities of frontal curves can be found in [6, 12].

Let $\gamma\colon (\mathbb{R},0) \to \mathbb{R}^2$ be a frontal curve. We assume that the component functions of γ are not flat at 0, that is, there exists a non-negative integer k such that $\gamma'(0) = \cdots = \gamma^{(k-1)}(0) = 0$ and $\gamma^{(k)}(0) \neq 0$. Then there exist functions $l, \theta\colon (\mathbb{R},0) \to \mathbb{R}$ such that

$$\gamma'(t) = l(t)(\cos\theta(t), \sin\theta(t)) = l(t)\boldsymbol{e}(t), \qquad \boldsymbol{n}(t) = (-\sin\theta(t), \cos\theta(t)),$$

where $\boldsymbol{n}$ is a unit normal vector to γ (see [3, 10]). We consider $\boldsymbol{e}$ as a *unit tangent vector* to γ. Hence the set of singular points $S(\gamma)$ on γ is as follows: $S(\gamma) = l^{-1}(0)$. Moreover, the line $\{\gamma(t_0) + u\boldsymbol{e}(t_0) \mid u \in \mathbb{R}\}$ is referred to as the *tangent line* to γ at $t_0 \in (\mathbb{R},0)$.

Using the above formulations, we obtain the following Frenet-type formula (cf. [3, 10]):

$$\boldsymbol{e}'(t) = \theta'(t)\boldsymbol{n}(t), \qquad \boldsymbol{n}'(t) = -\theta'(t)\boldsymbol{e}(t). \tag{2.2}$$

We assume $0 \in l^{-1}(0) = S(\gamma)$. Then γ is a front at 0 if and only if $\theta'(0) \neq 0$. By Fact 2.4, we obtain the following characterization of singularities using l and θ:

PROPOSITION 2.5 (cf. [10]). *Let* $\gamma\colon (\mathbb{R},0)\to\mathbb{R}^2$ *be a frontal. Then*

(1) γ *is a* $3/2$*-cusp at* 0 *if and only if* $l(0)=0$ *and* $l'(0)\theta'(0)\neq 0$;
(2) γ *is a* $4/3$*-cusp at* 0 *if and only if* $l(0)=l'(0)=0$ *and* $l''(0)\theta'(0)\neq 0$;
(3) γ *is a* $5/2$*-cusp at* 0 *if and only if* $l(0)=0$, $\theta'(0)=0$, $l'(0)\neq 0$ *and* $l'(0)\theta'''(0)-l''(0)\theta''(0)\neq 0$;
(4) γ *is a* $5/3$*-cusp at* 0 *if and only if* $l(0)=l'(0)=\theta'(0)=0$ *and* $l''(0)\theta''(0)\neq 0$.

Proof. By (2.2), we observe that

$$\begin{aligned}
\gamma'(t)&=l(t)\boldsymbol{e}(t),\\
\gamma''(t)&=l'(t)\boldsymbol{e}(t)+l(t)\theta'(t)\boldsymbol{n}(t),\\
\gamma'''(t)&=(l''(t)-l(t)\theta'(t)^2)\boldsymbol{e}(t)+(2l'(t)\theta'(t)+l(t)\theta''(t))\boldsymbol{n}(t),\\
\gamma^{(4)}(t)&=(l'''(t)-3l'(t)\theta'(t)^2-3l(t)\theta'(t)\theta''(t))\boldsymbol{e}(t)\\
&\qquad+(3l''(t)\theta'(t)+3l'(t)\theta''(t)+l(t)\theta'''(t)-l(t)\theta'(t)^3)\boldsymbol{n}(t).
\end{aligned}$$

Thus we obtain (1) and (2) from Fact 2.4(1)–(2). Now, we prove (3). We assume that $\theta'(0)=0$, that is, γ is not a front at 0. Then $\gamma''(0)=l'(0)\boldsymbol{e}(0)$ and $\gamma'''(0)=l''(0)\boldsymbol{e}(0)$. Thus γ is an A-type frontal curve if and only if $l'(0)\neq 0$, and hence we have

$$\gamma'''(0)=\frac{l''(0)}{l'(0)}\gamma''(0).$$

By the above calculation, we obtain $\gamma^{(4)}(0)=c_1\boldsymbol{e}(0)+3l'(0)\theta''(0)\boldsymbol{n}(0)$ and

$$\gamma^{(5)}(0)=c_2\boldsymbol{e}(0)+(6l''(0)\theta''(0)+4l'(0)\theta'''(0))\boldsymbol{n}(0),$$

where c_1,c_2 are some constants. Therefore we have

$$\det\left(\gamma''(0),3\gamma^{(5)}(0)-10\frac{l''(0)}{l'(0)}\gamma^{(4)}(0)\right)=12l'(0)(l'(0)\theta'''(0)-l''(0)\theta''(0)).$$

We obtain (3) from Fact 2.4(5), and (4) follows from the above calculations and Fact 2.4(6). ■

2.1. Invariants defined at cusps. Let $\gamma\colon(\mathbb{R},0)\to\mathbb{R}^2$ be an A-type frontal curve, that is, $\gamma'(0)=0$ and $\gamma''(0)\neq 0$. Then an invariant called the *cuspidal curvature* [14, 17] is defined by

$$\omega=\frac{\det(\gamma''(0),\gamma'''(0))}{|\gamma''(0)|^{5/2}}. \tag{2.3}$$

The cuspidal curvature measures the width of γ at 0. By Fact 2.4(1), γ is a 3/2-cusp at 0 if and only if $\omega\neq 0$. When $\omega=0$, we define the following invariants at 0 (see [7]):

$$\omega_r=\frac{\det(\gamma''(0),3\gamma^{(5)}(0)-10C\gamma^{(4)}(0))}{|\gamma''(0)|^{7/2}},\qquad b=\frac{\det(\gamma''(0),\gamma^{(4)}(0))}{|\gamma''(0)|^3}, \tag{2.4}$$

where C is a constant satisfying $\gamma'''(0) = C\gamma''(0)$. We call ω_r and b the *secondary cuspidal curvature* and the *bias* of the cusp, respectively. By Fact 2.4(5), γ is a 5/2-cusp at 0 if and only if $\omega_r \neq 0$. Similar to the cuspidal curvature ω, the secondary cuspidal curvature ω_r represents the width of a 5/2-cusp on γ. Moreover, the geometrical meaning of the bias b of γ is as follows [7, Proposition 2.2]: Let $\gamma\colon (\mathbb{R}, 0) \to \mathbb{R}^2$ be an A-type frontal curve with $\omega = 0$. We consider two images of γ:

$$\gamma_+ = \gamma([0, \varepsilon)), \qquad \gamma_- = \gamma((-\varepsilon, 0])$$

for sufficiently small $\varepsilon > 0$. The images γ_+ and γ_- lie on the same side of the tangent line to γ at $\gamma(0)$ if $b \neq 0$. Moreover, if $\omega_r \neq 0$ and $b = 0$, then γ_+ and γ_- lie on opposite sides of that tangent line (see Figure 1). The next section demonstrates that the bias plays an important role in the analysis of singularities on the pedal curves of A-type frontal curves.

Fig. 1. Left: a 5/2-cusp with vanishing bias. Right: a 5/2-cusp with non-vanishing bias. The dashed lines are tangent lines.

For other invariants defined at more degenerate singular points of a frontal curve, see [11, 12].

3. Pedal curves of frontal curves. Let $\gamma\colon (\mathbb{R}, 0) \to \mathbb{R}^2$ be a frontal and let $\boldsymbol{n}$ be its unit normal vector. Then we define the map $\mathrm{Pe}_\gamma\colon (\mathbb{R}, 0) \to \mathbb{R}^2$ by

$$\mathrm{Pe}_\gamma(t) = A(t)\boldsymbol{n}(t), \quad \text{where} \quad A(t) = \gamma(t) \cdot \boldsymbol{n}(t). \tag{3.1}$$

At a point $t_0 \in (\mathbb{R}, 0)$, $\mathrm{Pe}_\gamma(t_0)$ corresponds to the foot of a perpendicular from the origin to the tangent line to γ at $\gamma(t_0)$, even if t_0 is a singular point of γ (see Figure 2). Therefore, similar to the case of regular curves [2], we refer to the map Pe_γ given by (3.1) as the *pedal curve* of γ (with respect to the origin). Moreover, we refer to $\mathrm{Pe}_\gamma(t_0)$ as the *pedal point* of γ relative to $\gamma(t_0)$ with respect to the origin.

We investigate the singularities of the pedal curve, as described in the next subsections.

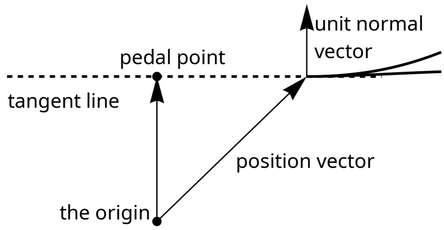

Fig. 2. Pedal point of a frontal curve with respect to the origin. The dashed line represents the tangent line to the curve at a singular point.

3.1. Pedal curves of frontal curves not passing through the origin. Here, we consider the case where a frontal $\gamma\colon (\mathbb{R},0) \to \mathbb{R}^2$ satisfies $\gamma(0) \neq 0$.

LEMMA 3.1. *Under these conditions,* 0 *is a singular point of* Pe_γ *if and only if* γ *is a frontal but not a front at* 0.

Proof. By direct calculations, we obtain

$$\begin{aligned}\mathrm{Pe}'_\gamma(t) &= -\theta'(t)(\gamma(t)\cdot \boldsymbol{e}(t))\boldsymbol{n}(t) - \theta'(t)(\gamma(t)\cdot \boldsymbol{n}(t))\boldsymbol{e}(t) \\ &= -\theta'(t)(A(t)\boldsymbol{e}(t) + B(t)\boldsymbol{n}(t)),\end{aligned} \tag{3.2}$$

where we set $B(t) = \gamma(t)\cdot \boldsymbol{e}(t)$. Since $\{\boldsymbol{e}(0), \boldsymbol{n}(0)\}$ provides an orthonormal basis for $\mathbb{R}^2$ and $\gamma(0) \neq 0$, $\gamma(0)$ can be represented as a linear combination of $\boldsymbol{e}(0)$ and $\boldsymbol{n}(0)$. Thus $(A(0), B(0)) \neq (0,0)$, and hence $\mathrm{Pe}'_\gamma(0) = 0$ if and only if $\theta'(0) = 0$. This shows the assertion. ∎

By this lemma, if γ is a front at 0, then Pe_γ is regular at 0. By (3.2), one can take the unit normal vector $\boldsymbol{n}_{\mathrm{Pe}_\gamma}$ to be

$$\boldsymbol{n}_{\mathrm{Pe}_\gamma}(t) = \frac{1}{\sqrt{A(t)^2 + B(t)^2}}(B(t)\boldsymbol{e}(t) - A(t)\boldsymbol{n}(t)),$$

where $A(t) = \gamma(t)\cdot \boldsymbol{n}(t)$ and $B(t) = \gamma(t)\cdot\boldsymbol{e}(t)$. Thus Pe_γ is a frontal curve (cf. [8, Proposition 3.2]). Moreover, we obtain the following:

PROPOSITION 3.2. *Let* $\gamma\colon (\mathbb{R},0) \to \mathbb{R}^2$ *be a frontal curve,* 0 *a singular point of* γ*, and* $\boldsymbol{n}$ *a unit normal vector to* γ*. Assume that* $\gamma(0) \neq 0$*. If* 0 *is a singular point on the pedal curve* Pe_γ*, then* Pe_γ *is a frontal but not a front at* 0.

Proof. We consider the derivative $\boldsymbol{n}'_{\mathrm{Pe}_\gamma}$ of $\boldsymbol{n}_{\mathrm{Pe}_\gamma}$. By direct calculations,

$$\begin{aligned}\boldsymbol{n}'_{\mathrm{Pe}_\gamma}(t) = &-\frac{A'(t)A(t) + B'(t)B(t)}{(A(t)^2 + B(t)^2)^{3/2}}(B(t)\boldsymbol{e}(t) - A(t)\boldsymbol{n}(t)) \\ &+ \frac{(B'(t) + \theta'(t)A(t))\boldsymbol{e}(t) - (A'(t) - \theta'(t)B(t))\boldsymbol{n}(t)}{\sqrt{A(t)^2 + B(t)^2}}.\end{aligned}$$

Since $A'(t) = -\theta'(t)B(t)$, $B'(t) = l(t) + \theta'(t)A(t)$ and $l(0) = \theta'(0) = 0$, we see that $\boldsymbol{n}'_{\mathrm{Pe}_\gamma}(0) = 0$. This implies that $(\mathrm{Pe}_\gamma, \boldsymbol{n}_{\mathrm{Pe}_\gamma})\colon (\mathbb{R}, 0) \to \mathbb{R}^2 \times \mathbb{S}^1$ is not an immersion at 0. This completes the proof. ∎

We consider the case where γ is of A-type. In other words, $\gamma'(0) = 0$ and $\gamma''(0) \neq 0$. In this case, we have the following properties.

PROPOSITION 3.3. *Let $\gamma\colon (\mathbb{R}, 0) \to \mathbb{R}^2$ be an A-type frontal curve which is not a $3/2$-cusp, and $\boldsymbol{n}$ a unit normal vector to γ. Assume that $\gamma(0) \neq 0$. Then the pedal curve Pe_γ is also of A-type if and only if the bias b of γ does not vanish.*

Proof. By (3.2),

$$\begin{aligned}(3.3)\quad \mathrm{Pe}''_\gamma(t) &= -\theta''(t)(A(t)\boldsymbol{e}(t) + B(t)\boldsymbol{n}(t)) \\ &\quad - \theta'(t)\big((A'(t) - \theta'(t)B(t))\boldsymbol{e}(t) + (B'(t) + \theta'(t)A(t))\boldsymbol{n}(t)\big).\end{aligned}$$

Since $\theta'(0) = 0$, we get

$$\mathrm{Pe}''_\gamma(0) = -\theta''(0)(A(0)\boldsymbol{e}(0) + B(0)\boldsymbol{n}(0)).$$

Therefore Pe_γ is of A-type if and only if $\theta''(0) \neq 0$. On the other hand,

$$b = \frac{3\theta''(0)}{|l'(0)|}$$

(cf. (2.4)). Hence we get the conclusion. ∎

When γ is an A-type frontal curve with nonvanishing bias b, we obtain the following:

THEOREM 3.4. *Let $\gamma\colon (\mathbb{R}, 0) \to \mathbb{R}^2$ be an A-type frontal curve which is not a $3/2$-cusp. Let $\boldsymbol{n}$ be a unit normal vector to γ. Assume that $\gamma(0) \neq 0$ and the bias b of γ does not vanish. Then the pedal curve Pe_γ is a $5/2$-cusp at 0 if and only if γ is also a $5/2$-cusp and $\gamma(0) \cdot \boldsymbol{n}(0) \neq 0$. Moreover, the bias b_{Pe_γ} does not vanish.*

Proof. By Proposition 3.3, Pe_γ is an A-type frontal curve. By (3.3), we observe that

$$\begin{aligned}(3.4)\quad \mathrm{Pe}'''_\gamma(t) &= -\theta'''(t)(A(t)\boldsymbol{e}(t) + B(t)\boldsymbol{n}(t)) \\ &\quad - 2\theta'(t)\big((A'(t) - \theta'(t)B(t))\boldsymbol{e}(t) + (B'(t) + \theta'(t)A(t))\boldsymbol{n}(t)\big) \\ &\quad - \theta'(t)\big((A'(t) - \theta'(t)B(t) - 2\theta'(t)B'(t) - \theta'(t)^2A(t))\boldsymbol{e}(t) \\ &\qquad + (B'(t) + \theta'(t)A(t) + 2\theta'(t)A'(t) - \theta'(t)^2B(t))\boldsymbol{n}(t)\big).\end{aligned}$$

Since $A'(0) = B'(0) = 0$, we have $\mathrm{Pe}'''_\gamma(0) = -\theta'''(0)(A(0)\boldsymbol{e}(0) + B(0)\boldsymbol{n}(0))$. Hence

$$\mathrm{Pe}'''_\gamma(0) = \frac{\theta'''(0)}{\theta''(0)}\,\mathrm{Pe}''_\gamma(0).$$

Furthermore, by straightforward calculations, we obtain

$$
\begin{aligned}
\mathrm{Pe}_\gamma^{(4)}(t) = {} & -\theta^{(4)}(t)(A(t)\boldsymbol{e}(t) + B(t)\boldsymbol{n}(t)) \\
& - 3\theta'''(t)\big((A'(t)-\theta'(t)B(t))\boldsymbol{e}(t)+(B'(t)+\theta'(t)A(t))\boldsymbol{n}(t)\big) \\
& - 3\theta''(t)\big((A''(t)-\theta''(t)B(t)-2\theta'(t)B'(t)-\theta'(t)^2A(t))\boldsymbol{e}(t) \\
& \quad + (B''(t) + \theta''(t)A(t) + 2\theta'(t)A'(t) - \theta'(t)^2B(t))\boldsymbol{n}(t)\big) \\
& - \theta'(t)\big((A'''(t) - \theta'''(t)B(t) - 3\theta''(t)B'(t) - 3\theta'(t)B''(t) \\
& \quad -3\theta'(t)\theta''(t)A(t) - 3\theta'(t)^2A'(t) + \theta'(t)^3B(t))\boldsymbol{e}(t) \\
& \quad +(B'''(t) + \theta'''(t)A(t) + 3\theta''(t)A'(t) + 3\theta'(t)A''(t) \\
& \quad -3\theta'(t)\theta''(t)B(t) - 3\theta'(t)^2B'(t) - \theta'(t)^3A(t))\boldsymbol{n}(t)\big).
\end{aligned} \tag{3.5}
$$

Since

$$
\begin{aligned}
A''(t) &= -\theta''(t)B(t) - \theta'(t)(l(t) + \theta'(t)A(t)), \\
B''(t) &= l'(t) + \theta''(t)A(t) - \theta'(t)^2B(t),
\end{aligned}
$$

we obtain

$$
A''(0) = -\theta''(0)B(0), \qquad B''(0) = l'(0) + \theta''(0)A(0).
$$

It follows that

$$
\begin{aligned}
\mathrm{Pe}_\gamma^{(4)}(0) = {} & (-\theta^{(4)}(0)A(0) + 6\theta''(0)B(0))\boldsymbol{e}(0) \\
& - (\theta^{(4)}(0)B(0) + 6\theta''(0)A(0) + 3l'(0)\theta''(0))\boldsymbol{n}(0).
\end{aligned}
$$

By (3.5), we obtain

$$
\begin{aligned}
\mathrm{Pe}_\gamma^{(5)}(0) = {} & -\theta^{(5)}(0)(A(0)\boldsymbol{e}(0) + B(0)\boldsymbol{n}(0)) \\
& - 6\theta'''(0)\big((A''(0) - \theta''(0)B(0))\boldsymbol{e}(0) + (B''(0) + \theta''(0)A(0))\boldsymbol{n}(0)\big) \\
& - 4\theta''(0)\big((A'''(0) - \theta'''(0)B(0))\boldsymbol{e}(0) + (B'''(0) + \theta'''(0)A(0))\boldsymbol{n}(0)\big).
\end{aligned}
$$

Since $A'''(0) = -\theta'''(0)B(0)$ and $B'''(0) = l''(0)+\theta'''(0)A(0)$, $\mathrm{Pe}_\gamma^{(5)}(0)$ can be written as

$$
\begin{aligned}
\mathrm{Pe}_\gamma^{(5)}(0) = {} & (-\theta^{(5)}(0)A(0) + 20\theta''(0)\theta'''(0)B(0))\boldsymbol{e}(0) \\
& - (\theta^{(5)}(0)B(0) + 20\theta''(0)\theta'''(0)A(0) + 6l'(0)\theta'''(0) + 4l''(0)\theta''(0))\boldsymbol{n}(0).
\end{aligned}
$$

Thus we observe that

$$
\begin{aligned}
& 3\,\mathrm{Pe}_\gamma^{(5)}(0) - 10\frac{\theta'''(0)}{\theta''(0)}\,\mathrm{Pe}_\gamma^{(4)}(0) \\
& \qquad = *(A(0)\boldsymbol{e}(0) + B(0)\boldsymbol{n}(0)) + 12(l'(0)\theta'''(0) - l''(0)\theta''(0))\boldsymbol{n}(0),
\end{aligned}
$$

where $*$ is a constant. By Fact 2.4(5), Pe_γ is a 5/2-cusp at 0 if and only if

$$\det\left(\mathrm{Pe}_\gamma''(0), 3\,\mathrm{Pe}_\gamma^{(5)}(0) - 10\frac{\theta'''(0)}{\theta''(0)}\,\mathrm{Pe}_\gamma^{(4)}(0)\right)$$
$$= -12A(0)\theta''(0)(l'(0)\theta'''(0) - l''(0)\theta''(0)) \neq 0.$$

Thus Pe_γ is a 5/2-cusp at 0 if and only if γ is also a 5/2-cusp at 0 from Proposition 2.5(3) and $A(0) \neq 0$. Moreover,

$$|\,\mathrm{Pe}_\gamma''(0)| = |\theta''(0)|\sqrt{A(0)^2 + B(0)^2},$$
$$\det(\mathrm{Pe}_\gamma''(0), \mathrm{Pe}_\gamma^{(4)}(0)) = 3\theta''(0)^2(A(0)^2 + B(0)^2 + l'(0)A(0)).$$

Therefore

$$b_{\mathrm{Pe}_\gamma} = \frac{A(0)^2 + B(0)^2 + l'(0)A(0)}{|\theta''(0)|(A(0)^2 + B(0)^2)^{3/2}}.$$

Since $A(0) \neq 0$, we have $A(0)^2 + B(0)^2 \neq 0$. Moreover, by changing $\boldsymbol{n}$ to $-\boldsymbol{n}$ if necessary, we see that $\mathrm{sgn}(l'(0)) = \mathrm{sgn}(A(0))$. Therefore $b_{\mathrm{Pe}_\gamma} \neq 0$. ∎

EXAMPLE 3.5. Let $\gamma\colon (\mathbb{R}, 0) \to \mathbb{R}^2$ be the frontal provided by

$$\gamma(t) = (1 + t^2, 1 + t^4 + t^5/5),$$

which is a 5/2-cusp at $t = 0$. Defining $l, \theta\colon (\mathbb{R}, 0) \to \mathbb{R}$ by

$$l(t) = t\sqrt{4 + t^4(4 + t)^2}, \qquad \theta(t) = \arctan\left(\tfrac{1}{2}t^2(4 + t)\right),$$

we see that

$$\gamma'(t) = l(t)\boldsymbol{e}(t) = l(t)(\cos\theta(t), \sin\theta(t))$$
$$= t\sqrt{4 + t^4(4 + t)^2}\left(\frac{2}{\sqrt{4 + t^4(4 + t)^2}}, \frac{t^2(4 + t^2)}{\sqrt{4 + t^4(4 + t)^2}}\right),$$
$$\boldsymbol{n}(t) = (-\sin\theta(t), \cos\theta(t)) = \left(-\frac{t^2(4 + t^2)}{\sqrt{4 + t^4(4 + t)^2}}, \frac{2}{\sqrt{4 + t^4(4 + t)^2}}\right).$$

Thus the pedal curve Pe_γ of γ is

$$\mathrm{Pe}_\gamma(t) = \left(\frac{t^2(4 + t)^2(-10 + 20t^2 + 5t^3 + 10t^4 + 3t^5)}{5(4 + t^4(4 + t)^2)},\right.$$
$$\left. - \frac{2(-10 + 20t^2 + 5t^3 + 10t^4 + 3t^5)}{5(4 + t^4(4 + t)^2)}\right).$$

As shown, Pe_γ is a 5/2-cusp at $t = 0$, with non-vanishing bias b_{Pe_γ} (see Figure 3).

Next, we consider the case that γ is an A-type frontal curve that is not a 3/2-cusp with vanishing bias b at the cusp. By Proposition 3.3, the pedal curve Pe_γ of γ is not an A-type frontal curve.

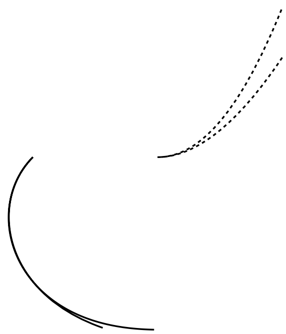

Fig. 3. Images of γ (dashed) and Pe_γ (solid). We can observe that both γ and Pe_γ are 5/2-cusps at $t = 0$.

THEOREM 3.6. *Let $\gamma\colon (\mathbb{R}, 0) \to \mathbb{R}^2$ be an A-type frontal curve that is not a 3/2-cusp. Assume that $\gamma(0) \neq 0$ and the bias b of γ vanishes at 0. Then Pe_γ is a 5/3-cusp at 0 if and only if γ is a 5/2-cusp and $\gamma(0) \cdot \boldsymbol{n}(0) \neq 0$.*

Proof. Based on the assumption, $l(0) = \theta'(0) = \theta''(0) = 0$ and $l'(0) \neq 0$. Thus from the proof of Theorem 3.4, $\mathrm{Pe}'_\gamma(0) = \mathrm{Pe}''_\gamma(0) = 0$. We have

$$\begin{aligned} \mathrm{Pe}'''_\gamma(0) &= -\theta'''(0)(A(0)\boldsymbol{e}(0) + B(0)\boldsymbol{n}(0)), \\ \mathrm{Pe}^{(4)}_\gamma(0) &= -\theta^{(4)}(0)(A(0)\boldsymbol{e}(0) + B(0)\boldsymbol{n}(0)), \\ \mathrm{Pe}^{(5)}_\gamma(0) &= -\theta^{(5)}(0)(A(0)\boldsymbol{e}(0) + B(0)\boldsymbol{n}(0)) - 6l'(0)\theta'''(0)\boldsymbol{n}(0). \end{aligned}$$

Therefore we obtain $\det(\mathrm{Pe}'''_\gamma(0), \mathrm{Pe}^{(4)}_\gamma(0)) = 0$ and

$$\det(\mathrm{Pe}'''_\gamma(0), \mathrm{Pe}^{(5)}_\gamma(0)) = 6A(0)l'(0)\theta'''(0)^2.$$

Hence $\det(\mathrm{Pe}'''_\gamma(0), \mathrm{Pe}^{(5)}_\gamma(0)) \neq 0$ if and only if $A(0)\theta'''(0) \neq 0$. This implies that Pe_γ is a 5/3-cusp at 0 if and only if $\theta'''(0) \neq 0$ by Fact 2.4(6). By contrast, in the above situation, γ is a 5/2-cusp at 0 if and only if $\theta'''(0) \neq 0$ by Proposition 2.5(3). Thus we have the conclusion. ∎

Now, consider the case where γ is not of A-type, that is, $\gamma'(0) = \gamma''(0) = 0$. In this case, we obtain the following:

PROPOSITION 3.7. *Let $\gamma\colon (\mathbb{R}, 0) \to \mathbb{R}^2$ be a frontal curve that is not of A-type. Let $\boldsymbol{n}$ be a unit normal vector to γ. Assume that $\gamma(0) \neq 0$. If γ is a 5/3-cusp at 0 and $\gamma(0) \cdot \boldsymbol{n}(0) \neq 0$, then Pe_γ is a 5/2-cusp at 0. Moreover, b_{Pe_γ} does not vanish.*

Proof. From Proposition 2.5(4), the conditions $l(0) = l'(0) = \theta'(0) = 0$ and $l''(0)\theta''(0) \neq 0$ hold. In particular, $\theta''(0) \neq 0$. Since

$$\mathrm{Pe}''_\gamma(0) = -\theta''(0)(A(0)\boldsymbol{e}(0) + B(0)\boldsymbol{n}(0)) \neq 0,$$

Pe_γ is an A-type frontal curve. From the proof of Theorem 3.4, we can observe that

$$\mathrm{Pe}_\gamma'''(0) = -\theta'''(0)(A(0)\boldsymbol{e}(0) + B(0)\boldsymbol{n}(0)) = \frac{\theta'''(0)}{\theta''(0)}\,\mathrm{Pe}_\gamma''(0),$$

and

$$3\,\mathrm{Pe}_\gamma^{(5)}(0) - 10\frac{\theta'''(0)}{\theta''(0)}\,\mathrm{Pe}_\gamma^{(4)}(0) = *(A(0)\boldsymbol{e}(0) + B(0)\boldsymbol{n}(0)) - 12l''(0)\theta''(0)\boldsymbol{n}(0),$$

where $*$ is a real constant. Therefore we obtain

$$\det\left(\mathrm{Pe}_\gamma''(0), 3\,\mathrm{Pe}_\gamma^{(5)}(0) - 10\frac{\theta'''(0)}{\theta''(0)}\,\mathrm{Pe}_\gamma^{(4)}(0)\right) = 12A(0)l''(0)\theta''(0)^2 \neq 0$$

by the assumption. This implies that Pe_γ is a 5/2-cusp at 0, based on Fact 2.4(5). Moreover, since

$$\mathrm{Pe}_\gamma^{(4)}(0) = -\theta^{(4)}(0)(A(0)\boldsymbol{e}(0) + B(0)\boldsymbol{n}(0)) + 6\theta''(0)(B(0)\boldsymbol{e}(0) - A(0)\boldsymbol{n}(0)),$$

we see that

$$\det(\mathrm{Pe}_\gamma''(0), \mathrm{Pe}_\gamma^{(4)}(0)) = 6\theta''(0)^2(A(0)^2 + B(0)^2) \neq 0.$$

From the definition of bias (see (2.4)), we have $b_{\mathrm{Pe}_\gamma} \neq 0$. Thus we get the assertion. ∎

3.2. Pedal curves of fronts passing through the origin. We consider the case where a frontal curve $\gamma\colon (\mathbb{R}, 0) \to \mathbb{R}^2$ passes through the origin, that is, $\gamma(0) = 0$. Consequently, $\mathrm{Pe}_\gamma(0) = 0 = \gamma(0)$. In this case, $\mathrm{Pe}_\gamma'(0) = 0$ whether γ is a front at 0 or not. Here, we examine the case where γ is a front at 0.

LEMMA 3.8. *Under the above assumptions, the pedal curve* Pe_γ *is not of* A*-type.*

Proof. Since $A(0) = A'(0) = B(0) = B'(0) = 0$, we obtain $\mathrm{Pe}_\gamma''(0) = 0$ using (3.3). Thus Pe_γ is not of A-type. ∎

PROPOSITION 3.9 (cf. [16]). *Let* $\gamma\colon (\mathbb{R}, 0) \to \mathbb{R}^2$ *be a* 3/2*-cusp at* 0. *Assume that* $\gamma(0) = 0$. *Then* Pe_γ *is a* 4/3*-cusp at* 0.

Proof. By Proposition 2.5(1), conditions $l(0) = 0$ and $l'(0)\theta'(0) \neq 0$ are satisfied. Moreover, by Lemma 3.8, Pe_γ is not of A-type. We calculate the third and fourth derivatives of Pe_γ at 0. Since $A(t) = \gamma(t) \cdot \boldsymbol{n}(t)$ and $B(t) = \gamma(t) \cdot \boldsymbol{e}(t)$, we have $A''(0) = 0$, $B''(0) = l'(0)$, $A'''(0) = -l'(0)\theta'(0)$ and $B'''(0) = l''(0)$. By (3.4) and (3.5),

$$\begin{aligned}\mathrm{Pe}_\gamma'''(0) &= -l'(0)\theta'(0)\boldsymbol{n}(0),\\ \mathrm{Pe}_\gamma^{(4)}(0) &= 4l'(0)\theta'(0)^2\boldsymbol{e}(0) - (l''(0)\theta'(0) + 3l'(0)\theta''(0))\boldsymbol{n}(0).\end{aligned}$$

Therefore we obtain

$$\det(\mathrm{Pe}_\gamma'''(0), \mathrm{Pe}_\gamma^{(4)}(0)) = 4l'(0)^2\theta'(0)^3 \neq 0.$$

This implies that Pe_γ is a 4/3-cusp at 0 by Fact 2.4(2). ■

Next, we assume that γ is a 4/3-cusp at 0. Then $l(0) = l'(0) = 0$ and $l''(0)\theta'(0) \neq 0$ by Proposition 2.5(2). In this case, we obtain the following:

THEOREM 3.10. *Let $\gamma\colon (\mathbb{R},0) \to \mathbb{R}^2$ be a 4/3-cusp at 0 and $\gamma(0) = 0$. Then the pedal curve Pe_γ is C^1-equivalent to a 5/4-cusp at 0. Moreover, Pe_γ is $\mathcal{A}$-equivalent to a 5/4-cusp at 0 if and only if $\tilde{P} = 0$, where*

$$\begin{aligned}\tilde{P} = {} & -477l''(0)^2\theta''(0)^2 + 1400l''(0)^2\theta'(0)^4 \\ & + 2l''(0)\theta'(0)(17l^{(3)}(0)\theta''(0) + 125\theta^{(3)}(0)l''(0)) \\ & + \theta'(0)^2(18l^{(3)}(0)^2 - 25l^{(4)}(0)l''(0)).\end{aligned} \tag{3.6}$$

In addition, Pe_γ is $\mathcal{A}$-equivalent to a $(5/4;+7)$-cusp (resp. $(5/4;-7)$-cusp) at 0 if and only if $\tilde{P} > 0$ (resp. $\tilde{P} < 0$).

Proof. By the proof of Proposition 3.9, we have $\mathrm{Pe}_\gamma'(0) = \mathrm{Pe}_\gamma''(0) = \mathrm{Pe}_\gamma'''(0) = 0$ and $\mathrm{Pe}_\gamma^{(4)}(0) = -l''(0)\theta'(0)\boldsymbol{n}(0) \neq 0$. We calculate $\mathrm{Pe}_\gamma^{(k)}(0)$ for $k = 5, 6, 7$. Under the above conditions, straightforward computations show that

$$\begin{aligned}
\mathrm{Pe}_\gamma^{(5)}(0) = {} & 5l''(0)\theta'(0)\boldsymbol{e}(0) - (l^{(3)}(0)\theta'(0) + 4l''(0)\theta''(0))\boldsymbol{n}(0), \\
\mathrm{Pe}_\gamma^{(6)}(0) = {} & 3\theta'(0)(2l^{(3)}(0)\theta'(0) + 13l''(0)\theta''(0))\boldsymbol{e}(0) \\
& + \big(-l^{(4)}(0)\theta'(0) + 16l''(0)\theta'(0)^3 \\
& \qquad - 5(l^{(3)}(0)\theta''(0) + 2\theta^{(3)}(0)l''(0))\big)\boldsymbol{n}(0), \\
\mathrm{Pe}_\gamma^{(7)}(0) = {} & 7\big(l^{(4)}(0)\theta'(0)^2 + 12l''(0)\theta''(0)^2 - 6l''(0)\theta'(0)^4 \\
& \qquad + \theta'(0)(8l^{(3)}(0)\theta''(0) + 15\theta^{(3)}(0)l''(0))\big)\boldsymbol{e}(0) \\
& + \big(-l^{(5)}(0)\theta'(0) - 6l^{(4)}(0)\theta''(0) - 15l^{(3)}(0)\theta^{(3)}(0) \\
& \qquad + 22l^{(3)}(0)\theta'(0)^3 - 20\theta^{(4)}(0)l''(0) + 204l''(0)\theta'(0)^2\theta''(0)\big)\boldsymbol{n}(0),
\end{aligned}$$

where $A(0) = B(0) = A'(0) = B'(0) = A''(0) = B''(0) = A'''(0) = 0$, $B'''(0) = l''(0) \neq 0$,

$$\begin{aligned}
& A^{(4)}(0) = -l''(0)\theta'(0), \quad A^{(5)}(0) = -4l''(0)\theta''(0) - l^{(3)}(0)\theta'(0), \\
& A^{(6)}(0) = -l^{(4)}(0)\theta'(0) - 5l^{(3)}(0)\theta''(0) - 10\theta^{(3)}(0)l''(0) + l''(0)\theta'(0)^3, \\
& B^{(4)}(0) = l^{(3)}(0), \quad B^{(5)}(0) = l^{(4)}(0) - l''(0)\theta'(0), \\
& B^{(6)}(0) = l^{(5)}(0) - l^{(3)}(0)\theta'(0)^2 - 9l''(0)\theta'(0)\theta''(0).
\end{aligned}$$

Thus
$$X = \det(\mathrm{Pe}_\gamma^{(4)}(0), \mathrm{Pe}_\gamma^{(5)}(0)) = 5l''(0)^2\theta'(0)^2 \neq 0.$$
Hence Pe_γ is C^1-equivalent to a 5/4-cusp at 0 by Fact 2.4(3). Moreover, by the above calculations, the constant P as in (2.1) is
$$P = 21l''(0)^2\theta'(0)^4\tilde{P},$$
where $\tilde{P}$ is the constant given by (3.6). Thus we have the assertions by Fact 2.4(3)–(4). ∎

Example 3.11. Let $\gamma\colon \mathbb{R} \to \mathbb{R}^2$ be the curve defined by $\gamma(t) = (t^3, t^4)$. Then γ is a 4/3-cusp at 0. A unit normal vector $\boldsymbol{n}(t)$ to $\gamma(t)$ is
$$\boldsymbol{n}(t) = \left(-\frac{4t}{\sqrt{9+16t^2}}, \frac{3}{\sqrt{9+16t^2}}\right).$$
Thus the pedal curve Pe_γ of γ is
$$\mathrm{Pe}_\gamma(t) = \left(\frac{4t^5}{9+16t^2}, -\frac{3t^4}{9+16t^2}\right).$$
Since $\mathrm{Pe}_\gamma'(0) = \mathrm{Pe}_\gamma''(0) = \mathrm{Pe}_\gamma'''(0) = 0$ and
$$\det(\mathrm{Pe}_\gamma^{(4)}(0), \mathrm{Pe}_\gamma^{(5)}(0)) = \frac{1280}{3} \neq 0,$$
Pe_γ is C^1-equivalent to a 5/4-cusp at 0. However, the constant P as in (2.1) is
$$P = \frac{1835008000}{9} > 0.$$
Thus Pe_γ is not $\mathcal{A}$-equivalent to $t \mapsto (t^4, t^5)$ at 0. But Pe_γ is $\mathcal{A}$-equivalent to $t \mapsto (t^4, t^5 + t^7)$ at 0 (see Figure 4). Indeed, set
$$\tilde{t} = \frac{t}{(9+16t^2)^{1/4}}.$$

This gives a new parameter for Pe_γ. By the binomial theorem, there exists a function germ $\psi(\tilde{t})$ such that
$$t = \sqrt{3}\,\tilde{t} + \frac{4}{\sqrt{3}}\tilde{t}^{\,3} + \tilde{t}^{\,4}\psi(\tilde{t}\,).$$

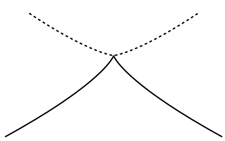

Fig. 4. Images of γ (dashed) and Pe_γ (solid).

Therefore we see that

$$\mathrm{Pe}_\gamma(\tilde{t}\,) = \left(4\sqrt{3}\,\tilde{t}^{\,5} + \frac{16}{\sqrt{3}}\tilde{t}^{\,7} + 4\tilde{t}^{\,8}\psi(\tilde{t}\,), -3\tilde{t}^{\,4}\right)$$

Thus Pe_γ is $\mathcal{A}$-equivalent to $t \mapsto (t^4, t^5 + t^7)$ at 0.

Acknowledgements. The author thanks Yoshiki Matsushita for valuable comments. He is also grateful to the referees for kind suggestions and comments.

Funding. This work was supported by the JSPS KAKENHI Grant numbers JP22K13914 and JP22KK0034.

REFERENCES

[1] V. Balestro, H. Martini and R. Teixeira, *On Legendre curves in normed planes*, Pacific J. Math. 297 (2018), 1–27.
[2] J. W. Bruce and P. J. Giblin, *Curves and Singularities*, 2nd ed., Cambridge Univ. Press, Cambridge, 1992.
[3] T. Fukunaga and M. Takahashi, *Existence and uniqueness for Legendre curves*, J. Geom. 104 (2013), 297–307.
[4] T. Fukunaga and M. Takahashi, *Evolutes of fronts in the Euclidean plane*, J. Singul. 10 (2014), 92–107.
[5] T. Fukunaga and M. Takahashi, *Evolutes and involutes of frontals in the Euclidean plane*, Demonstr. Math. 48 (2015), 147–166.
[6] Y. Hattori, A. Honda and T. Morimoto, *Bour's theorem for helicoidal surfaces with singularities*, Differential Geom. Appl. 99 (2025), art. 102248, 23 pp.
[7] A. Honda and K. Saji, *Geometric invariants of* 5/2*-cuspidal edges*, Kodai Math. J. 42 (2019), 496–525.
[8] S. Izumiya and N. Takeuchi, *Evolutoids and pedaloids of plane curves*, Note Mat. 39 (2019), no. 2, 12–23.
[9] Y. Li and D. Pei, *Pedal curves of frontals in the Euclidean plane*, Math. Methods Appl. Sci. 41 (2018), 1988–1997.
[10] L. F. Martins, K. Saji, S. P. dos Santos and K. Teramoto, *Singular surfaces of revolution with prescribed unbounded mean curvature*, An. Acad. Brasil. Ciênc. 91 (2019), no. 3, art. e20170865, 10 pp.
[11] L. F. Martins, K. Saji, S. P. dos Santos and K. Teramoto, *Boundedness of geometric invariants near a singularity which is a suspension of a singular curve*, Rev. Un. Mat. Argentina 67 (2024), 475–502.
[12] Y. Matsushita, *Classifications of cusps appearing on plane curves*, arXiv:2402.12166v1 (2024).
[13] I. R. Porteous, *Geometric Differentiation*, 2nd ed., Cambridge Univ. Press, Cambridge, 2001.
[14] S. Shiba and M. Umehara, *The behavior of curvature functions at cusps and inflection points*, Differential Geom. Appl. 30 (2012), 285–299.
[15] K. Teramoto, *Parallels, evolutes and certain lifts of frontal curves in the Euclidean plane*, Bull. Belg. Math. Soc. Simon Stevin 32 (2025), 302–331.

[16] O. O. Tuncer, H. Ceyhan, İ. Gök and F. N. Ekmekci, *Notes on pedal and contrapedal curves of fronts in the Euclidean plane*, Math. Methods Appl. Sci. 41 (2018), 5096–5111.

[17] M. Umehara, K. Saji and K. Yamada, *Differential Geometry of Curves and Surfaces with Singularities*, World Sci., Hackensack, NJ, 2022.

Keisuke Teramoto
Department of Mathematics
Yamaguchi University
753-8512 Yamaguchi, Japan
E-mail: kteramoto@yamaguchi-u.ac.jp