GEOMETRIC PROPERTIES OF CAUSTICS OF PSEUDO-SPHERICAL SURFACES

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Abstract. We deal with pseudo-spherical surfaces admitting certain singularities and their caustics. In particular, we give characterizations of a cuspidal butterfly, cuspidal lips and cuspidal beaks singularities on a pseudo-spherical surface. Moreover, characterizations of certain singularities on the caustics of a pseudo-spherical surface are given. Furthermore, when the caustic has a cuspidal edge singularity, we investigate geometric invariants defined at that point.

1. Introduction

A pseudo-spherical surface is an immersed surface in the Euclidean 3-space $\mathbb{R}^3$ whose Gaussian curvature is negative constant. It is known that a pseudo-spherical surface corresponds to a solution to the sine-Gordon equation as in (3.1) (cf. [21]). In general, a pseudo-spherical surface can be extended to a pseudo-spherical front which admits certain singularities and has a well-defined smooth unit normal vector even at singular points ([16]). Cuspidal edges and swallowtails are known to be generic singularities of pseudo-spherical fronts ([4, Theorem 1.1]). Furthermore, a cuspidal lips, a cuspidal beaks and a cuspidal butterfly can appear as singularities in a generic one-parameter family of pseudo-spherical fronts ([3, Theorem 3.3]). Note that D_4-singularities do not appear as singularities of pseudo-spherical fronts ([3, Proposition 3.1]).

At each singular point of a pseudo-spherical front, we can define a caustic with respect to the unbounded principal curvature near such a point ([19,20]). In [20], the condition for singularities of caustics to be a cuspidal edge is given in terms of the corresponding solution of the sine-Gordon equation. Moreover, explicit representations for the Gaussian and mean curvature of caustics for pseudo-spherical fronts are stated by using a solution to the sine-Gordon equation ([20, Theorem 4.5]). It is known that when the initial pseudo-spherical front has a cuspidal edge as a singularity, then the corresponding caustic is regular. In that case, the condition for the caustic to be a minimal surface is given. Besides, in [8], it is shown that pseudo-spherical fronts whose caustics are minimal surfaces form a family of Dini surfaces and those minimal surfaces are the catenoid and helicoid family. While the case where caustics are regular is well studied, it seems that we do not have enough knowledge about the geometric information of caustics at singular points. Thus it is natural to consider relation between the solution of the sine-Gordon equation and the geometric properties of the caustic at singular points.

In this paper, we study relation between singularities of pseudo-spherical fronts and geometric properties of their caustics by using solutions to the sine-Gordon equation. In particular, we deal with the case that a caustic has a cuspidal edge singular point. At a cuspidal edge, we can define the singular curvature, the cuspidal curvature and the cuspidal torsion ([11, 12, 15]). Especially, it is known that the singular curvature is an intrinsic invariant of a front at a cuspidal edge ([15]). Our aim is to clarify behavior of the singular

Date: September 3, 2025.
2020 *Mathematics Subject Classification.* 53A05, 53A55, 57R45.
Key words and phrases. pseudo-spherical front, caustic, singularity, cuspidal edge, geometric invariants.

curvature defined at a cuspidal edge on the caustic which relates to convexity or concavity at the edge (see Figure 2).

In order to do this, we first investigate singularities of pseudo-spherical fronts. We give conditions for pseudo-spherical fronts to be a cuspidal lips, cuspidal beaks and cuspidal butterfly in terms of the solution to sine-Gordon equation (Proposition 3.3). According to the established criteria for singularities of pseudo-spherical fronts, we observe that when a pseudo-spherical front is characterized as a swallowtail, a cuspidal lips or a cuspidal beaks, then the caustic is a cuspidal edge at the corresponding point (Proposition 4.2). We finally examine geometric invariants at a cuspidal edge of the caustic. We give explicit representations of the singular curvature, cuspidal curvature and cuspidal torsion in terms of the solution to the sine-Gordon equation which corresponds to the initial pseudo-spherical front (Theorem 4.4). As a consequence of the expression, we show that if a pseudo-spherical front is a cuspidal lips (resp. cuspidal beaks), then its caustic is a cuspidal edge with positive (resp. negative) singular curvature at that point (Corollary 4.6). This implies that a cuspidal edge on the caustic corresponding to a cuspidal lips (resp. a cuspidal beaks) is curved convexly (resp. concavely).

2. Preliminaries

In this section, we recall some notions and fundamental facts on fronts. See [1,2,5,10,22] for details.

2.1. **Fronts.** Let $U \subset \mathbb{R}^2$ be an open set and $f\colon U \to \mathbb{R}^3$ a C^∞ map, where $\mathbb{R}^3$ is the Euclidean 3-space. Then we call f a (*wave*) *front* if there exists a C^∞ map $\nu\colon U \to \mathbb{S}^2(\subset \mathbb{R}^3)$ such that

- $\left\langle df_q(\boldsymbol{v}), \nu(q)\right\rangle = 0$ for any $q \in U$ and $\boldsymbol{v} \in T_q\mathbb{R}^2$, and
- the pair $(f, \nu)\colon U \to \mathbb{R}^3 \times \mathbb{S}^2$ gives an immersion,

where $\mathbb{S}^2$ is the standard unit sphere in $\mathbb{R}^3$ and $\langle\cdot,\cdot\rangle$ is the canonical scalar product on $\mathbb{R}^3$. The map ν is called a *unit normal vector* to f. A point $p \in U$ is said to be a *singular point* of f if $\operatorname{rank} df_p < 2$, namely f is not an immersion at p. We denote the set of singular points of f by $S(f)$.

For a front $f\colon U \to \mathbb{R}^3$, we set a function $\lambda\colon U \to \mathbb{R}$ by

$$\lambda(u, v) = \det(f_u, f_v, \nu)(u, v) \quad (f_u = \partial f/\partial u, f_v = \partial f/\partial v), \tag{2.1}$$

where det is the determinant and (u, v) is some local coordinate system on U. We call λ the *signed area density function* of f. Moreover, a function $\tilde{\lambda}$ satisfying $\lambda = \alpha \cdot \tilde{\lambda}$, where α is some non-zero function on U, is called an *identifier of singularities* of f. Obviously, we see that $S(f) = \lambda^{-1}(0) = \tilde{\lambda}^{-1}(0)$. A singular point $p \in S(f)$ is said to be *non-degenerate* if $(\lambda_u, \lambda_v) \neq (0, 0)$ at p, equivalently $(\tilde{\lambda}_u, \tilde{\lambda}_v) \neq (0, 0)$ at p. Otherwise, it is called *degenerate*. If $p \in S(f)$ is a non-degenerate singular point of f, then there exist a neighborhood $V(\subset U)$ of p and a regular curve $\gamma\colon (-\varepsilon, \varepsilon) \to V$ such that $\gamma(0) = p$ and $\lambda(\gamma(t)) = 0$ ($|t| < \varepsilon$). We call the curve γ a *singular curve* for f. Moreover, if $p \in S(f)$ is non-degenerate, then $\operatorname{rank} df_p = 1$. On the other hand, if $p \in S(f)$ is a (co)rank one singular point, that is, $\operatorname{rank} df_p = 1$ holds, then there exists a non-zero vector field η on U such that $\langle \eta_p \rangle_{\mathbb{R}} = \ker df_p$. We call such a vector field η a *null vector field* for f.

Remark 2.1. If $p \in S(f)$ is a non-degenerate singular point of f, then it holds that $df_{\gamma(t)}(\eta_{\gamma(t)}) = 0$ for each $t \in (-\varepsilon, \varepsilon)$, where γ is a singular curve for f through p ([10]).

2.2. **Singularities of fronts.** Here we present criteria for certain singularities of fronts. We first recall types of singularities which appear in this paper.

Definition 2.2. Let $f\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a C^∞ map germ. Then

(1) f at 0 is a *cuspidal edge* if f is $\mathcal{A}$-equivalent to the germ $(u,v) \mapsto (u, v^2, v^3)$ at 0;
(2) f at 0 is a *swallowtail* if f is $\mathcal{A}$-equivalent to the germ $(u,v) \mapsto (u, 4v^3+2uv, 3v^4+uv^2)$ at 0;
(3) f at 0 is a *cuspidal butterfly* if f is $\mathcal{A}$-equivalent to the germ $(u,v) \mapsto (u, 5v^4+2uv, 4v^5+u^2v)$ at 0;
(4) f at 0 is a *cuspidal lips* if f is $\mathcal{A}$-equivalent to the germ $(u,v) \mapsto (u, v^3+u^2v, 3v^4+2u^2v^2)$ at 0;
(5) f at 0 is a *cuspidal beaks* if f is $\mathcal{A}$-equivalent to the germ $(u,v) \mapsto (u, v^3-u^2v, 3v^4-2u^2v^2)$ at 0.

Here two map germs $f_1, f_2 \colon (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ are *$\mathcal{A}$-equivalent* if there exist diffeomorphism germs $\varphi\colon (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$ on the source and $\Phi\colon (\mathbb{R}^3,0) \to (\mathbb{R}^3,0)$ on the target such that $f_2 = \Phi \circ f_1 \circ \varphi^{-1}$.

We note that these are rank one singularities of fronts. Moreover, a cuspidal edge, swallowtail and cuspidal butterfly are non-degenerate singular points, but a cuspidal lips and cuspidal beaks are degenerate ones (see Figure 1). For these singularities, the following useful criteria are known.

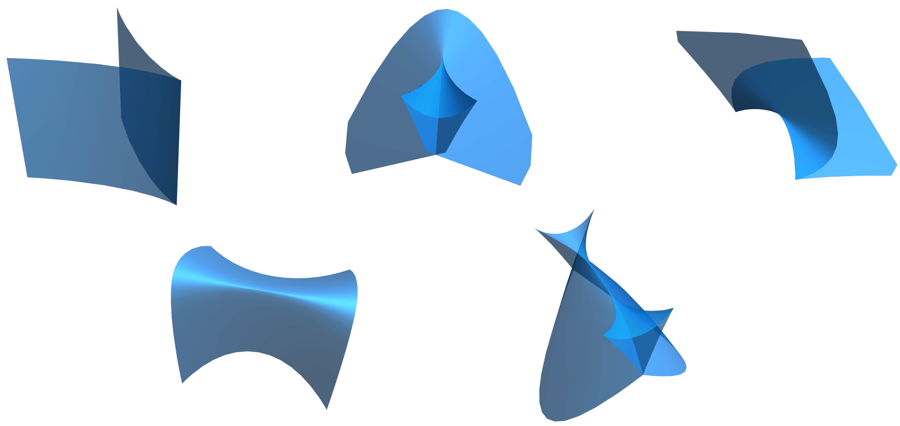

FIGURE 1. From the top left to the bottom right: cuspidal edge, swallowtail, cuspidal butterfly, cuspidal lips and cuspidal beaks

Theorem 2.3 (cf. [6, 7, 10, 14])**.** *Let $f\colon U \to \mathbb{R}^3$ be a front and $p \in S(f)$ a rank one singular point of f. Then as a map germ,*

(1) *f at p is a cuspidal edge if and only if $\eta\lambda(p) \neq 0$;*
(2) *f at p is a swallowtail if and only if $d\lambda(p) \neq 0$, $\eta\lambda(p) = 0$ and $\eta\eta\lambda(p) \neq 0$;*
(3) *f at p is a cuspidal butterfly if and only if $d\lambda(p) \neq 0$, $\eta\lambda(p) = \eta\eta\lambda(p) = 0$ and $\eta\eta\eta\lambda(p) \neq 0$;*
(4) *f at p is a cuspidal lips if and only if $d\lambda(p) = 0$ and $\det \mathcal{H}(\lambda)(p) > 0$;*
(5) *f at p is a cuspidal beaks if and only if $d\lambda(p) = 0$, $\eta\eta\lambda(p) \neq 0$ and $\det \mathcal{H}(\lambda)(p) < 0$.*

Here λ is the signed area density function of f, $d\lambda$ is the exterior derivative of λ, $\eta\lambda$ is the directional derivative of λ in the direction η and $\mathcal{H}(\lambda)(p)$ is the Hessian matrix of λ at p.

While the above criteria are denoted by the signed area density function λ, we can replace λ to an identifier of singularities $\tilde{\lambda}$ of a front f (cf. [22]).

2.3. **Geometric invariants defined at a cuspidal edge.** We recall geometric invariants defined at a cuspidal edge. We mean by a *gemetric invariant* of pseudo-spherical fronts an invariant of them under the Euclidean motion on the target up to diffeomorphisms on the source. For details, see [11, 12, 15, 22]. Let $f\colon U \to \mathbb{R}^3$ be a front and ν a unit normal vector to f, and $p \in U$ a cuspidal edge singular point of f. Let $\gamma\colon (-\varepsilon, \varepsilon) \to U$ be a singular curve for f through $p = \gamma(0)$. Then we can take a pair of vector fields (ξ, η) on U such that

- ξ is tangent to γ,
- η is a null vector field of f, and
- the pair (ξ, η) is linearly independent.

We call such a pair an *adapted pair* of vector fields ([11]). Using an adapted pair of vector fields, we can define the following geometric invariants along γ ([11, 12, 15]):

$$\kappa_s(t) = \varepsilon_\gamma \frac{\det(\xi f, \xi\xi f, \nu)}{|\xi f|^3}(\gamma(t)), \quad \kappa_\nu(t) = \frac{\langle \xi\xi f, \nu\rangle}{|\xi f|^2}(\gamma(t)),$$
$$\kappa_c(t) = \frac{|\xi f|^{3/2} \det(\xi f, \eta\eta f, \eta\eta\eta f)}{|\xi f \times \eta\eta f|^{5/2}}(\gamma(t)),$$
$$\kappa_t(t) = \frac{\det(\xi f, \eta\eta f, \xi\eta\eta f)}{|\xi f \times \eta\eta f|^2}(\gamma(t)) - \frac{\langle \xi f, \eta\eta f\rangle \det(\xi f, \eta\eta f, \xi\xi f)}{|\xi f|^2 |\xi f \times \eta\eta f|^2}(\gamma(t)),$$

where $\varepsilon_\gamma = \mathrm{sgn}(\eta\lambda \cdot \det(\xi, \eta))$ along γ. We call $\kappa_s, \kappa_\nu, \kappa_c$ and κ_t the *singular curvature*, *limiting normal curvature*, *cuspidal curvature* and *cuspidal torsion*, respectively. It is known that the Gaussian curvature of a front f is bounded near a cuspidal edge p if and only if κ_ν vanishes identically along γ (cf. [12, Theorem 3.9]). Under this condition, if the Gaussian curvature K is non-negative near a cuspidal edge, then the singular curvature κ_s is non-positive at the cuspidal edge ([15, Theorem 3.1]) (see Figure 2).

(A) $\kappa_s > 0$ (B) $\kappa_s < 0$

FIGURE 2. Cuspidal edges with non-zero κ_s.

3. Pseudo-spherical fronts

We investigate pseudo-spherical fronts in $\mathbb{R}^3$. We begin with the definition of pseudo-spherical fronts.

Definition 3.1 (cf. [16, 23])**.** Let $D \subset \mathbb{R}^2$ be a simply connected domain. Then a front $f\colon D \to \mathbb{R}^3$ is said to be *pseudo-spherical* if there exists an open and dense subset O of D such that the restriction $f|_O\colon O \to \mathbb{R}^3$ is a pseudo-spherical surface, namely f is an immersion with the constant negative Gaussian curvature on O.

By normalization, we may assume that the Gaussian curvature of a pseudo-spherical front is -1. In the following, we suppose the condition. In this case, we note that a front $f\colon D \to \mathbb{R}^3$ is pseudo-spherical if and only if $\det(f_u, f_v, \nu) = -\det(\nu_u, \nu_v, \nu)$ on an open dense subset O, where ν is a unit normal vector to f.

Let $f\colon D \to \mathbb{R}^3$ be a pseudo-spherical front and ν its unit normal vector, where D is a simply connected domain of $\mathbb{R}^2$. Then there exist local coordinates (u, v) on D and C^∞

function $\theta\colon D \to \mathbb{R}$ satisfying the *sine-Gordon equation*

$$\theta_{uu} - \theta_{vv} = \sin\theta \tag{3.1}$$

such that the first, second and third fundamental forms of f are given by

$$\mathrm{I} = \cos^2\frac{\theta}{2}du^2 + \sin^2\frac{\theta}{2}dv^2, \quad \mathrm{II} = \frac{1}{2}\sin\theta(du^2 - dv^2), \quad \mathrm{III} = \sin^2\frac{\theta}{2}du^2 + \cos^2\frac{\theta}{2}dv^2, \tag{3.2}$$

respectively, and vice versa ([16, Theorem 2.11]). We remark that in [16], they use the asymptotic Chebychev coordinates (x, y). When we change coordinates $(x, y) \mapsto (x + y, x - y) = (u, v)$, then we have the above formulations. We also remark that the function θ measures the angle between asymptotic lines at each point. By (3.2), the first fundamental form I degenerates on $\theta^{-1}(\pi\mathbb{Z})$, which implies that $S(f) = \theta^{-1}(\pi\mathbb{Z})$.

On the other hand, for a pseudo-spherical front f corresponding to the solution θ of (3.1), one can take C^∞ maps $\boldsymbol{e}_1, \boldsymbol{e}_2\colon D \to \mathbb{R}^3$ satisfying

$$f_u = \cos\frac{\theta}{2}\boldsymbol{e}_1, \quad f_v = \sin\frac{\theta}{2}\boldsymbol{e}_2, \quad \langle \boldsymbol{e}_i, \boldsymbol{e}_j \rangle = \delta_{ij}, \quad \nu = \boldsymbol{e}_1 \times \boldsymbol{e}_2, \tag{3.3}$$

where δ_{ij} is the Kronecker delta (cf. [16, 20]). We note that the triple $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \nu\}$ forms an orthonormal frame along f. By the Gauss and Weingarten formulas (cf. [9, Theorem 3.8.1]), we have

$$\begin{aligned}
f_{uu} &= -\frac{\theta_u}{2}\sin\frac{\theta}{2}\boldsymbol{e}_1 + \frac{\theta_v}{2}\cos\frac{\theta}{2}\boldsymbol{e}_2 + \frac{1}{2}\sin\theta\nu, \\
f_{uv} &= -\frac{\theta_v}{2}\sin\frac{\theta}{2}\boldsymbol{e}_1 + \frac{\theta_u}{2}\cos\frac{\theta}{2}\boldsymbol{e}_2 (= f_{vu}), \\
f_{vv} &= -\frac{\theta_u}{2}\sin\frac{\theta}{2}\boldsymbol{e}_1 + \frac{\theta_v}{2}\cos\frac{\theta}{2}\boldsymbol{e}_2 - \frac{1}{2}\sin\theta\nu, \\
\nu_u &= -\sin\frac{\theta}{2}\boldsymbol{e}_1, \quad \nu_v = \cos\frac{\theta}{2}\boldsymbol{e}_2.
\end{aligned} \tag{3.4}$$

Moreover, it holds that

$$(\boldsymbol{e}_1)_u = \frac{\theta_v}{2}\boldsymbol{e}_2 + \sin\frac{\theta}{2}\nu, \quad (\boldsymbol{e}_1)_v = \frac{\theta_u}{2}\boldsymbol{e}_2, \quad (\boldsymbol{e}_2)_u = -\frac{\theta_v}{2}\boldsymbol{e}_1, \quad (\boldsymbol{e}_2)_v = -\frac{\theta_u}{2}\boldsymbol{e}_1 - \cos\frac{\theta}{2}\nu$$

(see [20, (3.7)]). We will use these formulas to analyze the caustic.

3.1. **Singularities of pseudo-spherical fronts.** We deal with singularities of a pseudo-spherical front. Let $f\colon D \to \mathbb{R}^3$ be a pseudo-spherical front associated to the solution θ of the sine-Gordon equation (3.1). By (3.3) and (2.1), we see that

$$\lambda = \frac{1}{2}\sin\theta\left(= \cos\frac{\theta}{2}\sin\frac{\theta}{2}\right).$$

Let Z_1 and Z_2 be $Z_1 = \{(2n+1)\pi \mid n \in \mathbb{Z}\}$ and $Z_2 = \{2n\pi \mid n \in \mathbb{Z}\}$, respectively. Then we have $S(f) = \theta^{-1}(Z_1) \cup \theta^{-1}(Z_2)$ and $\theta^{-1}(Z_1) \cap \theta^{-1}(Z_2) = \emptyset$. If we take $p \in \theta^{-1}(Z_1)$ (resp. $p \in \theta^{-1}(Z_2)$), then

$$f_u = 0, \quad f_v \neq 0 \quad (\text{resp. } f_u \neq 0, \quad f_v = 0)$$

at p by (3.3). Thus $\eta_1 = \partial_u$ (resp. $\eta_2 = \partial_v$) can be taken as a null vector for f around $p \in \theta^{-1}(Z_1)$ (resp. $p \in \theta^{-1}(Z_2)$).

Fact 3.2 ([20, Proposition 3.2])**.** *Under the above situation, we have the following.*

(1) *Suppose $p \in \theta^{-1}(Z_1)$. Then*
   (a) *f at p is a cuspidal edge if and only if $\theta_u \neq 0$ at p;*
   (b) *f at p is a swallowtail if and only if $\theta_u = 0$, $\theta_v \neq 0$ and $\theta_{uu} \neq 0$ at p.*
(2) *Suppose $p \in \theta^{-1}(Z_2)$. Then*

(a) f *at* p *is a cuspidal edge if and only if* $\theta_v \neq 0$ *at* p;
(b) f *at* p *is a swallowtail if and only if* $\theta_v = 0$, $\theta_u \neq 0$ *anf* $\theta_{vv} \neq 0$ *at* p.

Note that a cuspidal edge and a swallowtail are generic singularities of pseudo-spherical fronts ([4]). On the other hand, Brander and Tari [3] pointed out that a pseudo-spherical front may have a cuspidal lips, a cuspidal beaks and a cuspidal butterfly as singularities. By Theorem 2.3, we have the following characterizations for these singularities in terms of the solution to the sine-Gordon equation (3.1).

Proposition 3.3. *Let* $f\colon D \to \mathbb{R}^3$ *be a pseudo-spherical front corresponding to the solution* θ *to the sine-Gordon equation* (3.1).

(1) *Suppose* $p \in \theta^{-1}(Z_1)$. *Then*
  (a) f *at* p *is a cuspidal butterfly if and only if* $\theta_u = 0$, $\theta_v \neq 0$, $\theta_{uu} = 0$ *and* $\theta_{uuu} \neq 0$ *at* p;
  (b) f *at* p *is a cuspidal lips if and only if* $\theta_u = \theta_v = 0$ *and* $\theta_{uu}^2 > \theta_{uv}^2$ *at* p;
  (c) f *at* p *is a cuspidal beaks if and only if* $\theta_u = \theta_v = 0$, $\theta_{uu} \neq 0$ *and* $\theta_{uu}^2 < \theta_{uv}^2$ *at* p.
(2) *Suppose* $p \in \theta^{-1}(Z_2)$. *Then*
  (a) f *at* p *is a cuspidal butterfly if and only if* $\theta_u \neq 0$, $\theta_v = 0$, $\theta_{vv} = 0$ *and* $\theta_{vvv} \neq 0$ *at* p;
  (b) f *at* p *is a cuspidal lips if and only if* $\theta_u = \theta_v = 0$ *and* $\theta_{vv}^2 > \theta_{uv}^2$ *at* p;
  (c) f *at* p *is a cuspidal beaks if and only if* $\theta_u = \theta_v = 0$, $\theta_{vv} \neq 0$ *and* $\theta_{vv}^2 < \theta_{uv}^2$ *at* p.

Proof. We show (1). Assume $p \in \theta^{-1}(Z_1)$. Since the signed area density function satisfies $2\lambda = \sin\theta$, we have

$$(\lambda_u, \lambda_v) = \frac{1}{2}\cos\theta(\theta_u, \theta_v) = -\frac{1}{2}(\theta_u, \theta_v)$$

at p because $\cos\theta(p) = -1$. We first assume that $(\theta_u, \theta_v) \neq (0, 0)$ at p. Then f at p is a cuspidal butterfly if and only if $\eta_1\lambda = \eta_1\eta_1\lambda = 0$ and $\eta_1\eta_1\eta_1\lambda \neq 0$ at p by Theorem 2.3 (3), where $\eta_1 = \partial_u$ is a null vector field of f near p. By direct calculations, we see that

$$\eta_1\lambda = \frac{\theta_u}{2}\cos\theta, \quad \eta_1\eta_1\lambda = \frac{\theta_{uu}}{2}\cos\theta - \frac{\theta_u^2}{2}\sin\theta,$$
$$\eta_1\eta_1\eta_1\lambda = \frac{\theta_{uuu}}{2}\cos\theta - \frac{3}{2}\theta_{uu}\theta_u\sin\theta - \frac{\theta_u^3}{2}\cos\theta.$$

Since $\sin\theta = 0$ and $\cos\theta = -1$ at $p \in \theta^{-1}(Z_1)$, we have the assertion (a).

We next assume that $\theta_u = \theta_v = 0$ at p. Under these conditions, we investigate the Hessian $\det\mathcal{H}(\lambda)$ of λ at p. By a direct calculation, we have

$$\det\mathcal{H}(\lambda) = \frac{1}{4}\cos^2\theta(\theta_{uu}\theta_{vv} - \theta_{uv}^2) = \frac{1}{4}(\theta_{uu}\theta_{vv} - \theta_{uv}^2)$$

at $p \in \theta^{-1}(Z_1)$. On the other hand, by (3.1), we see that

$$\theta_{uu} - \theta_{vv} = 0$$

at $p \in \theta^{-1}(Z_1)$. Therefore we have assertions (b) and (c) by Theorem 2.3 (4) and (5), respectively.

For (2), one can show similarly. □

Remark 3.4. Since $\theta_{uu} = \theta_{vv}$ at $p \in S(f)$, we see that $\theta_{uu}(p) \neq 0$ (resp. $\theta_{uu}(p) = 0$) yields $\theta_{vv}(p) \neq 0$ (resp. $\theta_{vv}(p) = 0$). Moreover, if f at p is a cuspidal beaks, then $\theta_{uv} \neq 0$ at p.

3.2. **Principal curvatures.** We recall behavior of principal curvatures of a pseudo-spherical front f associated to the solution θ to (3.1). By (3.2), principal curvatures of f are given by

$$\kappa_1 = \tan\frac{\theta}{2}, \quad \kappa_2 = -\cot\frac{\theta}{2}$$

on $D \setminus S(f)$. It is known that for a point $p \in S(f)$, there exists an open neighborhood U of p such that one of the principal curvatures can be extended as a C^∞ function on U (see [13, 18]). Indeed, if $p \in \theta^{-1}(Z_1)$ (resp. $p \in \theta^{-1}(Z_2)$), then κ_2 (resp. κ_1) is a C^∞ function near p. Note that the other is unbounded near p. Let us set $\rho_i = 1/\kappa_i$ on $D \setminus S(f)$. Then one can see that ρ_1 (resp. ρ_2) can be considered as a C^∞ function near $p \in \theta^{-1}(Z_1)$ (resp. $p \in \theta^{-1}(Z_2)$) ([19, 20]). By using these properties, we define the caustic with respect to κ_i ($i = 1$ or 2) near a singular point $p \in \theta^{-1}(Z_i)$ for f in the next section.

4. Caustics of pseudo-spherical fronts

We investigate a caustic of a pseudo-spherical front associated to the unbounded principal curvature near singular points. In general, caustics of a surface are obtained as loci of centers of principal curvature spheres on the set of regular points (cf. [5]). Let f be a pseudo-spherical front corresponding to the solution θ to (3.1). Take $p \in \theta^{-1}(Z_1)$. Then we investigate the caustic given by

$$C_1 = f + \rho_1\nu \quad (\rho_1 = \cot(\theta/2)), \tag{4.1}$$

which is defined on a sufficiently small neighborhood of p (cf. [20]). For a point $q \in \theta^{-1}(Z_2)$, we can consider another caustic C_2 given by

$$C_2 = f + \rho_2\nu \quad (\rho_2 = -\tan(\theta/2)) \tag{4.2}$$

on a sufficiently small neighborhood of q.

4.1. **Singularities of caustics.** We consider singularities of caustics. We first deal with the caustic C_1 as in (4.1) defined near $p \in \theta^{-1}(Z_1)$. By (3.3) and (3.4), we have

$$\begin{aligned} (C_1)_u &= (\rho_1)_u\nu = -\frac{\theta_u}{2}\csc^2\frac{\theta}{2}\nu, \\ (C_1)_v &= \csc\frac{\theta}{2}\boldsymbol{e}_2 + (\rho_1)_v\nu = \csc\frac{\theta}{2}\left(\boldsymbol{e}_2 - \frac{\theta_v}{2}\csc\frac{\theta}{2}\nu\right). \end{aligned} \tag{4.3}$$

Thus $\boldsymbol{e}_1$ can be taken as a unit normal vector to C_1 and the signed area density function of C_1 is given as

$$\lambda^{C_1} = \frac{\theta_u}{2}\csc^3\frac{\theta}{2} \tag{4.4}$$

by (4.3). We note that C_1 is a front at p ([20, Lemma 4.2], see also [19]). Since $\csc(\theta/2) = \pm 1$ at $p \in \theta^{-1}(Z_1)$, $S(C_1) = (\theta_u)^{-1}(0)$ near $p \in \theta^{-1}(Z_1)$. Thus an identifier of singularities of C_1 is given by

$$\tilde{\lambda}^{C_1} = \theta_u. \tag{4.5}$$

Assume that $p \in \theta^{-1}(Z_1) \cap (\theta_u)^{-1}(0)$. Then we see that $\operatorname{rank} dC_1 = 1$ at p by (4.3). Moreover, one can take a null vector field η^{C_1} for C_1 as

$$\eta^{C_1} = \partial_u \tag{4.6}$$

by (4.3) again.

For the caustic C_2 as in (4.2) defined near $p \in \theta^{-1}(Z_2)$, we observe that

$$(C_2)_u = \sec\frac{\theta}{2}\left(\boldsymbol{e}_1 - \frac{\theta_u}{2}\sec\frac{\theta}{2}\nu\right), \quad (C_2)_v = -\frac{\theta_v}{2}\sec^2\frac{\theta}{2}\nu.$$

Therefore $\boldsymbol{e}_2$ can be taken as a unit normal vector to C_2. Moreover, C_2 is a front at p (see [19, 20]). The signed area density function λ^{C_2} of C_2 is

$$\lambda^{C_2} = \frac{\theta_v}{2}\sec^3\frac{\theta}{2}.$$

Since $S(C_2) = (\theta_v)^{-1}(0)$ locally, rank $dC_2 = 1$ at p where $p \in \theta^{-1}(Z_2) \cap (\theta_v)^{-1}(0)$. Thus $\eta^{C_2} = \partial_v$ gives a null vector field for C_2. Then the following is known.

Fact 4.1 ([20, Proposition 4.4]). *Let $f\colon D \to \mathbb{R}^3$ be a pseudo-spherical front and θ a corresponding solution to* (3.1).

(1) *Assume that $p \in \theta^{-1}(Z_1)$. Then C_1 at p is a cuspidal edge if and only if $\theta_u(p) = 0$ and $\theta_{uu}(p) \neq 0$.*
(2) *Assume that $p \in \theta^{-1}(Z_2)$. Then C_2 at p is a cuspidal edge if and only if $\theta_v(p) = 0$ and $\theta_{vv}(p) \neq 0$.*

By this characterization for a cuspidal edge on the caustic, we get the following.

Proposition 4.2. *If a pseudo-spherical front f at $p \in \theta^{-1}(Z_1)$ (resp. $p \in \theta^{-1}(Z_2)$) is one of a swallowtail, a cuspidal lips or a cuspidal beaks, then the caustic C_1 (resp. C_2) at p is a cuspidal edge. Conversely, if the caustic C_1 (resp. C_2) of a pseudo-spherical front f at $p \in \theta^{-1}(Z_1)$ (resp. $p \in \theta^{-1}(Z_2)$) is a cuspidal edge and $\theta_{uu}(p) \neq \pm\theta_{uv}(p)$, then f is one of a swallowtail, a cuspidal lips or a cuspidal beaks.*

Proof. Assume that a pseudo-spherical front f at $p \in \theta^{-1}(Z_1)$ is one of a swallowtail, a cuspidal lisp or a cuspidal beaks. Then by Fact 3.2 and Proposition 3.3, it holds that $\theta_u(p) = 0$ and $\theta_{uu}(p) \neq 0$. Thus the caustic C_1 at p is a cuspidal edge. For the case of $p \in \theta^{-1}(Z_2)$, one can show in a similar way.

Conversely, we assume that the caustic C_1 of a pseudo-spherical front f at $p \in \theta^{-1}(Z_1)$ is a cuspidal edge and $\theta_{uu}(p) \neq \pm\theta_{uv}(p)$. Then by Fact 4.1, we see that $\theta_u(p) = 0$ and $\theta_{uu}(p) \neq 0$. Hence f at p is one of a swallowtail, a cuspidal lips or a cuspidal beaks by Fact 3.2 and Proposition 3.3. For the case of C_2, the assertion can be shown in a similar fashion. □

We next examine the case where a pseudo-spherical front f is a cuspidal butterfly at $p \in S(f)$.

Proposition 4.3. *Let $f\colon D \to \mathbb{R}^3$ be a pseudo-spherical front and θ a corresponding solution to* (3.1).

(1) *Suppose that f at $p \in \theta^{-1}(Z_1)$ is a cuspidal butterfly. Then the caustic C_1 as in* (4.1) *defined near p satisfies the following*:
   (a) *C_1 at p is a swallowtail if and only if $\theta_{uv} \neq 0$ at p;*
   (b) *C_1 at p is a cuspidal lips if and only if $\theta_{uv} = 0$ and $\theta_{uuu}^2 > \theta_{uuv}^2$ at p;*
   (c) *C_1 at p is a cuspidal beaks if and only if $\theta_{uv} = 0$ and $\theta_{uuu}^2 < \theta_{uuv}^2$ at p.*
(2) *Suppose that f at $p \in \theta^{-1}(Z_2)$ is a cuspidal butterfly. Then the caustic C_2 as in* (4.2) *defined near p satisfies the following*:
   (a) *C_2 at p is a swallowtail if and only if $\theta_{uv} \neq 0$ at p;*
   (b) *C_2 at p is a cuspidal lips if and only if $\theta_{uv} = 0$ and $\theta_{vvv}^2 > \theta_{uvv}^2$ at p;*
   (c) *C_2 at p is a cuspidal beaks if and only if $\theta_{uv} = 0$ and $\theta_{vvv}^2 < \theta_{uvv}^2$ at p.*

Proof. We consider the case (1). By Proposition 3.3, conditions $\theta_u = 0$, $\theta_v \neq 0$, $\theta_{uu} = 0$ and $\theta_{uuu} \neq 0$ are satisfied at p. By (4.5), it holds that $(\tilde{\lambda}_u^{C_1}, \tilde{\lambda}_v^{C_1}) = (\theta_{uu}, \theta_{uv})$. Since a null vector field can be taken as $\eta^{C_1} = \partial_u$ (cf. (4.6)), we see that

$$\eta^{C_1}\tilde{\lambda}^{C_1} = \theta_{uu}, \quad \eta^{C_1}\eta^{C_1}\tilde{\lambda}^{C_1} = \theta_{uuu}.$$

Hence we have the first assertion by Theorem 2.3 (2).

We assume that $\tilde{\lambda}^{C_1}_u = \tilde{\lambda}^{C_1}_v = 0$ at p. This implies that $\theta_{uu} = \theta_{uv} = 0$ at p. Under these conditions, the Hessian of $\tilde{\lambda}^{C_1}$ at p can be calculated as

$$\det \mathcal{H}(\tilde{\lambda}^{C_1})(p) = \theta_{uuu}(p)\theta_{uvv}(p) - \theta_{uuv}(p)^2.$$

On the other hand, by (3.1), we see that

$$\theta_{uuu} - \theta_{uvv} = \theta_u \cos\theta.$$

Thus $\theta_{uvv} = \theta_{uuu}$ at p because $\theta_u(p) = 0$. Hence the Hessian can be rewritten as

$$\det \mathcal{H}(\tilde{\lambda}^{C_1})(p) = \theta_{uuu}(p)^2 - \theta_{uuv}(p)^2.$$

Therefore we have the second and third assertions by Theorem 2.3 (4) and (5).

For (2), one can show similarly by using $\tilde{\lambda}^{C_2} = \theta_v$ and $\eta^{C_2} = \partial_v$. □

4.2. **Geometric invariants at a cuspidal edge of the caustic.** We finally investigate geometric invariants of C_1 defined at a cuspidal edge.

Theorem 4.4. *Let $f\colon D \to \mathbb{R}^3$ be a pseudo-spherical front associated to the solution θ of* (3.1).

(1) *Suppose that the caustic C_1 defined near $p \in \theta^{-1}(Z_1)$ is a cuspidal edge at p. Then the singular curvature $\kappa_s^{C_1}$, cuspidal curvature $\kappa_c^{C_1}$ and cuspidal torsion $\kappa_t^{C_1}$ of C_1 are given as*

$$\kappa_s^{C_1} = \frac{4(\theta_{uu}^2 - \theta_{uv}^2)}{|\theta_{uu}|(4 + \theta_v^2 \csc^2(\theta/2))^{3/2}},$$
$$\kappa_c^{C_1} = -\operatorname{sgn}(\theta_{uu})\frac{(4 + \theta_v^2 \csc^2(\theta/2))^{3/4}\sin^2(\theta/2)}{|\theta_{uu}|^{1/2}}, \qquad \kappa_t^{C_1} = \frac{\theta_{uv}\sin^2(\theta/2)}{\theta_{uu}}$$

on $S(C_1)$ near p.

(2) *Suppose that the caustic C_2 defined near $p \in \theta^{-1}(Z_2)$ is a cuspidal edge at p. Then the singular curvature $\kappa_s^{C_2}$, cuspidal curvature $\kappa_c^{C_2}$ and cuspidal torsion $\kappa_t^{C_2}$ of C_2 are given as*

$$\kappa_s^{C_2} = \frac{4(\theta_{vv}^2 - \theta_{uv}^2)}{|\theta_{vv}|(4 + \theta_u^2 \sec^2(\theta/2))^{3/2}},$$
$$\kappa_c^{C_2} = -\operatorname{sgn}(\theta_{vv})\frac{(4 + \theta_u^2 \sec^2(\theta/2))^{3/4}\cos^2(\theta/2)}{|\theta_{vv}|^{1/2}}, \qquad \kappa_t^{C_2} = \frac{\theta_{uv}\cos^2(\theta/2)}{\theta_{vv}}$$

on $S(C_2)$ near p.

Proof. We show the case (1). By the assumption, the corresponding solution θ to the sine-Gordon equation (3.1) satisfies $\theta_{uu} \neq 0$ at p. Since the signed area density function λ^{C_1} of C_1 is given by (4.4), a vector field defined as

$$X = -\theta_{uv}\partial_u + \theta_{uu}\partial_v$$

is tangent to the set of singular points $S(C_1)$. We denote by Y a null vector field $\eta^{C_1} = \partial_u$. Since $\theta_{uu} \neq 0$ at p, the pair (X, Y) is linearly independent at p. Moreover, we see that

$$\operatorname{sgn}(Y\lambda^{C_1} \cdot \det(X, Y)) = \operatorname{sgn}\left(-\theta_{uu}^2 \csc^3\frac{\theta}{2}\right) = -\operatorname{sgn}\left(\csc\frac{\theta}{2}\right) \tag{4.7}$$

on $S(C_1)$ near p.

We calculate directional derivatives of C_1 in the direction X. By straightforward calculations, we have

$$
\begin{aligned}
XC_1 &= -\theta_{uv}(C_1)_u + \theta_{uu}(C_1)_v, \\
XXC_1 &= \theta_{uv}^2(C_1)_{uu} - 2\theta_{uv}\theta_{uu}(C_1)_{uv} + \theta_{uu}^2(C_1)_{vv} \\
&\quad + (\theta_{uv}\theta_{uuv} - \theta_{uu}\theta_{uvv})(C_1)_u + (\theta_{uu}\theta_{uuv} - \theta_{uv}\theta_{uuu})(C_1)_v.
\end{aligned} \tag{4.8}
$$

On the other hand, by (3.1), (4.1) and (4.3), it holds that

$$
\begin{aligned}
(C_1)_{uu} &= \frac{\theta_u}{2}\csc\frac{\theta}{2}\boldsymbol{e}_1 - \frac{\theta_{uu} - \theta_u^2\cot(\theta/2)}{2}\csc^2\frac{\theta}{2}\nu, \\
(C_1)_{uv} &= -\frac{\theta_u}{2}\cot\frac{\theta}{2}\csc\frac{\theta}{2}\boldsymbol{e}_2 - \frac{\theta_{uv} - \theta_u\theta_v\cot(\theta/2)}{2}\csc^2\frac{\theta}{2}\nu, \\
(C_1)_{vv} &= -\frac{\theta_u}{2}\csc\frac{\theta}{2}\boldsymbol{e}_1 - \theta_v\cot\frac{\theta}{2}\csc\frac{\theta}{2}\boldsymbol{e}_2 - \frac{1}{2}\csc^2\frac{\theta}{2}\left(\theta_{vv} - \theta_v^2\cot(\theta/2) + \sin\theta\right)\nu \\
&= -\frac{\theta_u}{2}\csc\frac{\theta}{2}\boldsymbol{e}_1 - \theta_v\cot\frac{\theta}{2}\csc\frac{\theta}{2}\boldsymbol{e}_2 - \frac{1}{2}\csc^2\frac{\theta}{2}\left(\theta_{uu} - \theta_v^2\cot(\theta/2)\right)\nu.
\end{aligned} \tag{4.9}
$$

By (4.3), (4.8) and (4.9), we have

$$
\begin{aligned}
XC_1 &= \theta_{uu}\left(\csc\frac{\theta}{2}\boldsymbol{e}_2 - \frac{\theta_v}{2}\csc^2\frac{\theta}{2}\nu\right), \\
XXC_1 &= -\theta_v\theta_{uu}^2\cot\frac{\theta}{2}\csc\frac{\theta}{2}\boldsymbol{e}_2 + \frac{\theta_{uu}}{2}\csc^2\frac{\theta}{2}\left(\theta_{uv}^2 - \theta_{uu}^2 + \theta_v^2\theta_{uu}\cot\frac{\theta}{2}\right)\nu \\
&\quad + (\theta_{uu}\theta_{uuv} - \theta_{uv}\theta_{uuu})\left(\csc\frac{\theta}{2}\boldsymbol{e}_2 - \frac{\theta_v}{2}\csc^2\frac{\theta}{2}\nu\right)
\end{aligned}
$$

on $S(C_1)$. Thus we get

$$
\begin{aligned}
|XC_1| &= \frac{|\theta_{uu}|}{2}\left|\csc\frac{\theta}{2}\right|\sqrt{4 + \theta_v^2\csc^2\frac{\theta}{2}}, \\
\det(XC_1, XXC_1, \boldsymbol{e}_1) &= \frac{\theta_{uu}^2}{2}\csc^3\frac{\theta}{2}\left(\theta_{uv}^2 - \theta_{uu}^2\right)
\end{aligned} \tag{4.10}
$$

on $S(C_1)$. By (4.7) and (4.10), the singular curvature $\kappa_s^{C_1}$ of C_1 can be calculated as

$$
\kappa_s^{C_1} = \frac{4(\theta_{uu}^2 - \theta_{uv}^2)}{|\theta_{uu}|(4 + \theta_v^2\csc^2(\theta/2))^{3/2}}
$$

on $S(C_1)$.

We next consider the cuspidal curvature $\kappa_c^{C_1}$. Since $Y = \partial_u$ gives a null vector field, we have $YYC_1 = (C_1)_{uu}$. Along $S(C_1) = (\theta_u)^{-1}(0)$, we obtain

$$
XC_1 \times YYC_1 = -\frac{\theta_{uu}^2}{2}\csc^3\frac{\theta}{2}\boldsymbol{e}_1 \tag{4.11}
$$

by (4.3) and (4.9). Thus we get

$$
|XC_1 \times YYC_1| = \frac{\theta_{uu}^2}{2}\left|\csc\frac{\theta}{2}\right|^3
$$

on $S(C_1)$. To obtain the cuspidal curvature, we calculate $YYYC_1$ on $S(C_1)$. Since $Y = \partial_u$, we have

$$
YYYC_1 = (C_1)_{uuu} = -\frac{\theta_{uuu}}{2}\csc^2\frac{\theta}{2}\nu + \theta_{uu}\csc\frac{\theta}{2}\boldsymbol{e}_1
$$

on $S(C_1)$. By (4.11), we see that

$$\det(XC_1, YYC_1, YYYC_1) = -\frac{\theta_{uu}^3}{2}\csc^4\frac{\theta}{2}$$

on $S(C_1)$. Hence it holds that

$$\kappa_c^{C_1} = -\operatorname{sgn}(\theta_{uu})\frac{(4+\theta_v^2\csc^2(\theta/2))^{3/4}}{|\theta_{uu}|^{1/2}\csc^2(\theta/2)} = -\operatorname{sgn}(\theta_{uu})\frac{(4+\theta_v^2\csc^2(\theta/2))^{3/4}\sin^2(\theta/2)}{|\theta_{uu}|^{1/2}}$$

on $S(C_1)$.

We finally consider the cuspidal torsion $\kappa_t^{C_1}$ along $S(C_1)$. By the above calculation, we have

$$\det(XC_1, YYC_1, XXC_1) = 0$$

on $S(C_1)$. Thus the cuspidal torsion $\kappa_t^{C_1}$ can be calculated as

$$\kappa_t^{C_1} = \frac{\det(XC_1, YYC_1, XYYC_1)}{|XC_1 \times YYC_1|^2}$$

on $S(C_1)$. By direct computations, we see that

$$XYYC_1 = -\theta_{uv}(C_1)_{uuu} + \theta_{uu}(C_1)_{uuv}.$$

By (4.3), it holds that

$$(C_1)_{uuv} = \frac{1}{4}\csc\frac{\theta}{2}\left(2\theta_{uv} - \theta_u\theta_v\cot\frac{\theta}{2}\right)\boldsymbol{e}_1 + A\boldsymbol{e}_2 + B\nu,$$

where A and B are some functions. Thus we have

$$\begin{aligned} XYYC_1 &= -\theta_{uu}\theta_{uv}\csc\frac{\theta}{2}\boldsymbol{e}_1 + \frac{\theta_{uu}\theta_{uv}}{2}\csc\frac{\theta}{2}\boldsymbol{e}_1 + \tilde{A}\boldsymbol{e}_2 + \tilde{B}\nu \\ &= -\frac{\theta_{uu}\theta_{uv}}{2}\csc\frac{\theta}{2}\boldsymbol{e}_1 + \tilde{A}\boldsymbol{e}_2 + \tilde{B}\nu \end{aligned}$$

on $S(C_1)$, where $\tilde{A}$ and $\tilde{B}$ are some functions on $S(C_1)$. Therefore it follows that

$$\kappa_t^{C_1} = \frac{4\theta_{uu}^3\theta_{uv}\csc^4(\theta/2)}{4\theta_{uu}^4\csc^6(\theta/2)} = \frac{\theta_{uv}}{\theta_{uu}\csc^2(\theta/2)} = \frac{\theta_{uv}\sin^2(\theta/2)}{\theta_{uu}}$$

on $(\theta_u)^{-1}(0)$. Thus we obtain the assertion.

For (2), one can show in a similar way by using $X = \theta_{vv}\partial_u - \theta_{uv}\partial_v$ and $Y = \partial_v$. □

We remark that for the caustic C_i ($i = 1$ or 2) defined near $p \in \theta^{-1}(Z_i)$, it holds that

$$4(\kappa_t^{C_i})^2 + \kappa_s^{C_i}(\kappa_c^{C_i})^2 = -4K^{C_i}$$

on $S(C_i)$, where $K^{C_1} = -\sin^4(\theta/2)$ and $K^{C_2} = -\cos^4(\theta/2)$, respectively (cf. [12, Remark 3.19], [17, Theorem 3.8] and [20, Theorem 4.5]).

Example 4.5. Let $\theta\colon \mathbb{R}^2 \to \mathbb{R}$ be a function given by

$$\theta(u, v) = -4\arctan(v\operatorname{sech} u),$$

which satisfies (3.1). Then the corresponding pseudo-spherical front f is *Kuen's surface* (cf. [15]) (see Figure 3). The set of singular points of f is

$$S(f) = \{v = 0\} \cup \{v = \pm\cosh u\} = S_1 \cup S_2^{\pm}.$$

We notice that $\theta(u, 0) = 0$ and $\theta(u, \pm\cosh u) = \mp\pi$. Thus $S_1 = \theta^{-1}(0)$ and $S_2^{\pm} = \theta^{-1}(\mp\pi)$. Since $\theta_u(u, 0) = 0$ and $\theta_v(u, 0) = -\operatorname{sech} u \neq 0$, f is a cuspidal edge at $(u, 0) \in S_1$. On the other hand, it holds that

$$\theta_u(u, \pm\cosh u) = \pm 2\tanh u, \quad \theta_v(u, \pm\cosh u) = -2\operatorname{sech} u \neq 0.$$

Thus we have $\theta_u(0,\pm1) = 0$ and $\theta_v(0,\pm1) = -2 \neq 0$, where $(0,\pm1) \in S_2$. Moreover, we get $\theta_{uu}(u, \pm\cosh u) = \pm2\,\mathrm{sech}^2\,u$. Thus we see that $\theta_{uu}(0,\pm1) = \pm2 \neq 0$. This implies that f at $(0,\pm1) \in S_2^{\pm}$ is a swallowtail. Hence the caustics $C^{\pm}$ defined near $(0,\pm1)$ are cuspidal edges at $(0,\pm1)$, respectively. In addition, we obtain $\theta_{uv}(0,\pm1) = 0$. Hence the singular curvatures of the caustics $C^{\pm}$ are positive at $(0,\pm1)$ by Theorem 4.4. In particular, it holds that

$$\kappa_s^{C^{\pm}} = \frac{1}{2\sqrt{2}} > 0$$

at $(0,\pm1)$ (see Figure 3 again).

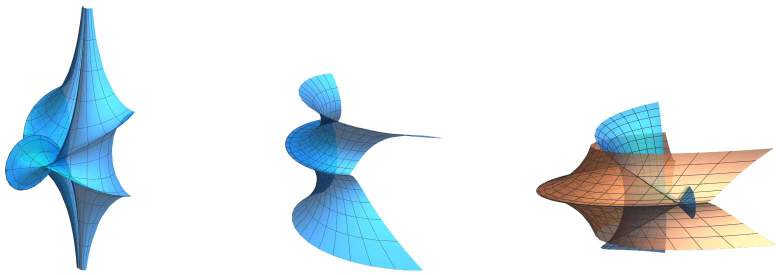

FIGURE 3. Kuen's surface (left), its caustic C^- defined near the point $(0,-1)$ (middle) and both of them (right).

When the initial pseudo-spherical front has either a cuspidal lips or a cuspidal beaks, we have the following.

Corollary 4.6. *Let $f\colon D \to \mathbb{R}^3$ be a pseudo-spherical front associated to the solution θ of the sine-Gordon equation* (3.1).

(1) *If f at $p \in \theta^{-1}(Z_i)$ $(i = 1$ or $2)$ is a cuspidal lips, then $\kappa_s^{C_i} > 0$ at p.*
(2) *If f at $p \in \theta^{-1}(Z_i)$ $(i = 1$ or $2)$ is a cuspidal beaks, then $\kappa_s^{C_i} < 0$ and $\kappa_t^{C_i} \neq 0$ at p.*

Proof. By Proposition 3.3 and Theorem 4.4, we have the assertions for the singular curvature. When f at p is a cuspidal beaks, it holds that $\theta_{uv}(p) \neq 0$ by Proposition 3.3. Thus by Theorem 4.4, $\kappa_t^{C_i} \neq 0$ at p. □

Acknowledgements. The author would like to thank Professors Shintaro Akamine, Kentaro Saji and Masatomo Takahashi for fruitful discussions and comments. He is also grateful to the referee for valuable comments and suggestions. This work was supported by JSPS KAKENHI Grant Numbers JP22K13914 and JP22KK0034.

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