博士論文

On direct sum decompositions of lifting modules
(lifting 加群の直和分解について)

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柴田　義大

山口大学大学院創成科学研究科

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Introduction

The origin of the ring theory can be traced back to the early 19th century. In particular, the field matured around the 1920s when the structural theorem of semisimple rings was reported.

In 1953, Eckmann and Schopf [8] reported that every module over an arbitrary ring is embedded in an injective module as an essential submodule. Such injective module is called the injective hull of the module, and it is used to study the structural properties of various modules. Conversely, the projective cover, which is a dual of the injective hull, does not exist for all modules over any ring. In fact, such example is easily constructed over the integer ring $\mathbb{Z}$. Therefore, in 1960, Bass [4] considered a ring whose any (finitely generated) module has the projective cover, and named it a (semi)perfect ring. Three years later, Mares [28] considered "a projective module of which any factor module has the projective cover", as a generalization of semiperfect rings, and named it a semiperfect module. On the other hand, Harada [13] and Oshiro [31] focused on the remarkable property of semiperfect rings "all idempotents of the factor ring by the radical are lifted to idempotents of the original ring", and considered the property "all direct summands of the factor module by the radical are lifted to direct summands of the original module" from the viewpoint of modules. In 1983, Oshiro [31] considered "all direct summands of the factor module by any submodule are lifted to direct summands of the original module" and introduced semiperfect modules, quasi-semiperfect modules and lifting modules, as generalizations of semiperfect modules proposed by Mares. After that, Mohamed and Müller [29] called the Oshiro's (quasi-)semiperfect modules as (quasi-)discrete modules to distinguish semiperfect modules by Mares and those by Oshiro. The findings obtained from the aforementioned studies have been applied to structural research of various rings, such as (semi)perfect rings, quasi-Frobenius rings, Harada rings and Nakayama rings. Many researchers both in Japan and abroad are continuing the study on the aforementioned ones owing to its importance. This study focuses on the direct sum decomposition of lifting modules.

In general, a direct sum of lifting modules is not always lifting. For (quasi-)discrete modules which are types of lifting modules, it is characterized by the relative projectivity that these modules are closed under direct sums (see [6, 26.22 and 27.4]). Although the relative projectivity implies a condition that a direct sum of lifting modules is lifting, but the converse does not hold. Therefore, Harada and Tozaki considered that a direct sum of lifting modules being lifting might be characterized by some kind of projectivity which is weaker than the relative projectivity, and introduced the almost projectivity as follows: A module M is called an almost N-projective for a module N if, for any module X, any homomorphism $f : M \to X$, and any epimorphism $g : N \to X$; (i) there exists a homomorphism $h : M \to N$ such that $f = gh$ or (ii) there exist a nonzero direct summand N' of N and a homomorphism $h' : N' \to M$ such that $fh' = g|_{N'}$.

$$\begin{array}{cccc} & & M & \\ & h \swarrow & \downarrow f & \\ N & \overset{g}{\to} & X & \to 0 \end{array} \qquad \text{or} \qquad \begin{array}{cccc} N' & \overset{h'}{\to} & M & \\ \hookdownarrow & & \downarrow f & \\ N & \overset{g}{\to} & X & \to 0 \end{array}$$

In 1990, Baba and Harada [3] reported the following result associated with direct sums of lifting modules using almost projectivity.

Theorem. Let $M_1, M_2, \ldots, M_n$ be lifting modules whose their endomorphism rings are local. Then $\oplus_{i=1}^n M_i$ is lifting if and only if M_i is almost M_j-projective for distinct $i, j \in \{1, 2, \ldots, n\}$.

After the almost projectivity was introduced by Harada and Tozaki, Baba introduced the almost injectivity as the dual of the almost projectivity. Following that, in 2002, Hanada, Kuratomi and Oshiro [12] introduced the generalized injectivity, which is more precise than the almost injectivity, as a necessary and sufficient condition for a direct sum of extending modules with the finite internal exchange property to be extending with the finite internal exchange property. In 2004, Mohamed and Müller [30] introduced the generalized projectivity as follows as the dual of the generalized injectivity, and investigated a direct sum of lifting modules under some kind of conditions. A module M is called generalized N-projective for a module N if, for any module X, any homomorphism $f : M \to X$, and any epimorphism $g : N \to X$, there exist decompositions $M = M_1 \oplus M_2$, $N = N_1 \oplus N_2$, a homomorphism $h_1 : M_1 \to N_1$, and an epimorphism $h_2 : N_2 \to M_2$ such that $gh_1 = f|_{M_1}$ and $fh_2 = g|_{N_2}$.

$$\begin{array}{ccccccccc} M_1 & \oplus & M_2 & & = & & M & & \\ h_1 \downarrow & & h_2 \uparrow & & & & \downarrow f & & \\ N_1 & \oplus & N_2 & = & N & \stackrel{g}{\to} & X & \to 0 & \end{array}$$

In 2005, Kuratomi [24] obtained the following results:

Theorem. Let $M_1, M_2, \ldots, M_n$ be lifting modules with the finite internal exchange property. Then $\oplus_{i=1}^{n} M_i$ is lifting with the finite internal exchange property if and only if M_i is generalized M/M_i-projective for each $i = 1, 2, \ldots, n$.

Almost of known extending modules satisfy the finite internal exchange property, but not all known extending modules satisfy the finite internal exchange property. However, the existence of lifting modules which do not satisfy the finite internal exchange property has not been confirmed for a long time.

In chapter 1, we solve the following problem negatively:

Does any lifting module satisfy the finite internal exchange property? $\cdots$ (#)

In 1969, Warfield [33] reported that, for an indecomposable module M, if $M^2 = M \times M$ satisfies the finite internal exchange property, then the endomorphism ring of M is local. Hence, if we can confirm an indecomposable module whose its endomorphism ring is not local and the square of the module is lifting, then we can make an example of a lifting module which does not satisfy the finite internal exchange property. Now, we introduce a certain projectivity, which is different from the almost projectivity and the generalized projectivity, as a necessary and sufficient condition for the square of a specific lifting module to be lifting:

Theorem A. Let U be a uniserial module. Then, the following are equivalent:

(a) U^2 is lifting,

(b) for any module X, any homomorphism f from U to X, any epimorphism g from U to X, (i) there exists an endomorphism h of U, such that $f = gh$ or (ii) there exists a submodule N of U and an epimorphism h' from N to U, such that $fh' = g|_N$.

$$\begin{array}{ccc} & & U \\ {}^{\exists}h \swarrow & \circlearrowright & \downarrow f \\ U & \overset{g}{\twoheadrightarrow} & X \end{array} \quad \text{or} \quad \begin{array}{ccc} {}^{\exists}N & \overset{{}^{\exists}h'}{\twoheadrightarrow} & U \\ \downarrow & \circlearrowright & \downarrow f \\ U & \overset{g}{\twoheadrightarrow} & X \end{array}$$

Let p and q be distinct prime numbers, put a semiperfect ring $R = \begin{pmatrix} \mathbb{Z}_{(p)} & \mathbb{Q} \\ 0 & \mathbb{Z}_{(q)} \end{pmatrix}$ and a right ideal $L = \begin{pmatrix} 0 & \mathbb{Z}_{(q)} \\ 0 & \mathbb{Z}_{(q)} \end{pmatrix}$ of R, and consider $U = R/L$. Then U is uniserial and its endomorphism ring is not local (see [9]). Therefore, based on the finding reported by Warfield, U^2 does not satisfy the finite internal exchange property. Moreover, U^2 is lifting according to Theorem A, and hence the problem (#) is negatively solved over a semiperfect ring. On the other hand, Kikumasa and Kuratomi [21] proved that "any lifting module over a right artinian ring satisfies the external exchange property (and thus the finite internal exchange property)". Hence, we should consider the problem (#) over a right perfect ring or a semiprimary ring, which are stronger than a semiperfect ring and weaker than a right artinian ring.

In chapter 2, we first introduce the concepts of a dual square free module and a factor square full module, and give their fundamental properties. In 2021, Kuratomi [26] has showed that any lifting module over a right perfect ring is a direct sum of a dual square free module and factor square full module such that they have no isomorphic nonzero factor modules, and any dual square free lifting module over a right perfect ring is quasi-discrete. In the end of this chapter, we apply these results to give the necessary and sufficient condition for a lifting module over a right perfect ring to satisfy the finite internal exchange property.

In chapter 3, we consider when is a direct sum of lifting modules over a right perfect ring to be lifting. As aforementioned, the almost projectivity and the generalized projectivity are strongly associated with the structure of lifting modules. In this chapter, we first give new characterizations of these projectivities by the projective covers as follows:

Theorem B. Let M and N be modules over a right perfect ring and let (P, φ) and (Q, ψ) be projective covers of M and N, respectively. Then the following conditions are equivalent, respectively:

(1) (a) M is almost N-projective,

(b) for any $\alpha \in \mathrm{Hom}_R(P, Q)$, either $\alpha(\mathrm{Ker}\,\varphi) \subseteq \mathrm{Ker}\,\psi$, or there exist $P' \leq_{\oplus} P$ and $Q' \leq_{\oplus} Q$ such that $0 \neq \psi(Q') \leq_{\oplus} N$, $\alpha|_{P'} : P' \to Q'$ is an isomorphism and $(\alpha|_{P'})^{-1}(\mathrm{Ker}\,\psi|_{Q'}) \subseteq \mathrm{Ker}\,\varphi|_{P'}$.

(2) (a) M is generalized N-projective,

(b) for any $\alpha \in \mathrm{Hom}_R(P, Q)$, there exist decompositions $P = P_1 \oplus P_2$ and $Q = Q_1 \oplus Q_2$ such that $\alpha(P_1) \subseteq Q_1$, $\alpha(\mathrm{Ker}\,\varphi|_{P_1}) \subseteq \mathrm{Ker}\,\psi|_{Q_1}$, $\alpha|_{P_2} : P_2 \to Q_2$ is an isomorphism, $(\alpha|_{P_2})^{-1}(\mathrm{Ker}\,\psi|_{Q_2}) \subseteq \mathrm{Ker}\,\varphi|_{P_2}$, $M = \varphi(P_1) \oplus \varphi(P_2)$ and $N = \psi(Q_1) \oplus \psi(Q_2)$.

After that, using Theorem B, we investigate the relationship between the almost projectivity and the generalized projectivity, and conditions for these projectivities to close under direct sums, over a right perfect ring. Finally, we give a condition for a direct sum of lifting modules over a right perfect ring to be lifting.

In the Appendix, we provide details about im-small projective, im-closed projective and im-summand projective related to almost projective and generalized projective, and rings whose factor square full modules are closed under essential extensions or essential submodules.

序論

非可換環論の起源は 19 世紀初頭にまで遡ることができるが, 分野として成熟するのは半単純環の構造定理が報告された 1920 年代頃である.

1953 年, Eckmann と Schopf [8] により, “ 任意の環上のすべての加群はある移入加群に稠密に含まれる ” という驚くべき結果が示された. このような移入加群は, その加群の移入包絡とよばれ, 様々な加群の構造研究に活用されている. 一方で, 移入包絡の双対である射影被覆は, 任意の環上のすべての加群において存在するわけではない. 実際, 射影被覆をもたない加群の例は, 整数環 $\mathbb{Z}$ 上で容易に構成される. そこで 1960 年に Bass [4] は, 任意の (有限生成) 加群が射影被覆をもつ環を考察し, これを (半) 完全環と名付けた. その 3 年後 Mares [28] は, 半完全環の一般化として “ 任意の剰余加群が射影被覆をもつ射影加群 ” を考察し, 半完全加群と名付けた. 一方で Harada [13] や Oshiro [31] は, 半完全環のもつ顕著な性質 “ 根基による剰余環の冪等元がもとの環の冪等元に持ち上がる ” を加群の視点で “ 根基による剰余加群の直和因子がもとの加群の直和因子に持ち上がる ” とみることで加群のある種の lifting 性を考察した. 1983 年には Oshiro [31] は, “ 任意の部分加群による剰余加群の直和因子がもとの加群の直和因子に持ち上がる ” という状況を考察し, Mares の半完全加群をさらに一般化させる形で, 一般の加群の中で半完全加群, 準半完全加群, lifting 加群なるものを導入した. その後 Mohamed と Müller [29] は, Mares と Oshiro による半完全加群を区別するために, Oshiro による (準) 半完全加群を (準) 離散加群とよんだ. これらの加群の研究は, (半) 完全環や準フロベニウス環, 原田環, 中山環など様々な環の構造研究に応用され, その重要性から国内外の多くの研究者らによって連綿と研究が続けられている. 本論文では, lifting 加群の直和分解に焦点を当てた研究を行う.

一般に, lifting 加群の直和は lifting 加群になるとは限らない. ある種の lifting 加群である (準) 離散加群については, その直和がまた (準) 離散加群になることが相互射影性により特徴づけられる (see [6, 26.22 and 27.4]). しかし lifting 加群においてその直和が lifting になるという状況を考えたとき, 相互射影性では条件が強すぎて必要十分条件には成りえない. そこで lifting 加群の直和がまた lifting 加群になるための必要十分条件が, 相互射影性よりも弱い何らかの射影性で与えられるのでは

ないか, という視点から, Harada と Tozaki [16] により, almost projectivity なる射影性が導入された. この射影性は次で定義される: M, N を加群とするとき, M が almost N-projective であるとは, 任意の加群 X, M から X への任意の準同型 f, N から X への任意の全射準同型 g に対し, (i) M から N への準同型 h で $f = gh$ をみたすものが存在する, もしくは (ii) N の非零直和因子 N' と N' から M への準同型 h' が存在し $fh' = g|_{N'}$ をみたす, のいずれかが成り立つときをいう.

$$\begin{array}{ccccc} & & M & & \\ & h \swarrow & \downarrow f & & \\ N & \xrightarrow{g} & X & \to 0 & \end{array} \qquad \text{or} \qquad \begin{array}{cccc} N' & \xrightarrow{h'} & M & \\ \hookdownarrow & & \downarrow f & \\ N & \xrightarrow{g} & X & \to 0 \end{array}$$

1990 年, Baba と Harada [3] はこの almost projectivity を用いて lifting 加群の直和に関する次の結果を示した:

定理. $M_1, M_2, \ldots, M_n$ を自己準同型環が局所環である lifting 加群とするとき, $\oplus_{i=1}^n M_i$ が lifting 加群であることと, 相異なる $i, j \in \{1, 2, \ldots, n\}$ に対し M_i が almost M_j-projective であることが同値である.

Harada と Tozaki により almost projectivity が導入されたのち, 双対である almost injectivity も Baba [2] により導入され, 上の定理の双対が得られている. その後 2002 年に Hanada, Kuratomi, Oshiro [12] は, 有限内部交換性をみたす extending 加群の直和がまた有限内部交換性をみたす extending 加群になるための必要十分条件として, almost injectivity よりもさらに精密な generalized injectivity なる移入性を導入した. 2004 年に Mohamed と Müller [30] は generalized injectivity の双対として次の generalized projectivity を導入し, ある条件下で lifting 加群の直和の研究を行った. M, N を加群とするとき, M が generalized N-projective であるとは, 任意の加群 X, M から X への任意の準同型 f, N から X への任意の全射準同型 g に対し, M の直和分解 $M = M_1 \oplus M_2$, N の直和分解 $N = N_1 \oplus N_2$, M_1 から N_1 への準同型 h_1, N_2 から M_2 への全射準同型 h_2 が存在し $gh_1 = f|_{M_1}, fh_2 = g|_{N_2}$ をみ

たすときをいう.

$$
\begin{array}{ccccccccc}
M_1 & \oplus & M_2 & & = & & M & & \\
g_1\downarrow & & g_2\uparrow & & & & \downarrow f & & \\
N_1 & \oplus & N_2 & = & N & \overset{\pi}{\to} & N/X & \to 0 &
\end{array}
$$

そして 2005 年, Kuratomi [24] により次が得られた:

定理. $M_1, M_2, \ldots, M_n$ を有限内部交換性をみたす lifting 加群とするとき, $\oplus_{i=1}^n M_i$ が有限内部交換性をみたす lifting 加群であることと, 各 $i = 1, 2, \ldots, n$ に対し M_i が generalized M/M_i-projective であることが同値である.

既知の extending 加群の多くは有限内部交換性をみたすが, すべての extending 加群が有限内部交換性をみたすとは限らない. しかし, 有限内部交換性をみたさない lifting 加群は長らく存在が確認されていなかった.

第 1 章では, 以下の問題を否定的に解決する:

すべての lifting 加群は有限内部交換性をみたすか？ $\cdots$ (#)

1969 年に Warfield [33] は, 直既約加群 M に対し, $M^2 = M \times M$ が有限内部交換性をみたすならば M の自己準同型環は局所環である, という有限内部交換性に関する有用な結果を与えた. これにより, 自己準同型環が局所環でない直既約 lifting 加群の平方が lifting 加群であることが示されれば, それが有限内部交換性をみたさない lifting 加群の例となる. ここでは, ある種の lifting 加群の平方がまた lifting 加群となるための必要十分条件として, almost projectivity, generalized projectivity とは異なる射影性を導入する. 定理の主張は以下である:

定理 A. U を単列加群とするとき, 次が同値である:

(a) U^2 が lifting 加群である,

(b) 任意の加群 X, U から X への任意の準同型 f, U から X への任意の全射準同型 g に対し, (i) U の自己準同型 h が存在し $f = gh$ をみたす, もしくは (ii) U

の部分加群 N と N から U への全射準同型 h' が存在し $fh' = g|_N$ をみたす.

$$\begin{array}{ccccccccc} & & U & & & {}^{\exists}N & \overset{\exists h'}{\twoheadrightarrow} & U \\ & {}^{\exists}h \swarrow \circlearrowright & \downarrow f & \text{or} & & \hookdownarrow & \circlearrowright & \downarrow f \\ U & \overset{g}{\twoheadrightarrow} & X & & & U & \overset{g}{\twoheadrightarrow} & X \end{array}$$

p, q を相異なる素数とし, 環 R を $\begin{pmatrix} \mathbb{Z}_{(p)} & \mathbb{Q} \\ 0 & \mathbb{Z}_{(q)} \end{pmatrix}$ により定め, L を R の右イデアル $\begin{pmatrix} 0 & \mathbb{Z}_{(q)} \\ 0 & \mathbb{Z}_{(q)} \end{pmatrix}$ とする. このとき R は半完全環であり, $U = R/L$ とおくと U は自己準同型環が局所環でない単列加群であるため (see [9]), 上記 Warfield の結果より U^2 は有限内部交換性をみたさない. また定理 A から U^2 は lifting 加群であることがわかる. これによって，半完全環上で問題 (#) は否定的に解決される. 一方，Kikumasa と Kuratomi [21] は“右アルチン環上の lifting 加群は外部交換性を（よって有限内部交換性も）みたす”ことを示している. そこで新たに“右完全環上や半準素環上で問題 (#) は肯定的に解決されるか?”が浮上する.

第 2 章では, まず dual square free, factor square full なる加群を導入し, その基本性質を紹介する. 2021 年に Kuratomi [26] により, 右完全環上の lifting 加群が, 同型な非零剰余加群をもたない dual square free 加群と factor square full 加群の直和に分解されることと, 右完全環上の dual square free lifting 加群が準離散であることが示されている. 本章ではこれらの結果を応用し, 右完全環上の lifting 加群が有限内部交換性をみたすための必要十分条件を与える.

第 3 章では, 右完全環上の lifting 加群の直和がいつ lifting 加群になるかについて考察する. 上記の通り, almost projectivity, generalized projectivity は lifting 加群の構造に強く関連した射影性である. 本章ではまず, これら 2 つの射影性の射影被覆による次の特徴づけを与える.

定理 B. R を右完全環とし, M, N を加群, $(P, \varphi), (Q, \psi)$ をそれぞれ M, N の射影被覆とする. このとき, 次が同値である.

(1) (a) M は almost N-projective である,

(b) P から Q への任意の準同型 α に対し, (i) $\alpha(\operatorname{Ker}\varphi) \subseteq \operatorname{Ker}\psi$ が成り立つ, もしくは (ii) P の直和因子 P' と Q の直和因子 Q' が存在して, $0 \neq \psi(Q') \leq_{\oplus} N$, $\alpha|_{P'}$ により P' と Q' は同型, $(\alpha|_{P'})^{-1}(\operatorname{Ker}\psi|_{Q'}) \subseteq \operatorname{Ker}\varphi|_{P'}$ をみたす.

(2) (a) M は generalized N-projective である,

(b) P から Q への任意の準同型 α に対し, P の直和分解 $P = P_1 \oplus P_2$ と Q の直和分解 $Q = Q_1 \oplus Q_2$ が存在して, $\alpha(P_1) \subseteq Q_1$, $\alpha(\operatorname{Ker}\varphi|_{P_1}) \subseteq \operatorname{Ker}\psi|_{Q_1}$, $\alpha|_{P_2}$ により P_2 と Q_2 は同型, $(\alpha|_{P_1})^{-1}(\operatorname{Ker}\psi|_{Q_2}) \subseteq \operatorname{Ker}\varphi|_{P_2}$, $M = \varphi(P_1) \oplus \varphi(P_2)$, $N = \psi(Q_1) \oplus \psi(Q_2)$ をみたす.

その後, 定理 B を用いることで, これら 2 つの射影性の関係や直和で閉じるための条件を調べ, 最後に右完全環上の lifting 加群が直和で閉じるための必要十分条件を与える.

付録では, almost projective や generalized projective に関連する im-small projective, im-closed projective, im-summand projective について, また factor square full 加群が本質拡大や本質部分加群で閉じる環についての詳細を記載している。

0 Preliminaries

Throughout this dissertation, we consider an associative ring R with identity, and all modules considered are unitary right R-modules. $N \leq_{\oplus} M$ means that N is a direct summand of a module M. A submodule N of a module M is said to be *essential* in M (or an *essential submodule* of M) if $N \cap X$ is nonzero for any nonzero submodule X of M and we denote by $N \subseteq_e M$ in this case. A submodule N of a module M is said to be *small* in M (or a *small submodule* of M) if $N + X$ is proper for any proper submodule X of M and we denote by $N \ll M$ in this case. A homomorphism $f : M \to N$ is called a *small epimorphism* if f is onto and $\operatorname{Ker} f \ll M$.

Lemma 0.1 ([1, Propositions 5.16, 5.17 and 5.20])

(1) Let K and N be submodules of a module M with $K \subseteq N$, then

1. *$K \subseteq_e N$ and $N \subseteq_e M$ if and only if $K \subseteq_e M$.*
2. *$N \ll M$ if and only if $K \ll M$ and $N/K \ll M/K$.*

(2) Let K and N be submodules of a module M, then

1. *$K \subseteq_e M$ and $N \subseteq_e M$ if and only if $K \cap N \subseteq_e M$.*
2. *$K \ll M$ and $N \ll M$ if and only if $K + N \ll M$.*

(3) Let N_i be a submodule of a module M_i ($i = 1, 2, \ldots, n$). Then

1. *$N_i \subseteq_e M_i$ for each $i = 1, 2, \ldots, n$ if and only if $\oplus_{i=1}^n N_i \subseteq_e \oplus_{i=1}^n M_i$.*
2. *$N_i \ll M_i$ for each $i = 1, 2, \ldots, n$ if and only if $\oplus_{i=1}^n N_i \ll \oplus_{i=1}^n M_i$.*

(4) A submodule N of a module M is essential in M if and only if, for any nonzero element x of M, there exists an element r of R such that $0 \neq xr \in N$.

A module M is called *N-projective* for a module N if, for any module X, any homomorphism $f : M \to X$ and any epimorphism $g : N \to X$, there exists a homomorphism $h : M \to N$ such that $f = gh$. In particular, M is called *projective* if it is N-projective for any module N.

$$\begin{array}{cccc} & & M & \\ & h \swarrow \circlearrowright & \downarrow f & \\ N & \xrightarrow{g} & X & \to 0 \end{array}$$

Dually, a module M is called *N-injective* for a module N if, for any module X, any homomorphism $f : X \to M$ and any monomorphism $g : X \to N$, there exists a homomorphism $h : N \to M$ such that $f = hg$. In particular, M is called *injective* if it is N-injective for any module N.

$$\begin{array}{cccc} 0 \to & X & \xrightarrow{g} & N \\ & f \downarrow & \circlearrowright \swarrow h & \\ & M & & \end{array}$$

Lemma 0.2 ([29, Propositions 1.3, 1.5, 1.6, 4.31, 4.32 and 4.33]) *Let M, N and N_i be modules ($i = 1, 2, \ldots, n$). Then the following holds.*

(1) If M is N-projective, then it is N'-projective and N/N'-projective for any submodule N' of N.

(2) N-projective modules are closed under direct summands and direct sums.

(3) If M is N_i-projective for any $i \in \{1, 2, \ldots, n\}$, then M is $\oplus_{i=1}^{n} N_i$-projective.

(4) If M is N-injective, then it is N'-injective and N/N'-injective for any submodule N' of N.

(5) N-injective modules are closed under direct summands and direct products.

(6) If M is N_i-injective for any $i \in \{1, 2, \ldots, n\}$, then M is $\oplus_{i=1}^{n} N_i$-injective.

A module N is called an *essential extension* of a module M if M is isomorphic to an essential submodule of N. In particular, if N is injective, then it is said to be an *injective hull* of M. We denote the injective hull of a module M by $E(M)$. A pair (P, φ) of a module P and a small epimorphism $\varphi : P \to M$ is called a *small cover* of a module M. In particular, if N is projective, then a pair (P, φ) is said to be a *projective cover* of M. We also employ natural variations and abbreviations of this terminology; for example, we may well call P itself a projective cover of M.

Lemma 0.3 ([1, Theorem 18.10 and Lemma 17.17])

(1) Every module has an injective hull and it is unique up to isomorphism.

(2) If a module M has a projective cover, then it is unique up to isomorphism.

Lemma 0.4 *Let M_i be a module and (P_i, φ_i) a small cover (the projective cover) of M_i $(i = 1, 2, \ldots, n)$. Then $(\oplus_{i=1}^n P_i, \oplus_{i=1}^n \varphi_i)$ is a small cover (the projective cover) of $\oplus_{i=1}^n M_i$.*

Proof Clear. □

For a submodule N of a module M, K is called a *complement* of N in M if it is a maximal element in the set of all submodules K' of M with $N \cap K' = 0$. A submodule N of a module M is said to be *closed* in M (or a *closed submodule* of M) if N has no essential extensions in M. Let $K \subseteq N \subseteq M$, then K is called a *coessential submodule* of N in M if N/K is small in M/K and we write $K \subseteq_c^M N$ in this case. A submodule N of a module M is said to be *coclosed* in M (or a *coclosed submodule* of M) if N has no proper coessential submodules in M. Clearly, any direct summand of a module M is closed and coclosed in M.

Lemma 0.5 ([6, 1.9 and 1.10])

(1) For any submodule N of a module M, there exists a complement of N in M.

(2) Any complement of a submodule of a module M is closed in M.

Lemma 0.6 ([10, Proposition 1.4]) *A submodule N of a module M is closed in M if and only if $X/N \subseteq_e M/N$ for any essential submodule X of M containing N.*

Lemma 0.7 ([6, 3.2])

(1) For any submodules K and N of a module M with $K \subseteq N$, K is a coessential submodule of N in M if and only if $M = K + X$ holds for any submodule X of M with $M = N + X$.

(2) Let $M = M_1 \oplus M_2$ be a module and N a submodule of M with $M_1 \subseteq N$. Then $M_1 \subseteq_c^M N$ if and only if $N \cap M_2 \ll M_2$.

(3) Let $f : M \to N$ be an epimorphism. If $A \subseteq_c^M B$, then $f(A) \subseteq_c^M f(B)$. Moreover, $C \subseteq_c^N D$ if and only if $f^{-1}(C) \subseteq_c^M f^{-1}(D)$. In particular, suppose that f is a small epimorphism. If X is coclosed in M, then $f(X)$ is coclosed in N.

(4) $X \subseteq_c^M X + S$ for any submodule X of a module M and any small submodule S of M.

A direct sum decomposition $M = \oplus_I M_i$ is said to be *exchangeable* if, for any direct summand X of M, there exists $M_i' \subseteq M_i$ $(i \in I)$ such that $M = X \oplus (\oplus_I M_i')$. A module M is said to satisfy the *(finite) internal exchange property* if any (finite) direct sum decomposition $M = \oplus_I M_i$ is exchangeable. A ring R is called *local* if it has the maximum proper right (or left) ideal.

Lemma 0.8 ([6, 11.40]) *Let $M_1, M_2, \ldots, M_n$ be modules and put $M = \oplus_{i=1}^n M_i$. Then M satisfies the finite internal exchange property if and only if each M_i satisfies the finite internal exchange property and the decomposition $M = \oplus_{i=1}^n M_i$ is exchangeable.*

Lemma 0.9 (cf. [29, Theorem 1.21]) *An injective module satisfies the (finite) internal exchange property.*

Lemma 0.10 (cf. [33, Proposition 1]) *For an indecomposable module M, if $M^2 = M \times M$ satisfies the finite internal exchange property, then the endomorphism ring of M is local.*

For a direct sum decomposition $M = A \oplus B$ and a homomorphism $h : A \to B$, the set $\{a + h(a) \mid a \in A\}$ is called a *graph* of h and denoted by $\langle h \rangle$. It is clear that $M = \langle h \rangle \oplus B$, $A \cap \langle h \rangle = \operatorname{Ker} h$, and $M = A + \langle h \rangle$ if h is an epimorphism.

A family $\{X_i\}_{i \in I}$ of submodules of a module M is called a *local summand* of M if $\sum_I X_i$ is direct and $\sum_F X_i$ is a direct summand of M for any finite subset F of I. A module M is said to satisfy *LSS* if any local summand of M is a direct summand of M.

Lemma 0.11 ([29, Lemma 2.16 and Theorem 2.17])

(1) A module M satisfies LSS if and only if the union of any chain of direct summands of M is a direct summand of M.

(2) If a module satisfies LSS, then it has an indecomposable decomposition.

A module M is said to be *lifting* if, for any submodule N of M, there exists a direct summand X of M such that $X \subseteq_c^M N$. An indecomposable lifting module is called *hollow*. A module M is called *local* if it has a small maximal submodule. A lifting module M is called *quasi-discrete* if $A \cap B$ is a direct summand for any direct summands A and B with $M = A + B$.

Lemma 0.12 ([6, 2.15, 22.2, 22.3])

(1) Lifting modules and quasi-discrete modules are closed under direct summands.

(2) Hollow modules are closed under factor modules and small covers.

(3) A module M is hollow if and only if any proper submodule of M is small in M.

(4) Any local module is hollow.

(5) Any submodule of a lifting module M is a direct sum of a direct summand of M and a small submodule of M.

Lemma 0.13 ([31, Theorem 3.5] and [29, Theorem 4.15]) *A quasi-discrete module satisfies LSS and the internal exchange property.*

Lemma 0.14 ([6, Corollary 23.12 and 26.22]) *Let M_1 and M_2 be lifting modules and put $M = M_1 \oplus M_2$. If M_i is M_j-projective ($i \neq j$), then M is lifting and the decomposition $M = M_1 \oplus M_2$ is exchangeable. In particular, if M_1 and M_2 are quasi-discrete, then M is quasi-discrete if and only if M_i is M_j-projective ($i \neq j$).*

A module M is said to be *extending* (or *CS*) if, for any submodule N of M, there exists a direct summand X of M such that $N \subseteq_e X$. An indecomposable extending module is called *uniform*. A module M is said to be *uniserial* if its submodules are linearly ordered by the inclusion.

Lemma 0.15 ([29, 1.16, 2.17] and [6, 2.17])

(1) Extending modules are closed under direct summands.

(2) Uniform modules are closed under submodules and essential extensions.

(3) A module M is uniform if and only if any nonzero submodule of M is essential.

(4) Any injective module is extending.

(5) A module M is uniserial, if and only if any submodule of M is hollow, if and only if any factor module of M is uniform.

(6) Uniserial modules are closed under submodules and factor modules.

A submodule Y of a module M is called a *supplement* of a submodule X in M if $M = X + Y$ and $X \cap Y \ll Y$. A module M is called *amply supplemented* if, for any submodules X and Y of M with $M = X + Y$, Y contains a supplement of X in M.

Lemma 0.16 ([6, 20.22] and [29, Proposition 4.8])

(1) Amply supplemented modules are closed under factor modules.

(2) Any lifting module is amply supplemented.

Lemma 0.17 ([18, Lemma 1.6]) *Let M be a lifting module, N an amply supplemented module and $f : M \to N$ a homomorphism. Then there exists a decomposition $M = M_1 \oplus M_2$ such that $f(M_1)$ is coclosed in N and $f(M_2)$ is small in N.*

A ring R is called *right perfect (left perfect, semiperfect, resp.)* if, any right R-module (left R-module, finitely generated right R-module, resp.) has the projective cover.

Lemma 0.18 ([29, Theorem 4.41] and [1, Theorem 28.4])

(1) For a ring R, the following are equivalent:

(a) R is right perfect,

(b) any projective right R-module is lifting,

(c) any right R-module is amply supplemented.

(2) Any module over a right (left, resp.) perfect ring has a maximal (minimal, resp.) submodule.

Lemma 0.19 *Let M be a module over a right perfect ring and let (P, φ) be the projective cover of M. Then for any finite direct sum decomposition $M = \oplus_{i=1}^{n} M_i$ of M, there exists a direct sum decomposition $P = \oplus_{i=1}^{n} P_i$ such that each $(P_i, \varphi|_{P_i})$ is the projective cover of M_i.*

Proof Let $M = \oplus_{i=1}^{n} M_i$. Because P is lifting, there exist direct summands P_1 and P_1' of P such that $P_1 \subseteq_c^P \varphi^{-1}(M_1)$ and $P_1' \subseteq_c^P \varphi^{-1}(\oplus_{i=2}^{n} M_i)$. By Lemma 0.7 (1), $P = P_1 + P_1'$. Since P is quasi-discrete and $P_1 \cap P_1' \subseteq \operatorname{Ker} \varphi \ll P$, we see $P = P_1 \oplus P_1'$. By Lemma 0.7 (3), $(P_1, \varphi|_{P_1})$ and $(P_1', \varphi|_{P_1'})$ are the projective covers of M_1 and $\oplus_{i=2}^{n} M_i$, respectively. Inductively, we obtain a decomposition $P = \oplus_{i=1}^{n} P_i$ such that each $(P_i, \varphi|_{P_i})$ is the projective cover of M_i. □

An element e of R is called *idempotent* if $e = e^2$. An element c of R is called *central* if $cr = rc$ for any element r of R. An idempotent e of R is said to be *primitive* if eR is indecomposable. Elements $e_1, e_2, \dots, e_n$ of R is said to be *orthogonal* if $e_i e_j = 0$ for any distinct $i, j \in \{1, 2, \dots, n\}$. A set $\{e_1, e_2, \dots, e_n\}$ of elements of R is said to be *complete* if $e_1 + e_2 + \cdots + e_n = 1$. A set $\{e_1, e_2, \dots, e_n\}$ of primitive orthogonal idempotents of R is said to be *basic* if (i) $e_i R \not\cong e_j R$ for any distinct $i, j \in \{1, 2, \dots, n\}$ and (ii) for any primitive idempotent e of R, there exists $i \in \{1, 2, \dots, n\}$ such that $eR \cong e_i R$. A semiperfect ring R is called *basic* if any complete set of primitive orthogonal idempotents of R is basic.

Lemma 0.20 ([1, Corollaries 4.7, 7.4, 7.5 and Propositions 7.2, 7.6, 27.10])

(1) For any idempotents e_i and e_j of R, $e_i R e_j \cong \mathrm{Hom}_R(e_j R, e_i R)$ as the abelian group.

(2) For any idempotent e of R, eR is a direct summand of the right R-module R.

(3) For any central idempotent e of R, eRe forms a ring by the addition and the multiplication of R.

(4) For any elements $e_1, e_2, \dots, e_n$ of R,

(a) $\{e_1, e_2, \dots, e_n\}$ is a complete set of primitive orthogonal idempotents if and only if $R_R = e_1 R \oplus e_2 R \oplus \cdots \oplus e_n R$.

(b) $\{e_1, e_2, \dots, e_n\}$ is a complete set of primitive central orthogonal idempotents if and only if $R \cong e_1 R e_1 \times e_2 R e_2 \times \cdots \times e_n R e_n$ as the ring.

(5) A semiperfect ring has a complete set (and a basic set) of primitive orthogonal idempotents.

Lemma 0.21 ([1, Theorem 27.11]) *Let $\{e_1, e_2, \dots, e_n\}$ be a basic set of primitive orthogonal idempotents of R. Then for any projective module P, there exists a set Λ_i $(1, 2, \dots, n)$ such that $P \cong (e_1 R)^{(\Lambda_1)} \oplus (e_2 R)^{(\Lambda_2)} \oplus \cdots \oplus (e_n R)^{(\Lambda_n)}$.*

1 An example of a lifting module which does not satisfy the finite internal exchange property

The content of this chapter is described in [32].

In this chapter, we give a new characterization for the square of a uniserial module to be lifting by the certain projectivity different from the almost projectivity and the generalized projectivity, and make an example of a lifting module which does not satisfy the finite internal exchange property using the characterization.

First, we show the following lemmas:

Lemma 1.1 *Let A and B be modules and put $M = A \oplus B$. For any nonzero proper direct summand X of M, the following holds:*

(1) If A and B are hollow, then so is X.

(2) If A and B are uniform, then so is X.

Proof Let $p : M = A \oplus B \to A$ and $q : M = A \oplus B \to B$ be the canonical projections.

(1) Since A and B are hollow and X is non-small, X satisfies either $p(X) = A$ or $q(X) = B$. Without loss of generality, we can take X with $p(X) = A$. By $X \neq M$, we see $X \cap B \ll B$ because B is hollow. Since X is a proper direct summand of M, we obtain $\operatorname{Ker} p|_X = X \cap B \ll X$. Hence $(X, p|_X)$ is a small cover of A. By Lemma 0.12 (2), X is hollow.

(2) Since A and B are uniform and X is non-essential, X satisfies either $X \cap A = 0$ or $X \cap B = 0$. Without loss of generality, we can take X with $X \cap A = 0$. Then $q|_X : X \to B$ is a nonzero monomorphism. By Lemma 0.15, X is uniform. □

Now we give a key lemma for Theorems 1.3 and 1.4.

Lemma 1.2 *Let U be a uniserial module and put $M = U^2$, $U_1 = U \times 0$ and $U_2 = 0 \times U$. Then for any submodule N_1 of U_1 and any epimorphism h_1 from N_1 to U_2, $\langle h_1 \rangle$ is a direct summand of M.*

Proof If $N_1 = U_1$ or $\operatorname{Ker} h_1 = 0$, it is clear $M = \langle h_1 \rangle \oplus U_2$ or $M = \langle h_1 \rangle \oplus U_1$. We assume $N_1 \neq U_1$ and $\operatorname{Ker} h_1 \neq 0$, and take a submodule N_2 of U_2 which is a natural isomorphic image of N_1 and an epimorphism h_2 from N_2 to U_1. Now we prove $M = \langle h_1 \rangle \oplus \langle h_2 \rangle$.

First we show $M = \langle h_1 \rangle + \langle h_2 \rangle$. Let $\iota_i : h_i^{-1}(N_j) \to U_i$ $(i \neq j)$ be the inclusion. Then $\operatorname{Im} \iota_i = h_i^{-1}(N_j) \subseteq h_i^{-1}(U_j) = N_i \subsetneq U_i$ $(i \neq j)$. We define a homomorphism $h_i' : h_i^{-1}(N_j) \to U_i$ by $h_i'(x) = h_j h_i(x)$ for $x \in h_i^{-1}(N_j)$ $(i \neq j)$. Then h_i' is onto $(i = 1, 2)$. Since U_i is hollow, we obtain that $\iota_i - h_i' : h_i^{-1}(N_j) \to U_i$ is onto $(i \neq j)$. For any element $u_1 + u_2$ of M $(u_i \in U_i)$, there exists an element x_i of $h_i^{-1}(N_j)$ such that $(\iota_i - h_i')(x_i) = u_i$ $(i \neq j)$. Hence

$$
\begin{aligned}
u_1 + u_2 &= ((x_1 - h_2(x_2)) + h_1(x_1 - h_2(x_2))) + ((x_2 - h_1(x_1)) + h_2(x_2 - h_1(x_1))) \\
&\in \langle h_1 \rangle + \langle h_2 \rangle.
\end{aligned}
$$

Thus $M = \langle h_1 \rangle + \langle h_2 \rangle$.

Next we show $\langle h_1 \rangle \cap \langle h_2 \rangle = 0$. We see

$$
(\langle h_1 \rangle \cap \langle h_2 \rangle) \cap \operatorname{Ker} h_1 = (\langle h_1 \rangle \cap \langle h_2 \rangle) \cap (\langle h_1 \rangle \cap N_1) \subseteq \langle h_2 \rangle \cap N_1 = 0.
$$

Since $\langle h_1 \rangle \cong N_1$ is uniform and $\operatorname{Ker} h_1 \neq 0$, we obtain $\langle h_1 \rangle \cap \langle h_2 \rangle = 0$. $\square$

The following theorem is a characterization for the square of a uniserial module to be lifting.

Theorem 1.3 *Let U be a uniserial module and put $M = U^2$, $U_1 = U \times 0$ and $U_2 = 0 \times U$. Then the following conditions are equivalent:*

(a) M is lifting,

(b) for any module X, any homomorphism $f : U_1 \to X$ and any epimorphism $g : U_2 \to X$, one of the following holds:

(i) there exists a homomorphism $h : U_1 \to U_2$ such that $f = gh$,

(ii) there exist a submodule N of U_2 and an epimorphism $h : N \to U_1$ such that $g|_N = fh$,

$$\begin{array}{ccc} & & U \\ {}^{\exists}h & \swarrow \circlearrowleft & \downarrow f \\ U & \overset{g}{\twoheadrightarrow} & X \end{array} \qquad or \qquad \begin{array}{ccc} {}^{\exists}N & \overset{{}^{\exists}h}{\twoheadrightarrow} & U \\ \downarrow & \circlearrowleft & \downarrow f \\ U & \overset{g}{\twoheadrightarrow} & X \end{array}$$

(c) for any module X, any homomorphism $f : U_1 \to X$ and any epimorphism $g : U_2 \to X$, one of the following holds:

(i) there exists a homomorphism $h : U_1 \to U_2$ such that $f = gh$,

(ii) there exist a submodule K of $\operatorname{Ker} g$ and a monomorphism $h : U_1 \to U_2/K$ such that $g'h = f$, where $g' : U_2/K \to X$ is defined by $g'(\overline{u}) = g(u)$ for $\overline{u} \in U_2/K$.

$$\begin{array}{ccc} & & U \\ {}^{\exists}h & \swarrow \circlearrowleft & \downarrow f \\ U & \overset{g}{\twoheadrightarrow} & X \end{array} \qquad or \qquad \begin{array}{ccc} & & U \\ {}^{\exists}h & \nearrow \circlearrowleft & \downarrow f \\ U/{}^{\exists}K & \overset{g'}{\twoheadrightarrow} & X \end{array}$$

Proof Let $p_i : M = U_1 \oplus U_2 \to U_i$ be the canonical projection $(i = 1, 2)$.

(a) $\Rightarrow$ (b): Let $f : U_1 \to X$ be a nonzero homomorphism and $g : U_2 \to X$ an epimorphism. We define a homomorphism $\varphi : M \to X$ by $\varphi(u_1+u_2) = f(u_1)-g(u_2)$ for $u_i \in U_i$ $(i = 1, 2)$. Since M is lifting, there exists a direct summand A of M such that $A \subseteq_c^M \operatorname{Ker}\varphi$. Then $M = \operatorname{Ker}\varphi + U_2 = A + U_2$ because g is onto. So $p_1(A) = U_1$.

If $A\cap U_2 = 0$, we can define a homomorphism $h : U_1 = p_1(A) \to U_2$ by $h(p_1(a)) = p_2(a)$ for $a \in A$, and h satisfies $f = gh$. Therefore (i) holds.

Otherwise we see $A \cap U_1 = 0$ since U is uniform. Hence we can define an epimorphism $h : p_2(A) \to p_1(A) = U_1$ by $h(p_2(a)) = p_1(a)$ for $a \in A$, and h satisfies $g|_{p_2(A)} = fh$. Thus (ii) holds.

(b) $\Rightarrow$ (a): Let X be a submodule of M. We may assume that X is a proper non-small submodule of M. Since U_1 and U_2 are hollow with $U_1 \cong U_2$, we only consider the case $p_1(X) = U_1$. Then $M = X + U_2$. Let $\pi : M \to M/X$ be the natural epimorphism. Since $\pi|_{U_2}$ is onto, one of the following (i) or (ii) holds:

(i) there exists a homomorphism $h : U_1 \to U_2$ such that $\pi|_{U_1} = \pi|_{U_2} h$,

(ii) there exist a submodule N of U_2 and an epimorphism $h : N \to U_1$ such that $\pi|_N = \pi|_{U_1} h$.

$$
\begin{array}{ccc}
 & & U_1 \\
 & {}^{\exists}h \swarrow \circlearrowright & \downarrow \pi|_{U_1} \\
U_2 & \overset{\pi|_{U_2}}{\twoheadrightarrow} & X
\end{array}
\quad \text{or} \quad
\begin{array}{ccc}
{}^{\exists}N & \overset{{}^{\exists}h}{\twoheadrightarrow} & U_1 \\
\hookdownarrow & \circlearrowright & \downarrow \pi|_{U_1} \\
U_2 & \overset{\pi|_{U_2}}{\twoheadrightarrow} & X
\end{array}
$$

In either case, we see $\langle -h \rangle$ is a direct summand of M by Lemma 1.2, and $\langle -h \rangle \subseteq X$ by the commutativity of the diagram. Put $M = \langle -h \rangle \oplus T$ using a direct summand T of M. Since T is hollow by Lemma 1.1, we obtain $T \cap X \ll T$. Hence $\langle -h \rangle \subseteq_c^M X$ by Lemma 0.7 (2). Thus M is lifting.

(b) $\Rightarrow$ (c): It is enough to show (b)(ii) $\Rightarrow$ (c)(ii). For any homomorphism $f : U_1 \to X$ and any epimorphism $g : U_2 \to X$, we assume that there exist a submodule N of U_2 and an epimorphism $h : N \to U_1$ such that $g|_N = fh$. Then $\operatorname{Ker} h \subseteq \operatorname{Ker} g$, hence we can define an epimorphism $g' : U_2/\operatorname{Ker} h \to X$ by $g'(\overline{u}) = g(u)$ for $\overline{u} \in U_2/\operatorname{Ker} h$. Let $\overline{h} : N/\operatorname{Ker} h \to U_1$ be the natural isomorphism and $\iota : N/\operatorname{Ker} h \to U_2/\operatorname{Ker} h$ the inclusion, and put $h' = \iota \overline{h}^{-1}$. Clearly, h' is a monomorphism and $g'h' = f$.

(c) $\Rightarrow$ (b): We show (c)(ii) $\Rightarrow$ (b)(ii). For any homomorphism $f : U_1 \to X$ and any epimorphism $g : U_2 \to X$, we assume that there exist a submodule K of $\operatorname{Ker} g$ and a monomorphism $h : U_1 \to U_2/K$ such that $f = g'h$, where $g' : U_2/K \to X$ is defined by $g'(\overline{u}) = g(u)$ for $\overline{u} \in U_2/\operatorname{Ker} h$. We express $\operatorname{Im} h = N/K$. Let $\varphi : N/K \to U_1$ be the inverse map of h and $\pi : N \to N/K$ the natural epimorphism, and put $h' = \varphi\pi$. Then h' is onto and $g|_N = fh'$. $\square$

The following is the dual of Theorem 1.3.

Theorem 1.4 *Let U be a uniserial module and put $M = U^2$, $U_1 = U \times 0$ and $U_2 = 0 \times U$. Then the following conditions are equivalent:*

(a) M is extending,

(b) for any module X, any homomorphism $f : X \to U_2$ and any monomorphism $g : X \to U_1$, one of the following holds:

(i) there exists a homomorphism $h : U_1 \to U_2$ such that $f = hg$,

(ii) there exist a submodule K of U_1 and a monomorphism $h : U_2 \to U_1/K$ such that $hf = \pi g$, where π is the natural epimorphism from U_1 to U_1/K,

$$\begin{array}{ccc} X & \overset{g}{\rightarrowtail} & U_1 \\ f\downarrow & \circlearrowright\swarrow h & \\ U_2 & & \end{array} \quad \text{or} \quad \begin{array}{ccc} X & \overset{g}{\rightarrowtail} & U_1 \\ f\downarrow & \circlearrowright & \downarrow\pi \\ U_2 & \overset{h}{\rightarrowtail} & U_1/K \end{array}$$

(c) for any module X, any homomorphism $f : X \to U_2$ and any monomorphism $g : X \to U_1$, one of the following holds:

(i) there exists a homomorphism $h : U_1 \to U_2$ such that $f = hg$,

(ii) there exist a submodule N of U_1 containing $\operatorname{Im} g$ and an epimorphism $h : N \to U_2$ such that $f = hg$.

$$\begin{array}{ccc} X & \overset{g}{\rightarrowtail} & U_1 \\ f\downarrow & \circlearrowright\swarrow h & \\ U_2 & & \end{array} \quad \text{or} \quad \begin{array}{ccc} X & \overset{g}{\rightarrowtail} & N \subseteq U_1 \\ f\downarrow & \circlearrowright \swarrow h & \\ U_2 & & \end{array}$$

Proof Let $p_i : M = U_1 \oplus U_2 \to U_i$ be the canonical projection ($i = 1, 2$).

(a) $\Rightarrow$ (c): Let $f : X \to U_2$ be a nonzero homomorphism and $g : X \to U_1$ a monomorphism. We define a homomorphism $\varphi : X \to M$ by $\varphi(x) = g(x) + f(x)$ for $x \in X$. Since M is extending, there exists a direct summand A of M such that $\operatorname{Im} \varphi \subseteq_e A$. By $\operatorname{Im} \varphi \cap U_2 = 0$, $A \cap U_2 = 0$.

If $p_1(A) = U_1$, we can define a homomorphism $h : U_1 = p_1(A) \to U_2$ by $h(p_1(a)) = p_2(a)$ for $a \in A$, and h satisfies $f = hg$. Therefore (i) holds.

Otherwise, we see $p_2(A) = U_2$ since U is hollow. We see $\operatorname{Im} g \subseteq p_1(A)$, and we can define an epimorphism $h : p_1(A) \to p_2(A) = U_2$ by $h(p_1(a)) = p_2(a)$ for $a \in A$. Then h satisfies $f = hg$. Therefore (ii) holds.

(c) $\Rightarrow$ (a): Let X be a submodule of M. We may assume that X is a nonzero non-essential submodule of M. Since U_1 and U_2 are uniform with $U_1 \cong U_2$, , we only consider the case $X \cap U_2 = 0$ because U is uniform. Since $p_1|_X$ is a monomorphism, one of the following (i) or (ii) holds:

(i) there exists a homomorphism $h : U_1 \to U_2$ such that $p_2|_X = hp_1|_X$.

(ii) there exist a submodule N of U_1 containing $p_1(X)$ and an epimorphism $h : N \to U_2$ such that $p_2|_X = hp_1|_X$.

$$\begin{array}{ccc} X & \overset{p_1|_X}{\rightarrowtail} & U_1 \\ p_2|_X \downarrow & \circlearrowright \swarrow h & \\ U_2 & & \end{array} \qquad \text{or} \qquad \begin{array}{cccc} X & \overset{p_1|_X}{\rightarrowtail} & N & \subseteq U_1 \\ p_2|_X \downarrow & \circlearrowright \swarrow h & & \\ U_2 & & & \end{array}$$

In either case, $\langle h \rangle$ is a direct summand of M by Lemma 1.2, and $X \subseteq \langle h \rangle$ by commutativity of the diagram. Since $\langle h \rangle$ is uniform by Lemma 1.1, we obtain $X \subseteq_e \langle h \rangle$. Thus M is extending.

(c) $\Rightarrow$ (b): It is enough to show (c)(ii) $\Rightarrow$ (b)(ii). For any homomorphism $f : X \to U_2$ and any monomorphism $g : X \to U_1$, we assume that there exist a submodule N of U_1 containing $\operatorname{Im} g$ and an epimorphism $h : N \to U_2$ such that $f = hg$. Let $\overline{h} : N/\operatorname{Ker} h \to U_2$ be the natural isomorphism and $\iota : N/\operatorname{Ker} h \to U_1/\operatorname{Ker} h$ the inclusion, and put $h' = \iota\overline{h}^{-1}$. Then h' is a monomorphism and $h'f = \pi g$, where $\pi : U_1 \to U_1/\operatorname{Ker} h$ is the natural epimorphism.

(b) $\Rightarrow$ (c): We show (b)(ii) $\Rightarrow$ (c)(ii). For any homomorphism $f : X \to U_2$ and any monomorphism $g : X \to U_1$, we assume that there exist a submodule K of U_1 and a monomorphism $h : U_2 \to U_1/K$ such that $hf = \pi g$, where $\pi : U_1 \to U_1/K$ is the natural epimorphism. We express $\operatorname{Im} h = N/K$. Let $\varphi : N/K \to U_2$ be the inverse map of h and $\eta : N \to N/K$ the natural epimorphism, and put $h' = \varphi\eta$. Then we see $\operatorname{Im} g \subseteq N$, h' is an epimorphism and $f = h'g$. $\square$

Lifting modules do not necessarily satisfy the finite internal exchange property. At the end of this chapter, we make an example of a lifting module without the finite internal exchange property, using Theorem 1.3.

Example 1.5 *Let $\mathbb{Z}_{(p)}$ and $\mathbb{Z}_{(q)}$ be the localizations of $\mathbb{Z}$ at two distinct prime numbers p and q, respectively. We consider a semiperfect ring $R = \begin{pmatrix} \mathbb{Z}_{(p)} & \mathbb{Q} \\ 0 & \mathbb{Z}_{(q)} \end{pmatrix}$ and its right ideal $L = \begin{pmatrix} 0 & \mathbb{Z}_{(q)} \\ 0 & \mathbb{Z}_{(q)} \end{pmatrix}$, and put $U_R = R/L$. Then U is uniserial whose the endomorphism ring is not local (see [9]). According to the contraposition of Lemma 0.10, U^2 does not satisfy the finite internal exchange property. We show U^2 is lifting. For any nonzero homomorphism $f : U \to U/X$ where X is a submodule of U, we can take*

$$f(\overline{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}) = \overline{\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}} + X \qquad \text{for some } x \in \mathbb{Z}_{(p)}.$$

If $x \in \mathbb{Z}_{(q)}$, we can define an endomorphism h of U with $h(\overline{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}) = \overline{\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}}$, and h satisfies $\pi h = f$, where π is the natural epimorphism from U to U/X. Otherwise we can express $x = p^m \frac{1}{q^n} \frac{t}{s}$, where $m \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$ and $s, t \in \mathbb{Z} \setminus (p\mathbb{Z} \cup q\mathbb{Z})$. Put $N = \overline{\begin{pmatrix} p^m & 0 \\ 0 & 0 \end{pmatrix}} R$. We can define an epimorphism $h' : N \to U$ with $h'(\overline{\begin{pmatrix} p^m & 0 \\ 0 & 0 \end{pmatrix}}) = \overline{\begin{pmatrix} q^n \frac{s}{t} & 0 \\ 0 & 0 \end{pmatrix}}$, and h' satisfies $fh' = \pi|_N$, where π is the natural epimorphism from U to U/X. Thus U^2 is lifting by Theorem 1.3.

2 On a lifting module over a right perfect ring

In this chapter, we introduce the concepts of "dual square free" and "factor square full" modules and give these fundamental properties. After that we consider a condition for a lifting module over a right perfect ring to satisfy the finite internal exchange property using them.

First, we show that a lifting module over a right perfect ring satisfies LSS.

Proposition 2.1 *Let M and N be lifting modules and let $f : M \to N$ be an epimorphism. If M satisfies LSS, then so does N.*

Proof Since any direct summand of M is lifting with LSS, we may assume that $f : M \to N$ is a small epimorphism. Let $\{N_i\}_{i\in I}$ be a chain of direct summands of N. By Lemma 0.11 (2), we shall show that $\cup_I N_i$ is a direct summand of N. Consider the collection $\mathbb{X} = \{M' \leq_\oplus M \mid f(M') = \cup_K N_i$ for some $K \subseteq I\}$. Since M satisfies LSS, the set $\mathbb{X}$ is non-empty and inductive with respect to inclusion. By Zorn's Lemma, we obtain a maximal element M' of $\mathbb{X}$. Let $M = M' \oplus M''$ and let K be a subset of I with $f(M') = \cup_K N_i$.

Suppose $\cup_K N_i \neq \cup_I N_i$. Then there exists $a \in I \setminus K$ such that $\cup_K N_i \subsetneq N_a$. By $M' \subseteq f^{-1}(f(M')) \subsetneq f^{-1}(N_a)$, we see $f^{-1}(N_a) = M' \oplus (f^{-1}(N_a) \cap M'')$. Since M'' is lifting, there exists a direct summand M_1 of M'' such that $M_1 \subseteq_c^{M''} f^{-1}(N_a) \cap M''$. By $N_a = f(M') + f(f^{-1}(N_a) \cap M'')$, we see $f^{-1}(N_a) \cap M''$ is non-small in M'' and so $M_1 \neq 0$. By Lemma 0.7 (3), $f(M_1) \subseteq_c^N f(f^{-1}(N_a) \cap M'')$. Hence $f(M' \oplus M_1) \subseteq_c^N N_a$. As N_a is a direct summand of N, $f(M' \oplus M_1) = N_a$ which contradicts the maximality of M'. Thus $\cup_K N_i = \cup_I N_i$. By Lemma 0.7 (3), $f(M')$ is coclosed in N. Since N is lifting, we obtain $\cup_I N_i = \cup_K N_i = f(M') \leq_\oplus N$. □

Corollary 2.2 ([27, Theorem 3.3]) *Any lifting module over a right perfect ring satisfies LSS. Hence it has an indecomposable decomposition.*

Proof By Lemmas 0.11, 0.18, 0.13 and Proposition 2.1. □

2.1 Dual square free modules

The notion of square free modules was introduced by Camillo [5] in the study of distributive modules. It plays important roles in the several studies, such as quasi-continuous modules, and the exchange properties for modules.

After that, the notions of dual of square free modules were introduced by Ding-Ibrahim-Yousif-Zhou [7] as DSF-modules in the study of D4-modules in 2017, and by Kikumasa-Kuratomi [21] as d-square free modules in the study of H-supplemented modules in 2018, respectively. The definition of a DSF-module and a d-square free module are coincide (see [19]). In this section, we give some fundamental properties of dual square free modules.

Definition. A module M is called *DSF* if there are no proper submodules A and B of M such that $M = A + B$ and $M/A \cong M/B$. A module M is called *d-square free* if there are no epimorphisms from M to N^2 for some nonzero module N.

Now we shall show that they are equivalent. Assume that M is d-square free. Let $M = A+B$ and $M/A \cong M/B$. Clearly, $M/(A\cap B) = A/(A\cap B) \oplus B/(A\cap B) \cong M/B \oplus M/A \cong (M/A)^2$. Since M is d-square free, $M/A = 0$, and hence $A = M$. Thus M is DSF. Conversely, assume that M is DSF. Let $f : M \to N^2 = N_1 \oplus N_2$ be an epimorphism, where $N_1 \cong N_2$. Then $M = f^{-1}(N_1)+f^{-1}(N_2)$. Put $A_i = f^{-1}(N_i)$. Then we obtain $A_1 \cap A_2 = \operatorname{Ker} f$. Therefore $M/A_1 = (A_1 + A_2)/A_1 \cong A_2/\operatorname{Ker} f \cong f(A_2) = N_2 \cong N_1 \cong A_1/\operatorname{Ker} f \cong M/A_2$. Since M is DSF, $A_1 = M$. Hence $N_1 \cong M/A_1 = 0$. Thus M is d-square free. In this dissertation, we shall call them *dual square free.*

Proposition 2.3 *Dual square free modules are closed under factor modules and small covers.*

Proof It is clear that dual square free modules are closed under factor modules. We prove that they are closed under small covers.

Let M be a dual square free module and let (P, φ) be a small cover of M. Take submodules A and B of P with $P = A+B$ and $P/A \cong P/B$. We shall show $B = P$.

Consider the epimorphism η which is obtained by compositions of natural maps:

$$P \twoheadrightarrow P/A \cong P/B \twoheadrightarrow P/(B + \operatorname{Ker}\varphi).$$

Then $A \subseteq \operatorname{Ker}\eta$, $P = (B + \operatorname{Ker}\varphi) + \operatorname{Ker}\eta$ and $P/(B + \operatorname{Ker}\varphi) \cong P/\operatorname{Ker}\eta$. Next, we consider the epimorphism ρ which is obtained by compositions of natural maps:

$$P \twoheadrightarrow P/(B + \operatorname{Ker}\varphi) \cong P/\operatorname{Ker}\eta \twoheadrightarrow P/(\operatorname{Ker}\eta + \operatorname{Ker}\varphi).$$

Then $B + \operatorname{Ker}\varphi \subseteq \operatorname{Ker}\rho$, $P = \operatorname{Ker}\rho + (\operatorname{Ker}\eta + \operatorname{Ker}\varphi)$ and $P/\operatorname{Ker}\rho \cong P/(\operatorname{Ker}\eta + \operatorname{Ker}\varphi)$. Since $\operatorname{Ker}\varphi \subseteq \operatorname{Ker}\rho \cap (\operatorname{Ker}\eta + \operatorname{Ker}\varphi)$, there exists an epimorphism $g : M \to P/(\operatorname{Ker}\rho \cap (\operatorname{Ker}\eta + \operatorname{Ker}\varphi)) = \operatorname{Ker}\rho/(\operatorname{Ker}\rho \cap (\operatorname{Ker}\eta + \operatorname{Ker}\varphi)) \oplus (\operatorname{Ker}\eta + \operatorname{Ker}\varphi)/(\operatorname{Ker}\rho \cap (\operatorname{Ker}\eta + \operatorname{Ker}\varphi)) \cong P/(\operatorname{Ker}\eta + \operatorname{Ker}\varphi) \oplus P/\operatorname{Ker}\rho \cong (P/(\operatorname{Ker}\eta + \operatorname{Ker}\varphi))^2$. Since M is dual square free, we see $P/(\operatorname{Ker}\eta + \operatorname{Ker}\varphi) = 0$. Therefore $P = \operatorname{Ker}\eta + \operatorname{Ker}\varphi = \operatorname{Ker}\eta$. Since $P/(B + \operatorname{Ker}\varphi) \cong P/\operatorname{Ker}\eta = 0$, we see $P = B + \operatorname{Ker}\varphi = B$. Thus P is dual square free. □

Lemma 2.4 ([29, Theorem 4.24]) *Let A and B be direct summands of a quasi-discrete module M. If $A/X \cong B/Y$ where $X \ll A$ and $Y \ll B$, then $A \cong B$.*

Proposition 2.5 *Let $M = \oplus_I H_i$ be a quasi-discrete module, where each H_i is hollow and $\#I \geq 2$. Then M is dual square free if and only if $H_j \ncong H_k$ for any distinct $j, k \in I$.*

Proof We prove the contraposition: "there exist distinct $j, k \in I$ such that $H_j \cong H_k$ if and only if M is not dual square free".

($\Rightarrow$) Obvious.

($\Leftarrow$) If M is not dual square free, then there exist proper submodules A and B of M with $M = A + B$ and $M/A \cong M/B$. As M is lifting, there exist direct summands A' and B' of M such that $A' \subseteq_c^M A$ and $B' \subseteq_c^M B$. Then $M = A' + B'$ by Lemma 0.7 (1). Since M is quasi-discrete, we can express $A' = C \oplus (A' \cap B')$

and $B' = D \oplus (A' \cap B')$. Then $M = C \oplus (A' \cap B') \oplus D = C + B = A + D$ and

$$C/(C \cap B) \cong M/B \cong M/A \cong D/(D \cap A)$$

By $C \cap B \ll C$, $D \cap A \ll D$ and Lemma 2.4, there exists an isomorphism $g : C \to D$. Since $M = \oplus_I H_i$ is exchangeable, there exist nonempty disjoint subsets K and L of I and isomorphisms $\alpha : C \to \oplus_K H_i$ and $\beta : D \to \oplus_L H_i$ such that $M = (\oplus_K H_i) \oplus (A' \cap B') \oplus (\oplus_L H_i)$. Given $k \in K$. Since $\beta(D) = \oplus_L H_i$ is exchangeable, there exists $l \in L$ such that $\beta(D) = \beta g \alpha^{-1}(H_k) \oplus (\oplus_{L \setminus \{l\}} H_i)$. Thus $H_l \cong \beta g \alpha^{-1}(H_k) \cong H_k$. $\square$

Proposition 2.6 *A dual square free module over a right perfect ring is cyclic.*

Proof Let R be a right perfect ring with a basic set $F = \{e_1, e_2, \ldots, e_n\}$ of primitive orthogonal idempotents, M a dual square free module and (P, φ) the projective cover of M. By Lemma 0.21, there exists a set I_j $(j = 1, 2, \ldots, n)$ such that $P \cong (e_1R)^{(I_1)} \oplus (e_2R)^{(I_2)} \oplus \cdots \oplus (e_nR)^{(I_n)}$. By Propositions 2.3 and 2.5, P is isomorphic to $e_{i_1}R \oplus e_{i_2}R \oplus \cdots \oplus e_{i_k}R$ for distinct $e_{i_j} \in F$. Therefore P is isomorphic to a direct summand of R and hence M is cyclic. $\square$

2.2 Factor square full modules

In this section, we introduce the concept of factor square full modules and give its fundamental properties. The content of this section is described in [22].

Definition. A module M is called *factor square full* if, for any proper submodule X of M, there exist a proper submodule Y of M with $X \subseteq Y$ and an epimorphism $f : M \to (M/Y)^2$.

Proposition 2.7 *For any module M, the following are equivalent:*

(a) M is factor square full,

(b) for any proper submodule X of M, there exist a proper submodule Y of M and an epimorphism $f : M \to (M/Y)^2$ such that $X \subseteq Y$ and $pf : M \to M/Y$ is the natural epimorphism, where p is the canonical projection from $(M/Y)^2$ to M/Y $((\overline{a}, \overline{b}) \mapsto \overline{a})$,

(c) for any proper submodule X of M, there exist proper submodules Y and Z of M such that $X \subseteq Y$, $M = Y + Z$ and $M/Y \cong M/Z$.

If M is lifting, (a) – (c) are equivalent to:

(d) for any proper direct summand X of M, there exist proper submodules Y and Z of M such that $X \subseteq Y$, $M = Y + Z$ and $M/Y \cong M/Z$.

Proof $(b) \Rightarrow (a)$: Clear.

$(a) \Rightarrow (c)$: For any proper submodule X of M, by (a), there exist a proper submodule Y of M with $X \subseteq Y$ and an epimorphism $f : M \to (M/Y)^2$. Let $p_i : (M/Y)^2 = M/Y \oplus M/Y \to M/Y$ be the i-projection $(i = 1, 2)$. If $M = Y + \operatorname{Ker} p_1 f$, then (c) holds. Otherwise, by (a) again, there exist a proper submodule Y' of M with $Y + \operatorname{Ker} p_1 f \subseteq Y'$ and an epimorphism $g : M \to (M/Y')^2$. Let η be an epimorphism which is defined by compositions of maps as follows:

$$M \twoheadrightarrow M/\operatorname{Ker} p_2 f \cong M/Y \twoheadrightarrow M/Y'$$

Then $X \subseteq Y \subseteq Y'$, $M = Y' + \operatorname{Ker} \eta$ and $M/Y' \cong M/\operatorname{Ker} \eta$.

$(c) \Rightarrow (b)$: For any proper submodule X of M, by (c), there exist proper submodules Y and Z of M such that $X \subseteq Y$, $M = Y + Z$ and $M/Z \overset{\varepsilon}{\cong} M/Y$. Then we can define an epimorphism as follows:

$$\begin{array}{ccccccc} M & \twoheadrightarrow & M/Y \oplus M/Z & \cong & M/Y \oplus M/Y & = & (M/Y)^2 \\ m & \mapsto & (m + Y, m + Z) & \mapsto & (m + Y, \varepsilon(m + Z)) & & . \end{array}$$

Thus (b) holds.

$(c) \Rightarrow (d)$: Clear.

$(d) \Rightarrow (c)$: For any proper submodule N of M, by Lemma 0.12 (5), there exist a proper direct summand X of M and a small submodule S of M such that $N = X \oplus S$. By (d), there exist proper submodules Y and Z of M such that $X \subseteq Y$, $M = Y + Z$ and $M/Y \cong M/Z$. Then $Y + S$ is proper in M and there exists an epimorphism η by compositions of maps as follows:

$$M \twoheadrightarrow M/Z \cong M/Y \twoheadrightarrow M/(Y + S).$$

Clearly, $M = (Y + S) + \operatorname{Ker} \eta$ and $M/(Y + S) \cong M/\operatorname{Ker} \eta$. Thus (c) holds. □

Proposition 2.8 *Let M be a module over an arbitrary ring. Then $M^{(I)}, M^I$ are factor square full, where I is an index set which has at least two elements.*

Proof Suppose that I is a finite set, this is shown by the induction on the number of elements of I. In the case of $\#I = 2$, we put $M_1 = M \times 0$ and $M_2 = 0 \times M$, and let $p : M^2 = M_1 \oplus M_2 \to M_1$ be the canonical projection and $\alpha : M_1 \to M_2$ an isomorphism. Take a proper submodule X of M^2. If $p(X) \neq M_1$, we put $Y = p(X) \oplus M_2$ and $Z = M_1 \oplus \alpha(p(X))$. Then Y and Z are proper submodules of M^2, and they satisfy $X \subseteq Y$, $M^2 = Y + Z$ and $M^2/Y \cong M^2/Z$. Otherwise, $M^2 = X + M_2$. We put $W = \alpha^{-1}(X \cap M_2) \oplus M_2$. Then $M^2 = X + W$ and $M^2/X \cong M^2/W$. Therefore M^2 is factor square full by Proposition 2.7.

Assume that M^n is factor square full for $n \geq 2$, and we consider the case of $\#I = n + 1$. Put $M_1 = M \times 0 \times \cdots \times 0$, and let $q : M^{n+1} = M \times M^n \to M^n$ be the canonical projection and $\beta : M_1 \to M$ an isomorphism. Take a proper submodule X' of M^{n+1}. If $q(X') \neq M^n$, there exist proper submodules A and B of M^n such that $q(X') \subseteq A$, $M^n = A + B$ and $M^n/A \cong M^n/B$. Put $Y' = M \times A$ and $Z' = M \times B$. Then Y' and Z' are proper submodules of M^{n+1} and they satisfy $X' \subseteq Y'$, $M^{n+1} = Y' + Z'$ and $M^{n+1}/Y' \cong M^{n+1}/Z'$. Otherwise, $M^{n+1} = M_1 + X'$. Put $W' = M \times \beta(M_1 \cap X') \times M \times \cdots \times M$. Then $M^{n+1} = X' + W'$ and $M^{n+1}/X' \cong M^{n+1}/W'$. Therefore M^{n+1} is factor square full.

Suppose that I is an infinite set, we see that $M^{(I)} \cong (M^{(I)})^2$ and $M^I \cong (M^I)^2$ are factor square full by the above result. □

Proposition 2.9 *Factor square full modules are closed under small epimorphic images, small covers and finite direct sums.*

Proof First we show that factor square full modules are closed under small epimorphic images. Let M be a factor square full module and $f : M \to N$ a small epimorphism. Take a proper submodule X of N. Since $\operatorname{Ker} f \ll M$, $f^{-1}(X)$ is proper in M. As M is factor square full, by Proposition 2.7, there exist proper submodules Y and Z such that $f^{-1}(X) \subseteq Y$, $M = Y + Z$ and $M/Y \cong M/Z$. Since $\operatorname{Ker} f \subseteq Y$, we see that M/Y is isomorphic to $N/f(Y)$ by the induced map. We consider an epimorphism η which is defined by compositions of maps as follows:

$$N \twoheadrightarrow N/f(Y) \cong M/Y \cong M/Z \twoheadrightarrow N/f(Z).$$

Then $X \subseteq f(Y) \subseteq \operatorname{Ker}\eta \subsetneq N$, $N = \operatorname{Ker}\eta + f(Z)$ and $N/\operatorname{Ker}\eta \cong N/f(Z)$. By Proposition 2.7 again, N is factor square full.

Next we show that factor square full modules are closed under small covers. Let M be a factor square full module and $f : N \to M$ a small epimorphism. Take a proper submodule X of N. As $\operatorname{Ker} f \ll M$, $f(X)$ is proper in M. Since M is factor square full, by Proposition 2.7, there exist proper submodules Y and Z such that $f(X) \subseteq Y$, $M = Y + Z$ and $M/Y \cong M/Z$. Then $X \subseteq f^{-1}(Y) \subsetneq N$, $N = f^{-1}(Y)+f^{-1}(Z)$ and $N/f^{-1}(Y) \cong M/Y \cong M/Z \cong N/f^{-1}(Z)$. By Proposition 2.7 again, N is factor square full.

Finally we show that factor square full modules are closed under finite direct sums. It is enough to show that $A \oplus B$ is factor square full for any factor square full modules A and B. Put $M = A \oplus B$ and let $p : M = A \oplus B \to A$ be the canonical projection. Take a proper submodule X of M. If $p(X) \neq A$, since A is factor square full, there exist proper submodules A_1 and A_2 of A such that $p(X) \subseteq A_1$, $A = A_1 + A_2$ and $A/A_1 \cong A/A_2$. Put $Y = A_1 \oplus B$ and $Z = A_2 \oplus B$. Then $X \subseteq Y \subsetneq M$, $M = Y + Z$ and $M/Y \cong M/Z$. Otherwise, $M = X + B$. Since $X \cap B \neq B$ and B is factor square full, there exist proper submodules B_1 and B_2 of B such that $X \cap B \subseteq B_1$, $B = B_1 + B_2$ and $B/B_1 \cong B/B_2$. Then we see

$X+B_1 \neq M$. Let η be an epimorphism which is obtained by compositions of natural maps as follows:

$$M \twoheadrightarrow M/B_2 \cong M/B_1 \twoheadrightarrow M/(X+B_1).$$

We see $M = (X+B_1) + \operatorname{Ker}\eta$ and $M/\operatorname{Ker}\eta \cong M/(X+B_1)$. Thus M is factor square full. □

According to the following example, factor square full modules are not closed under neither essential extensions nor essential submodules. In Appendix 4.2, we provide details of rings whose factor square full modules are closed under essential extensions or essential submodules.

Example 2.10 *Let K be any field, and we consider*

$$R = \left\{ \begin{pmatrix} a & b & c & d \\ 0 & e & 0 & f \\ 0 & 0 & e & g \\ 0 & 0 & 0 & h \end{pmatrix} \middle| \, a,b,c,d,e,f,g,h \in K \right\}, \quad M_R = (0,K,K,K).$$

We put $N_1 = (0,1,0,0)R$ and $N_2 = (0,0,1,0)R$, then we can see

$$M = N_1 + N_2, \ N_1 \cap N_2 = (0,0,0,K) \ll M$$
$$\text{and} \ \ P(N_1) = P(N_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} R,$$

where $P(N_1)$ and $P(N_2)$ are projective covers of N_1 and N_2 respectively. We obtain $P(M) = P(N_1) \oplus P(N_2) \cong P(N_1)^2$. By Propositions 2.8 and 2.9, M is factor square full. Moreover M is indecomposable because it has simple essential socle $\operatorname{Soc}(M) = (0,0,0,K)$. *However its injective hull $E(M) = (K,K,K,K)$ and its essential submodule* $\operatorname{Soc}(M) = (0,0,0,K)$ *are not factor square full.*

Modules A and B are said to be *dual orthogonal* if there are no nonzero factor modules of A and B which are isomorphic. In general, factor square full modules are not closed under direct summands. For instance, H^2 is factor square full by Proposition 2.8 and H is not factor square full for any hollow module H. However, the following proposition is true.

Proposition 2.11 *Let $M = A \oplus B$ be a factor square full module. If A and B are dual orthogonal, then they are factor square full.*

Proof Let $p : M = A \oplus B \to A$ be the canonical projection. For any proper submodule X of A, there exist proper submodules Y and Z of M such that $X \oplus B \subseteq Y$, $M = Y + Z$ and $M/Y \cong M/Z$. Then $Y = p(Y) \oplus B$. Now we assume that $p(Z) = A$, then $M = Z + B$, and so

$$0 \neq A/p(Y) \cong M/Y \cong M/Z \cong B/(B \cap Z),$$

a contradiction that A and B are dual orthogonal. Hence $p(Z) \neq A$. We consider the nonzero epimorphism η which is obtained by compositions of natural maps:

$$M \twoheadrightarrow M/Y \cong M/Z \twoheadrightarrow M/(p(Z) \oplus B).$$

Then $\operatorname{Ker}\eta = p(\operatorname{Ker}\eta) \oplus B$, so we see $X \subseteq p(Y) \subseteq p(\operatorname{Ker}\eta) \subsetneq A$, $A = p(\operatorname{Ker}\eta) + p(Z)$ and $A/p(\operatorname{Ker}\eta) \cong M/\operatorname{Ker}\eta \cong M/(p(Z) \oplus B) \cong A/p(Z)$. Thus A is factor square full. □

Proposition 2.12 *Let $M = \oplus_I H_i$ be a quasi-discrete module, where each H_i is hollow and $\#I \geq 2$. Then M is factor square full if and only if, for any $j \in I$, there exists $k \in I \setminus \{j\}$ such that $H_k \cong H_j$.*

Proof ($\Rightarrow$) For any $j \in I$, $\oplus_{I \setminus \{j\}} H_i$ is a proper submodule of M. Since M is factor square full, there exist proper submodules X and Y of M such that $\oplus_{I \setminus \{j\}} H_i \subseteq X$, $M = X + Y$ and $M/X \cong M/Y$. Then X is a direct sum of a direct summand $\oplus_{I \setminus \{j\}} H_i$ of M and a small submodule $X \cap H_j$ of M, and hence $\oplus_{I \setminus \{j\}} H_i \subseteq_c^M X$

by Lemma 0.7 (4). Since M is lifting, there exists a decomposition $M = Z \oplus Z'$ such that $Z \subseteq_c^M Y$. Hence $M = X + Y = (\oplus_{I\setminus\{j\}} H_i) + Z$. As M is quasi-discrete, $(\oplus_{I\setminus\{j\}} H_i) \cap Z$ is a direct summand of M. Put $\oplus_{I\setminus\{j\}} H_i = K \oplus [(\oplus_{I\setminus\{j\}} H_i) \cap Z]$ and $Z = L \oplus [(\oplus_{I\setminus\{j\}} H_i) \cap Z]$. Then $M = K \oplus [(\oplus_{I\setminus\{j\}} H_i) \cap Z] \oplus L$, $M = K \oplus Z = K + Y$ and $M = (\oplus_{I\setminus\{j\}} H_i) \oplus L = X + L$. Hence

$$L/(X \cap L) \cong M/X \cong M/Y \cong K/(Y \cap K).$$

By Lemma 2.4, $H_j \cong L \cong K$. Since $\oplus_{I\setminus\{j\}} H_i$ is exchangeable, there exists $l \in I \setminus \{j\}$ such that $\oplus_{I\setminus\{j\}} H_i = H_l \oplus [(\oplus_{I\setminus\{j\}} H_i) \cap Z]$. Thus we see $H_l \cong K \cong H_j$.

($\Leftarrow$) For any proper direct summand X of M, as $M = \oplus_I H_i$ is exchangeable, there exists a nonempty subset K of I such that $M = X \oplus (\oplus_K H_i)$. Given $k \in K$. By the assumption, there exists $j \in I$ ($j \neq k$) such that $H_j \cong H_k$. Put $Y = X \oplus (\oplus_{K\setminus\{k\}} H_i)$. Then $X \subseteq Y \subsetneq M$, $M = Y + (\oplus_{I\setminus\{j\}} H_i)$ and $M/Y \cong H_k \cong H_l \cong M/(\oplus_{I\setminus\{j\}} H_i)$. Thus M is factor square full by Proposition 2.7. □

Proposition 2.13 *Factor square full modules over a right perfect ring are closed under direct sums.*

Proof Let $\{e_1, e_2, \ldots, e_n\}$ be a basic set of primitive orthogonal idempotents of R and M_i a factor square full module ($i \in I$), and put $M = \oplus_I M_i$. Then each projective cover $P(M_i)$ of M_i is factor square full by Proposition 2.9 ($i \in I$). By Proposition 2.12 and Lemma 0.21, we can express

$$P(M) = \oplus_I P(M_i) \cong \oplus_{k=1}^{n} (e_k R)^{(I_k)},$$

where $\#I_k = 0$ or $\#I_k \geq 2$ for any $k = 1, 2, \ldots, n$. By Proposition 2.12 again, we see that $P(M)$ is factor square full. Thus M is also factor square full by Proposition 2.9. □

2.3 The finite internal exchange property for lifting modules over a right perfect ring

In this section, we consider a condition for a lifting module over a right perfect ring to satisfy the finite internal exchange property.

Lemma 2.14 ([26, Theorem 2.1]) *Let M be a lifting module over a right perfect ring. Then M is a direct sum of a dual square free module A and a factor square full module B such that A and B are dual orthogonal.*

Lemma 2.15 ([26, Cororally 2.2]) *A dual square free lifting module over a right perfect ring is quasi-discrete.*

Lemma 2.16 ([29, Proposition 4.35]) *Let I be an arbitrary set. If M is finitely generated and N_i-projective for a module N_i ($i \in I$), then M is $\oplus_I N_i$-projective.*

Proposition 2.17 *Let M be a lifting module over a right perfect ring and let $M = A \oplus B$ be a decomposition such that A is dual square free and B is factor square full such that A and B are dual orthogonal (see Lemma 2.14). Then the decomposition $M = A \oplus B$ is exchangeable, and so it is unique up to isomorphism.*

Proof By Corollary 2.2, A and B have indecomposable decompositions $A = \oplus_I H_i$ and $B = \oplus_J L_j$, respectively. For any $i \in I$ and any $j \in J$, by Proposition 2.5 the projective cover $P(H_i) \oplus P(L_j)$ of $H_i \oplus L_j$ is dual square free, and so $H_i \oplus L_j$ is also dual square free by Proposition 2.3. Therefore $H_i \oplus L_j$ is quasi-discrete by Lemma 2.15. By Lemma 0.14, H_i and L_j are relative projective. Since H_i and L_j are cyclic, by Lemma 2.16, we see H_i is $\oplus_J L_j$-projective and L_j is $\oplus_I H_i$-projective. Moreover, by Lemma 0.2 (2), A is B-projective and B is A-projective. Thus the decomposition $M = A \oplus B$ is exchangeable by Lemma 0.14.

Let $M = C \oplus D$, where C is a dual square free module, D is a factor square full module, and C, D are dual orthogonal. Since $M = A \oplus B$ is exchangeable, there exists direct summands A' of A and B' of B such that $M = C \oplus A' \oplus B'$. As $D \cong A' \oplus B'$, by Proposition 2.11, A' is factor square full. Therefore $A' = 0$. Let

$B = B' \oplus B''$. As $C \cong A \oplus B''$ is dual square free, so is B''. On the other hand, since $C \cong A \oplus B''$ and $D \cong B'$ are dual orthogonal, by Proposition 2.11 again, B'' is factor square full. Hence $B'' = 0$. Thus $C \cong A$ and $D \cong B$. $\square$

In the end of this chapter, we give a necessary and sufficient condition for a lifting module over a right perfect ring to satisfy the finite internal exchange property.

Corollary 2.18 *Let M be a lifting module over a right perfect ring and let $M = A \oplus B$ be a decomposition such that A is dual square free and B is factor square full such that A and B are dual orthogonal (see Lemma 2.14). Then M satisfies the finite internal exchange property if and only if the factor square full part B satisfies the finite internal exchange property.*

Proof By Lemmas 0.8, 2.15 and Proposition 2.17. $\square$

3 Direct sums of lifting modules over a right perfect ring

The content of this chapter is described in [23].

The almost projectivity and the generalized projectivity play important roles to the study of a direct sum of lifting modules. In this chapter, we consider almost projective modules and generalized projective modules over a right perfect ring. First, we give new characterizations of these projectivities by projective covers. Using these characterizations, we study on direct sums of almost projective modules, a relationship between almost projective modules and generalized projective modules and a direct sum of lifting modules.

3.1 A characterization of almost N-projective modules and its applications

In this section, we give a characterization of almost N-projective modules by homomorphisms between their projective covers. Recall the definition of almost N-projective.

Definition. A module M is said to be *almost N-projective* if for any module X, any homomorphism $f : M \to X$ and any epimorphism $g : N \to X$, either there exists a homomorphism $h : M \to N$ such that $gh = f$ or there exist a nonzero direct summand N' of N and a homomorphism $h' : N' \to M$ such that $fh' = g|_{N'}$.

$$
\begin{array}{ccccc}
 & & M & & \\
 & h \swarrow & \downarrow f & & \\
N & \stackrel{g}{\to} & X & \to 0
\end{array}
\qquad \text{or} \qquad
\begin{array}{ccccc}
N' & \stackrel{h'}{\to} & M & & \\
\hookdownarrow & & \downarrow f & & \\
N & \stackrel{g}{\to} & X & \to 0
\end{array}
$$

Proposition 3.1 *Almost N-projective modules are closed under direct summands and direct sums for a module N.*

Proof Obvious. □

Lemma 3.2 *Let X be a module, M an amply supplemented module, N a lifting module, $f : M \to X$ a homomorphism and $g : N \to X$ a small epimorphism. Then for any direct summand N' of N and any homomorphism $h : N' \to M$ with $fh = g|_{N'}$, $h(N')$ is coclosed in M.*

Proof By Lemma 0.17, there exists a decomposition $N' = N_1 \oplus N_2$ such that $h(N_1)$ is coclosed in M and $h(N_2)$ is small in M. Then $fh(N_2)$ is small in X. On the other hand, by Lemma 0.7 (3), $fh(N_2) = g(N_2)$ is coclosed in X. Therefore $g(N_2) = 0$. Since $N_2 \subseteq \operatorname{Ker} g \ll N$ and $N_2 \leq_{\oplus} N$, $N_2 = 0$. Hence $h(N') = h(N_1)$ is coclosed in M. □

Now we shall give a new characterization of almost N-projective modules.

Theorem 3.3 *Let M and N be modules over a right perfect ring and let (P, φ) and (Q, ψ) be projective covers of M and N, respectively. Then the following conditions are equivalent:*

(a) M is almost N-projective,

(b) for any $\alpha \in \operatorname{Hom}_R(P, Q)$, either $\alpha(\operatorname{Ker}\varphi) \subseteq \operatorname{Ker}\psi$, or there exist $P' \leq_{\oplus} P$ and $Q' \leq_{\oplus} Q$ such that $\alpha|_{P'} : P' \to Q'$ is an isomorphism, $(\alpha|_{P'})^{-1}(\operatorname{Ker}\psi|_{Q'}) \subseteq \operatorname{Ker}\varphi|_{P'}$ and $0 \neq \psi(Q') \leq_{\oplus} N$.

Proof (*a*) ⇒ (*b*): Let $\alpha \in \operatorname{Hom}_R(P, Q)$, put $\overline{N} = N/\psi\alpha(\operatorname{Ker}\varphi)$ and let $\pi : N \to \overline{N}$ be the natural epimorphism. Then we can define a homomorphism $f : M \to \overline{N}$ by $f(\varphi(x)) = \pi\psi\alpha(x)$ $(x \in P)$, and we obtain the following diagram:

$$\begin{array}{ccc} P & \xrightarrow{\alpha} & Q \\ \varphi\downarrow & & \downarrow\psi \\ M & \circlearrowright & N \\ f\searrow & & \swarrow\pi \\ & \overline{N} & . \end{array}$$

Since M is almost N-projective, either there exists a homomorphism $g : M \to N$ such that $\pi g = f$ or there exist a nonzero direct summand N' of N and a homomorphism $h : N' \to M$ such that $fh = \pi|_{N'}$.

First, we consider the former case. For any $x \in P$, $\pi\psi\alpha(x) = f\varphi(x) = \pi g\varphi(x)$. Therefore we see $\psi\alpha(x) - g\varphi(x) \in \operatorname{Ker}\pi = \psi\alpha(\operatorname{Ker}\varphi)$. Take $k \in \operatorname{Ker}\varphi$ with $\psi\alpha(x) - g\varphi(x) = \psi\alpha(k)$. Then $\psi\alpha(x-k) = g\varphi(x) = g\varphi(x-k)$, and so $x - k \in \operatorname{Ker}(\psi\alpha - g\varphi)$. Hence we obtain

$$P = \operatorname{Ker}(\psi\alpha - g\varphi) + \operatorname{Ker}\varphi = \operatorname{Ker}(\psi\alpha - g\varphi).$$

For any $a \in \operatorname{Ker}\varphi$, $0 = (\psi\alpha - g\varphi)(a) = \psi\alpha(a)$. Thus $\alpha(\operatorname{Ker}\varphi) \subseteq \operatorname{Ker}\psi$.

Next, we consider the latter case. Put $M' = h(N')$. Since Q is lifting and N' is coclosed in N, there exists a direct summand Q^* of Q such that $Q^* \subseteq_c^Q \psi^{-1}(N')$, and hence $N' = \psi(Q^*)$. By Lemma 3.2, we see $M' = h\psi(Q^*)$ is coclosed in M.

Put $K = \operatorname{Ker}\left((h\psi\alpha - \varphi)|_{(\psi\alpha)^{-1}(N') \cap \varphi^{-1}(M')}\right)$. Next we prove $M' = \varphi(K)$. For any $m \in M'$, there exist $x \in \varphi^{-1}(M')$ and $n \in N'$ such that $\varphi(x) = m = h(n)$. Then $\pi(n) = fh(n) = f\varphi(x) = \pi\psi\alpha(x)$ and so $n - \psi\alpha(x) \in \operatorname{Ker}\pi = \psi\alpha(\operatorname{Ker}\varphi)$. Hence there exists $k \in \operatorname{Ker}\varphi$ such that $n - \psi\alpha(x) = \psi\alpha(k)$. Then $x + k \in (\psi\alpha)^{-1}(N')$. On the other hand, by $h\psi\alpha(x+k) = h(n) = \varphi(x) = \varphi(x+k)$, we see $x + k \in K$ and hence $m = \varphi(x) = \varphi(x+k) \in \varphi(K)$. Thus $M' = \varphi(K)$.

Since P is lifting, there exists a direct summand P' of P such that $P' \subseteq_c^P K$. As M' is coclosed in M, $M' = \varphi(K) = \varphi(P')$. Put $Q' = \alpha(P')$. Now we show that $(Q', \psi|_{Q'})$ is the projective cover of N'. Since P' and Q are lifting and $\varphi|_{P'} = h\psi\alpha|_{P'}$, by Lemma 3.2, $Q' = \alpha(P')$ is a direct summand of Q, and so Q' is projective. Moreover, since $\varphi|_{P'} = h\psi\alpha|_{P'} : P' \to M'$ is onto and $\operatorname{Ker} h \subseteq \operatorname{Ker} fh = \operatorname{Ker}\pi|_{N'} \ll N'$, $N' = \psi\alpha(P') + \operatorname{Ker} h = \psi\alpha(P') = \psi(Q')$. In addition, by $\operatorname{Ker}\psi \ll Q$, $\operatorname{Ker}\psi|_{Q'} \ll Q'$. Thus $(Q', \psi|_{Q'})$ is the projective cover of N'.

Since Q' is projective and $\operatorname{Ker}\alpha|_{P'} \subseteq \operatorname{Ker} h\psi\alpha|_{P'} = \operatorname{Ker}\varphi|_{P'} \ll P'$, we get $\operatorname{Ker}\alpha|_{P'} = 0$ and hence P' and Q' are isomorphic by the restricted map $\alpha|_{P'}$. Moreover, by $\varphi(\alpha|_{P'})^{-1}(\operatorname{Ker}\psi|_{Q'}) = h\psi\alpha(\alpha|_{P'})^{-1}(\operatorname{Ker}\psi|_{Q'}) = h\psi(\operatorname{Ker}\psi|_{Q'}) = 0$, we obtain

$$(\alpha|_{P'})^{-1}(\operatorname{Ker}\psi|_{Q'}) \subseteq \operatorname{Ker}\varphi|_{P'}.$$

$(b) \Rightarrow (a)$: Let X be a module, let $g : N \to X$ be an epimorphism and let $f : M \to X$ be a homomorphism. Since P is projective, there exists a homomorphism $\alpha : P \to Q$ such that $g\psi\alpha = f\varphi$. By (b), either (i) $\alpha(\operatorname{Ker}\varphi) \subseteq \operatorname{Ker}\psi$, or (ii) there exist $P_1 \leq_\oplus P$ and $Q_1 \leq_\oplus Q$ such that $\alpha|_{P_1} : P_1 \to Q_1$ is an isomorphism, $(\alpha|_{P_1})^{-1}(\operatorname{Ker}\psi|_{Q_1}) \subseteq \operatorname{Ker}\varphi|_{P_1}$ and $0 \neq \psi(Q_1) \leq_\oplus N$.

If the case (i) occurs, then we can define a homomorphism $h : M \to N$ by $h(\varphi(x)) = \psi\alpha(x)$, where $x \in P$. Then, for any $\varphi(x) \in M$, $f(\varphi(x)) = g\psi\alpha(x) = gh(\varphi(x))$.

If the case (ii) occurs, we can define a homomorphism $h' : \psi(Q_1) \to M$ by $h'(\psi(y)) = \varphi(\alpha|_{P_1})^{-1}(y)$, where $y \in Q_1$. Then, for any $\psi(y) \in \psi(Q_1)$, $fh'(\psi(y)) = f\varphi(\alpha|_{P_1})^{-1}(y) = g\psi\alpha(\alpha^{-1}(y)) = g(\psi(y))$. Thus M is almost N-projective. □

The following corollary means that an almost N-projective module is im-small N-projective for a module N. See Appendix 4.1 for details.

Corollary 3.4 (cf. Proposition 4.4) *Let M and N be modules over a right perfect ring and let (P, φ) and (Q, ψ) be projective covers of M and N, respectively. If M is almost N-projective, then for any homomorphism $\alpha : P \to Q$ with $\alpha(P) \ll Q$, $\alpha(\operatorname{Ker}\varphi) \subseteq \operatorname{Ker}\psi$.*

It is well known that M is $\oplus_{i=1}^{n} N_i$-projective if M is N_i-projective for any $i = 1, \ldots, n$. However this fact is not satisfied for almost relative projective modules in general [16]. This is one of differences between relative projective and relative almost projective. Harada and Mabuchi [15] and Harada [14] took note of this fact and they have filled this gap when each module is an indecomposable module with the certain conditions over a right (semi-) perfect ring. As an application of Theorem 3.3, we consider the case that each N_i is not necessarily indecomposable.

First we show the following two lemmas.

Lemma 3.5 *Let M and N be modules, let $f : M \to N$ be an epimorphism such that $M = M_1 \oplus M_2$ and $N = f(M_1) \oplus f(M_2)$, and let $g : M_1 \to M_2$ be a homomorphism. Then $N = f(\langle g \rangle) \oplus f(M_2)$ if and only if $g(\operatorname{Ker} f|_{M_1}) \subseteq \operatorname{Ker} f|_{M_2}$.*

Proof ($\Rightarrow$) For any $k \in \operatorname{Ker} f|_{M_1}$, $f(k + g(k)) = fg(k) \in f(\langle g \rangle) \cap f(M_2) = 0$. Hence $g(\operatorname{Ker} f|_{M_1}) \subseteq \operatorname{Ker} f|_{M_2}$.

($\Leftarrow$) For any $f(m_1 + g(m_1)) = f(m_2) \in f(\langle g \rangle) \cap f(M_2)$, $f(m_1) = f(m_2 - g(m_2)) \in f(M_1) \cap f(M_2) = 0$. By $g(\operatorname{Ker} f|_{M_1}) \subseteq \operatorname{Ker} f|_{M_2}$, $f(g(m_1)) = 0$. Thus $f(m_1 + g(m_1)) = 0$. □

Lemma 3.6 (cf. [29, Proposition 4.35]) *Let I be an arbitrary set, M a direct sum of finitely generated modules and N_i a module ($i \in I$). If M is almost $\oplus_F N_i$-projective for any finite subset F of I, then M is almost $\oplus_I N_i$-projective.*

Proof Obvious. □

Theorem 3.7 *Let M and N_i be modules over a right perfect ring ($i \in I$). If M is lifting and almost N_i-projective for any $i \in I$, and N_i is almost N_j-projective for any distinct $i, j \in I$, then M is almost $\oplus_I N_i$-projective.*

Proof Let (P, φ) and (Q_i, ψ_i) be the projective covers of M and N_i ($i \in I$), respectively, and put $Q = \oplus_I Q_i$, $N = \oplus_I N_i$ and $\psi = \oplus_I \psi_i$.

By Corollary 2.2, M is a direct sum of local modules. If any indecomposable direct summand of M is almost $\oplus_I N_i$-projective, then so is M. Hence we may assume that M is a local module. By Lemma 3.6, it is enough to show the case that I is a finite set.

Let $\alpha : P \to Q$ be a homomorphism. Let $q_k : Q = \oplus_I Q_i \to Q_k$ be the projection ($k \in I$). By Theorem 3.3, for any $k \in I$, either (i) $q_k\alpha(\operatorname{Ker}\varphi) \subseteq \operatorname{Ker}\psi_k$, or (ii) there exists $Q'_k \leq_\oplus Q_k$ such that $0 \neq \psi_k(Q'_k) \leq_\oplus N_k$, $P \cong Q'_k = q_k\alpha(P)$ (by $q_k\alpha$) and $(q_k\alpha)^{-1}(\operatorname{Ker}\psi_k|_{Q'_k}) \subseteq \operatorname{Ker}\varphi$. Note that $\alpha : P \to \alpha(P)$ and $q_k|_{\alpha(P)} : \alpha(P) \to Q'_k$ are isomorphisms in the case (ii).

If (i) holds for any $k \in I$, then

$$\alpha(\operatorname{Ker}\varphi) \subseteq \oplus_I (q_k \alpha(\operatorname{Ker}\varphi)) \subseteq \oplus_I \operatorname{Ker}\psi_k = \operatorname{Ker}\psi.$$

Otherwise, let I_1 be a collection of elements of I which hold (ii). Note that I_1 is nonempty. Suppose I_1 has at least two elements. For any distinct $k, l \in I_1$, $(q_l|_{\alpha(P)})(q_k|_{\alpha(P)})^{-1} : Q'_k \to Q'_l$ is an isomorphism. Since $\psi_k(Q'_k)$ is almost $\psi_l(Q'_l)$-projective, by Theorem 3.3, either $(q_l|_{\alpha(P)})(q_k|_{\alpha(P)})^{-1}(\operatorname{Ker}\psi_k|_{Q'_k}) \subseteq \operatorname{Ker}\psi_l|_{Q'_l}$, or $(q_k|_{\alpha(P)})(q_l|_{\alpha(P)})^{-1}(\operatorname{Ker}\psi_l|_{Q'_l}) \subseteq \operatorname{Ker}\psi_k|_{Q'_k}$. Thus we can take $j \in I_1$ such that $(q_j|_{\alpha(P)})^{-1}(\operatorname{Ker}\psi_j|_{Q'_j}) \subseteq (q_k|_{\alpha(P)})^{-1}(\operatorname{Ker}\psi_k|_{Q'_k})$ for any $k \in I_1$.

Put $f_i = (q_i|_{\alpha(P)})(q_j|_{\alpha(P)})^{-1} : Q'_j \to Q_i$ for any $i \in I \setminus \{j\}$ and $f = \sum_{I \setminus \{j\}} f_i$. Then $f(\operatorname{Ker}\psi_j|_{Q'_j}) \subseteq \operatorname{Ker}\psi|_{\oplus_{I\setminus\{j\}} Q_i}$ and $\alpha(P) = \langle f \rangle$. Note that if I_1 has only one element, then the above f can be defined in the same way. By Lemma 3.5, $\psi(\alpha(P))$ is a direct summand of N and isomorphic to a direct summand of N_j. Since M is almost $\psi(\alpha(P))$-projective, by Theorem 3.3, either $\alpha(\operatorname{Ker}\varphi) \subseteq \operatorname{Ker}\psi|_{\alpha(P)}$, or $P \cong \alpha(P)$ (by α) and $\alpha^{-1}(\operatorname{Ker}\psi|_{\alpha(P)}) \subseteq \operatorname{Ker}\varphi$. By Theorem 3.3 again, we obtain that M is almost N-projective. □

3.2 A characterization of generalized N-projective modules and its applications

In this section, we first give a characterization of generalized projective modules by homomorphisms between their projective covers. In addition, as its application, we consider a condition for a direct sum of lifting modules to be lifting over a right perfect ring. Recall the definition of generalized N-projective.

Definition. A module M is said to be *generalized N-projective* for a module N if, for any module X, any homomorphism $f : M \to X$ and any epimorphism $g : N \to X$, there exist direct sum decompositions $M = M_1 \oplus M_2$ and $N = N_1 \oplus N_2$, a homomorphism $h_1 : M_1 \to N_1$ and an epimorphism $h_2 : N_2 \to M_2$ such that

$f|_{M_1} = gh_1$ and $g|_{N_2} = fh_2$.

$$\begin{array}{ccccccc} M_1 & \oplus & M_2 & & = & M & \\ h_1 \downarrow & & h_2 \uparrow & & & \downarrow f & \\ N_1 & \oplus & N_2 & = & N \stackrel{g}{\to} & X & \to 0 \end{array} .$$

Proposition 3.8 *Let M and N be modules over any ring. If M is generalized N-projective, then M is generalized A-projective for any submodule A of N.*

Proof Let A and X be submodules of N with $X \subseteq A$, $\pi : N \to N/X$ the natural epimorphism and $f : M \to A/X$ a homomorphism. Then

$$\begin{array}{cccc} & & M & \\ & & \downarrow f & \\ A & \stackrel{\pi|_A}{\to} & A/X & \to 0 \\ \cap\!| & & \cap\!| & \\ N & \stackrel{\pi}{\to} & N/X & \to 0 \end{array} .$$

Since M is generalized N-projective, there exist decompositions $M = M_1 \oplus M_2$ and $N = N_1 \oplus N_2$, a homomorphism $h_1 : M_1 \to N_1$ and an epimorphism $h_2 : N_2 \to M_2$ such that $\pi h_1 = f|_{M_1}$ and $fh_2 = \pi|_{N_2}$. For any $n \in N_2$, $n + X = \pi(n) = fh_2(n) \in A/X$ and hence $N_2 \subseteq A$. So we see $A = (A \cap N_1) \oplus N_2$. Similarly, for any $m \in M_1$, by $h_1(m) + X = \pi h_1(m) = f(m) \in A/X$, $h_1(M_1) \subseteq A \cap N_1$. Thus we obtain the following diagram:

$$\begin{array}{ccccccc} M_1 & \oplus & M_2 & & = & M & \\ h_1 \downarrow & & h_2 \uparrow & & & \downarrow f & \\ (A \cap N_1) & \oplus & N_2 & = & A \stackrel{\pi|_A}{\to} & A/X & \to 0 \end{array} .$$

Therefore M is generalized A-projective. □

Theorem 3.9 *Let M and N be modules over a right perfect ring and let (P, φ) and (Q, ψ) be projective covers of M and N, respectively. Then the following conditions* *are equivalent:*

(a) M *is generalized* N*-projective,*

(b) for any $\alpha \in \operatorname{Hom}_R(P, Q)$*, there exist decompositions* $P = P_1 \oplus P_2$ *and* $Q = Q_1 \oplus Q_2$ *such that* $\alpha(P_1) \subseteq Q_1$*,* $\alpha(\operatorname{Ker}\varphi|_{P_1}) \subseteq \operatorname{Ker}\psi|_{Q_1}$*,* $\alpha|_{P_2} : P_2 \to Q_2$ *is an isomorphism,* $(\alpha|_{P_2})^{-1}(\operatorname{Ker}\psi|_{Q_2}) \subseteq \operatorname{Ker}\varphi|_{P_2}$*,* $M = \varphi(P_1) \oplus \varphi(P_2)$ *and* $N = \psi(Q_1) \oplus \psi(Q_2)$*.*

Proof $(a) \Rightarrow (b)$: Let $\alpha \in \operatorname{Hom}_R(P, Q)$, put $\overline{N} = N/\psi\alpha(\operatorname{Ker}\varphi)$ and let $\pi : N \to \overline{N}$ be the natural epimorphism. Then we can define a homomorphism $f : M \to \overline{N}$ by $f(\varphi(x)) = \pi\psi\alpha(x)$ $(x \in P)$. Since M is generalized N-projective, there exist decompositions $M = M_1 \oplus M_2$ and $N = N_1 \oplus N_2$, a homomorphism $h_1 : M_1 \to N_1$ and an epimorphism $h_2 : N_2 \to M_2$ such that $f|_{M_1} = \pi h_1$ and $\pi|_{N_2} = f h_2$. Put $K_2 = \operatorname{Ker}\big((h_2\psi\alpha - \varphi)|_{(\psi\alpha)^{-1}(N_2)\cap\varphi^{-1}(M_2)}\big)$. Since P is lifting, there exists a direct summand P_2 of P such that $P_2 \subseteq_c^P K_2$. Put $Q_2 = \alpha(P_2)$. By the same way as in the proof of $(a) \Rightarrow (b)$ in Theorem 3.3, we obtain $M_2 = \varphi(K_2) = \varphi(P_2)$, $(Q_2, \psi|_{Q_2})$ is the projective cover of N_2,

$$P_2 \overset{\alpha|_{P_2}}{\cong} Q_2 \leq_\oplus Q \quad \text{and} \quad (\alpha|_{P_2})^{-1}(\operatorname{Ker}\psi|_{Q_2}) \subseteq \operatorname{Ker}\varphi|_{P_2}.$$

Since Q is quasi-discrete, by Lemma 0.19, we can take a direct summand Q_1 of Q such that $Q = Q_1 \oplus Q_2$ and $(Q_1, \psi|_{Q_1})$ is the projective cover of N_1. Now we take any $x \in P$ and express $\alpha(x)$ in $Q = Q_1 \oplus Q_2$ as $\alpha(x) = y_1 + y_2$ where $y_i \in Q_i$ $(i = 1, 2)$. By $y_2 \in Q_2 = \alpha(P_2)$, there exists $x_2 \in P_2$ such that $\alpha(x_2) = y_2$. Hence $\alpha(x - x_2) = y_1 \in Q_1$. So we see $x \in P_2 + \alpha^{-1}(Q_1)$ and hence $P = \alpha^{-1}(Q_1) + P_2$. By $\alpha(\alpha^{-1}(Q_1) \cap P_2) \subseteq Q_1 \cap Q_2 = 0$ and $\operatorname{Ker}\alpha|_{P_2} = 0$, $\alpha^{-1}(Q_1) \cap P_2 = 0$. Hence $P = \alpha^{-1}(Q_1) \oplus P_2$. Put $P_1 = \alpha^{-1}(Q_1)$. Then

$$P = P_1 \oplus P_2 \quad \text{and} \quad \alpha(P_1) \subseteq Q_1.$$

Put $K_1 = \operatorname{Ker}\big((\psi\alpha - h_1\varphi)|_{P_1\cap\varphi^{-1}(M_1)}\big)$. Now we shall show that $M_1 = \varphi(K_1)$. Given $m_1 \in M_1$. As φ is onto, there exists $x \in \varphi^{-1}(M_1)$ with $\varphi(x) = m_1$. Then

$\pi h_1\varphi(x) = f\varphi(x) = \pi\psi\alpha(x)$ and hence $h_1\varphi(x) - \psi\alpha(x) \in \operatorname{Ker}\pi = \psi\alpha(\operatorname{Ker}\varphi)$. So there exists $k \in \operatorname{Ker}\varphi$ such that $h_1\varphi(x) - \psi\alpha(x) = \psi\alpha(k)$. Now we express x and k in $P = P_1 \oplus P_2$ as $x = x_1 + x_2$ and $k = k_1 + k_2$ ($x_i, k_i \in P_i$), respectively. Then $h_1\varphi(x) - \psi\alpha(x_1 + k_1) = \psi\alpha(x_2 + k_2) \in N_1 \cap N_2 = 0$. By $\psi\alpha(x_2 + k_2) = 0$, $\varphi(x_2 + k_2) = h_2\psi\alpha(x_2 + k_2) = 0$ and hence $x_2 + k_2 \in \operatorname{Ker}\varphi$. In addition, by $h_1\varphi(x) - \psi\alpha(x_1 + k_1) = 0$, $\psi\alpha(x_1 + k_1) = h_1\varphi(x) = h_1\varphi((x_1 + k_1) + (x_2 + k_2) - k) = h_1\varphi(x_1 + k_1)$ and hence $(\psi\alpha - h_1\varphi)(x_1 + k_1) = 0$. Moreover, $m_1 = \varphi(x) = \varphi((x_1 + k_1) + (x_2 + k_2) - k) = \varphi(x_1 + k_1) \in \varphi(K_1)$. Thus $M_1 = \varphi(K_1)$.

By $\varphi(P) = M = M_1 \oplus M_2 = \varphi(K_1) \oplus \varphi(P_2)$, we see $P = (K_1 \oplus P_2) + \operatorname{Ker}\varphi = K_1 \oplus P_2$ and so $K_1 = P_1$. Hence $(P_1, \varphi|_{P_1})$ is the projective cover of M_1.

Thus we obtain

$$\begin{array}{ccc} P_1 & \overset{\alpha|_{P_1}}{\rightarrow} & Q_1 \\ \varphi|_{P_1}\downarrow & \circlearrowright & \downarrow\psi|_{Q_1} \\ M_1 & \overset{h_1}{\rightarrow} & N_1 \\ f|_{M_1} \searrow & \circlearrowright & \swarrow \pi|_{N_1} \\ & \pi(N_1) & . \end{array}$$

By $\psi\alpha(\operatorname{Ker}\varphi|_{P_1}) = h_1\varphi(\operatorname{Ker}\varphi|_{P_1}) = 0$,

$$\alpha(\operatorname{Ker}\varphi|_{P_1}) \subseteq \operatorname{Ker}\psi|_{Q_1}.$$

Thus (b) holds.

$(b) \Rightarrow (a)$: Let X be a module, $g : N \to X$ an epimorphism and $f : M \to X$ a homomorphism. Since P is projective, there exists a homomorphism $\alpha : P \to Q$ such that $g\psi\alpha = f\varphi$. By (b), there exist decompositions $P = P_1 \oplus P_2$ and $Q = Q_1 \oplus Q_2$ such that $\alpha(P_1) \subseteq Q_1$, $\alpha(\operatorname{Ker}\varphi|_{P_1}) \subseteq \operatorname{Ker}\psi$, $\alpha|_{P_2} : P_2 \to Q_2$ is an isomorphism, $(\alpha|_{P_2})^{-1}(\operatorname{Ker}\psi|_{Q_2}) \subseteq \operatorname{Ker}\varphi|_{P_2}$, $M = \varphi(P_1) \oplus \varphi(P_2)$ and $N = \psi(Q_1) \oplus \psi(Q_2)$. Then we can define a homomorphism $h_1 : \varphi(P_1) \to \psi(Q_1)$ by $h_1(\varphi(x)) = \psi\alpha(x)$ and an epimorphism $h_2 : \psi(Q_2) \to \varphi(P_2)$ by $h_2(\psi(y)) = \varphi(\alpha|_{P_2})^{-1}(y)$, where $x \in P_1$ and $y \in Q_2$. Clearly $gh_1 = f|_{\varphi(P_1)}$ and $fh_2 = g|_{\psi(Q_2)}$, and hence we see M is generalized N-projective. $\square$

Corollary 3.10 (cf. [30, Proposition 3.7] and [24, Proposition 2.3]) *Let M and N be modules over a right perfect ring. If M is generalized N-projective, then M' is generalized N-projective for any direct summand M' of M.*

Proof Let (P, φ) and (Q, ψ) be the projective covers of M and N respectively, and let $M = M' \oplus M''$. By Lemma 0.19, there exists a decomposition $P = P' \oplus P''$ such that $(P', \varphi|_{P'})$ and $(P'', \varphi|_{P''})$ are the projective covers of M' and M'', respectively. Let $p : P = P' \oplus P'' \to P'$ be the projection and take a homomorphism $\alpha : P' \to Q$. By Theorem 3.9, there exist decompositions $P = P_1 \oplus P_2$ and $Q = Q_1 \oplus Q_2$ such that $\alpha p(P_1) \subseteq Q_1$, $\alpha p(\operatorname{Ker} \varphi|_{P_1}) \subseteq \operatorname{Ker} \psi|_{Q_1}$, $\alpha p|_{P_2} : P_2 \to Q_2$ is an isomorphism, $(\alpha p|_{P_2})^{-1}(\operatorname{Ker} \psi|_{Q_2}) \subseteq \operatorname{Ker} \varphi|_{P_2}$, $M = \varphi(P_1) \oplus \varphi(P_2)$ and $N = \psi(Q_1) \oplus \psi(Q_2)$. Since $\alpha p|_{P_2}$ is an isomorphism, we see $P'' \subseteq \operatorname{Ker} \alpha \oplus P'' = \operatorname{Ker} \alpha p \subseteq P_1$. Hence we obtain

$$P' = p(P_1) \oplus p(P_2).$$

Given $k = p(x_1) \in \operatorname{Ker} \varphi|_{p(P_1)}$, where $x_1 \in P_1$. Then $k = x_1 - (1-p)(x_1) \in P_1 + P'' = P_1$ and hence $\operatorname{Ker} \varphi|_{p(P_1)} \subseteq \operatorname{Ker} \varphi|_{P_1}$ and $\alpha(k) = \alpha p(x_1) \in \alpha p(\operatorname{Ker} \varphi|_{P_1}) \cap Q_1 \subseteq \operatorname{Ker} \psi \cap Q_1 = \operatorname{Ker} \psi|_{Q_1}$. Thus

$$\alpha(\operatorname{Ker} \varphi|_{p(P_1)}) \subseteq \operatorname{Ker} \psi|_{Q_1}.$$

By $\alpha(p(P_2)) = Q_2$,

$$\alpha|_{p(P_2)} : p(P_2) \to Q_2 \text{ is an isomorphism.}$$

Now we show $(\alpha|_{p(P_2)})^{-1}(\operatorname{Ker} \psi|_{Q_2}) \subseteq \operatorname{Ker} \varphi|_{p(P_2)}$. By $M' \cap M'' = 0$, we obtain $p((\alpha p|_{P_2})^{-1}(\operatorname{Ker} \psi|_{Q_2})) \subseteq p(P_2) \cap \operatorname{Ker} \varphi = \operatorname{Ker} \varphi|_{p(P_2)}$. Hence

$$(\alpha|_{p(P_2)})^{-1}(\operatorname{Ker} \psi|_{Q_2}) \subseteq \operatorname{Ker} \varphi|_{p(P_2)}.$$

Next we prove $M' = \varphi(p(P_1)) \oplus \varphi(p(P_2))$. Given $\varphi(p(x_1)) = \varphi(p(x_2)) \in \varphi(p(P_1)) \cap \varphi(p(P_2))$ where $x_i \in P_i$ $(i = 1, 2)$ and we express x_i in $P = P' \oplus P''$ as $x_i = x_i' + x_i''$ where $x_i' \in P'$ and $x_i'' \in P''$ $(i = 1, 2)$. By $\varphi(x_1' - x_2') = \varphi(p(x_1)) - \varphi(p(x_2)) = 0$,

$\varphi(x_2) = \varphi(x_1) - \varphi(x_1'' - x_2'') \in \varphi(P_1) \cap \varphi(P_2) = 0$. So $\varphi(p(x_2)) = \varphi(x_2') = -\varphi(x_2'') \in \varphi(P') \cap \varphi(P'') = M' \cap M'' = 0$. Hence we see $\varphi(p(P_1)) \cap \varphi(p(P_2)) = 0$ and so

$$M' = \varphi(p(P_1)) \oplus \varphi(p(P_2)).$$

$$\begin{array}{ccccclc}
P' & = & p(P_1) & \oplus & p(P_2) & & M' = \varphi(p(P_1)) \oplus \varphi(p(P_2)) \\
 & & \downarrow \alpha|_{p(P_1)} & & \wr\!\uparrow (\alpha|_{p(P_2)})^{-1} & & \\
Q & = & Q_1 & \oplus & Q_2 & , & N = \psi(Q_1) \oplus \psi(Q_2)
\end{array}$$

By Theorem 3.9 again, M' is generalized N-projective. $\square$

Clearly, a generalized N-projective module is almost N-projective for any module N. However even case that M and N are indecomposable modules over an artinian ring, the converse does not hold. For example, let K be a field, $R = \begin{pmatrix} K & 0 & K \\ 0 & K & K \\ 0 & 0 & K \end{pmatrix}$, $M = (K, K, K)$ and $N = (K, 0, K)$. Then right R-module M has the following submodule lattice:

$$\begin{array}{ccccc}
 & & M & & \\
 & / & & \backslash & \\
N = (K, 0, K) & & & & (0, K, K) \\
 & \backslash & & / & \\
 & & (0, 0, K) & & \\
 & & | & & \\
 & & (0, 0, 0) & & .
\end{array}$$

Let $\pi : N \to N/(0, 0, K)$ be the natural epimorphism. Since $(1, 1, 0)$ is a generator of M, any nonzero homomorphism $f : M \to N/(0, 0, K)$ is defined by $f(1, 1, 0) = \overline{(a, 0, 0)}$ for some $a \in K$. Since the length of N is less than that of M, there are no epimorphisms from N to M. In addition, by $\mathrm{Hom}_R(M, N) = 0$, M is not generalized N-projective. On the other hand, for a homomorphism $h : N \to M$ defined by

$h((1,0,0)) = (a^{-1},0,0)$, we see $fh = \pi$. Hence M is almost N-projective.

Now we consider a condition for an almost N-projective module to be generalized N-projective.

Lemma 3.11 ([29, Lemma 4.22]) *Let M be a quasi-discrete module and let M_i be an indecomposable summand of M $(i \in I)$ such that $M = \sum_I M_i$ and $M \neq \sum_{I\setminus\{k\}} M_i$ for any $k \in I$. Then $M = \oplus_I M_i$.*

Theorem 3.12 *Let M and N be lifting modules over a right perfect ring. Then M is almost N-projective if and only if M is generalized N-projective.*

Proof Let (P, φ) and (Q, ψ) be the projective covers of M and N respectively, and let $\alpha : P \to Q$ be a homomorphism. Put $\Gamma = \{X \leq_\oplus P \mid \alpha(\operatorname{Ker}\varphi|_X) \subseteq \operatorname{Ker}\psi\}$. Since P satisfies LSS, by Zorn's Lemma, there exists a maximal element X of Γ. As M is lifting and P is quasi-discrete, by Lemma 0.19, we obtain decompositions

$$P = X \oplus P' \quad \text{and} \quad M = \varphi(X) \oplus \varphi(P').$$

Then $\operatorname{Ker}\varphi = \operatorname{Ker}\varphi|_X \oplus \operatorname{Ker}\varphi|_{P'}$. Since P' is lifting and the maximality of X, we see $\operatorname{Ker}\alpha|_{P'} \ll P'$. By Lemma 0.17, there exists a decomposition $P' = A \oplus B$ such that $\alpha(A) \leq_\oplus Q$ and $\alpha(B) \ll Q$. By Corollary 3.4, we see $\alpha(\operatorname{Ker}\varphi|_B) \subseteq \operatorname{Ker}\psi$. By the maximality of X, we see $B = 0$ and hence $\alpha(P') = \alpha(A) \leq_\oplus Q$. As $\alpha(P')$ is projective and $\operatorname{Ker}\alpha|_{P'} \ll P'$, $\alpha|_{P'} : P' \to \alpha(P')$ is an isomorphism. By Lemmas 0.7 (1) and 2.2, $\psi(\alpha(P'))$ is a direct summand of N and it has an indecomposable decomposition $\psi(\alpha(P')) = \oplus_I N_i$. Then there exists a direct summand Q_i of $\alpha(P')$ such that $(Q_i, \psi|_{Q_i})$ is the projective cover of N_i $(i \in I)$, by Lemma 3.11, $\alpha(P') = \oplus_I Q_i$. Put $P_i = (\alpha|_{P'})^{-1}(Q_i)$ and $M_i = \varphi(P_i)$ $(i \in I)$. Since M_i is almost N_i-projective for any $i \in I$ and the maximality of X, we see $(\alpha|_{P_i})^{-1}(\operatorname{Ker}\psi|_{\alpha(P_i)}) \subseteq \operatorname{Ker}\varphi|_{P_i}$. By $\operatorname{Ker}\psi|_{\alpha(P')} = \oplus_I \operatorname{Ker}\psi|_{Q_i}$,

$$(\alpha|_{P'})^{-1}(\operatorname{Ker}\psi|_{\alpha(P')}) \subseteq \operatorname{Ker}\varphi|_{P'}.$$

Since N is lifting and R is right perfect, there exists a direct summand Q' of Q such that

$$Q = Q' \oplus \alpha(P') \quad \text{and} \quad N = \psi(Q') \oplus \psi(\alpha(P')).$$

Then $\operatorname{Ker}\psi = \operatorname{Ker}\psi|_{Q'} \oplus \operatorname{Ker}\psi|_{\alpha(P')}$. Let $q_1 : Q = Q' \oplus \alpha(P') \to Q'$ and $q_2 : Q = Q' \oplus \alpha(P') \to \alpha(P')$ be the projections, and put $f = -(\alpha|_{P'})^{-1}q_2\alpha|_X$. Then $f(\operatorname{Ker}\varphi|_X) = -(\alpha|_{P'})^{-1}q_2\alpha(\operatorname{Ker}\varphi|_X) \subseteq -(\alpha|_{P'})^{-1}(\operatorname{Ker}\psi|_{\alpha(P')}) \subseteq \operatorname{Ker}\varphi|_{P'}$. By Lemma 3.5, we obtain

$$P = \langle f \rangle \oplus P' \quad \text{and} \quad M = \varphi(\langle f \rangle) \oplus \varphi(P').$$

By $\alpha(\langle f \rangle) \subseteq (\alpha - q_2\alpha)(X) \subseteq Q'$, $\alpha|_{\langle f \rangle} : \langle f \rangle \to Q'$ is a homomorphism. Given $k + f(k) \in \operatorname{Ker}\varphi|_{\langle f \rangle}$. By $k \in \operatorname{Ker}\varphi|_X$, $\alpha(k + f(k)) = q_1\alpha(k) \in q_1(\operatorname{Ker}\psi) = \operatorname{Ker}\psi|_{Q'}$ and so

$$\alpha(\operatorname{Ker}\varphi|_{\langle f \rangle}) \subseteq \operatorname{Ker}\psi|_{Q'}.$$

By Theorem 3.9, M is generalized N-projective. □

Lemma 3.13 ([24, Theorem 3.1]) *Let M_1 and M_2 be lifting modules and put $M = M_1 \oplus M_2$. Then, M is lifting and the decomposition $M = M_1 \oplus M_2$ is exchangeable, if and only if M_i' is generalized M_j-projective for any direct summand M_i' of M_i $(i \neq j)$.*

The following result is a consequence of Theorems 3.12 and 3.7, Corollary 3.10 and Lemma 3.13.

Proposition 3.14 *Let A, B_1 and B_2 be lifting modules over a right perfect ring. Assume that B_i is generalized B_j-projective for distinct $i, j \in \{1, 2\}$.*

(1) If A is generalized B_i-projective $(i = 1, 2)$, then A is generalized $B_1 \oplus B_2$-projective.

(2) If B_i is generalized A-projective $(i = 1, 2)$, then $B_1 \oplus B_2$ is generalized A-projective.

Lemma 3.15 ([24, Theorem 3.7]) *Let $M_1, M_2, \ldots, M_n$ be lifting modules with the finite internal exchange property and put $M = \oplus_{i=1}^n M_i$. Then M is lifting with the finite internal exchange property if and only if M_k and $\oplus_{i \neq k} M_i$ are generalized relative projective for each $k = 1, 2, \ldots, n$.*

Finally, we give conditions for a direct sum of lifting modules to be lifting over a right perfect ring.

Corollary 3.16 *Let $M_1, M_2, \ldots, M_n$ be lifting modules (lifting modules with the finite internal exchange property, resp.) over a right perfect ring and put $M = \oplus_{i=1}^n M_i$. Then the following conditions are equivalent:*

(a) (i) M is lifting, and

(ii) the decomposition $M = \oplus_{i=1}^n M_i$ is exchangeable (M satisfies the finite internal exchange property, resp.),

(b) M_i is generalized M_j-projective for any distinct $i, j \in \{1, 2, \ldots, n\}$,

(c) M_i is almost M_j-projective for any distinct $i, j \in \{1, 2, \ldots, n\}$.

Proof By Theorem 3.12, Proposition 3.14, Lemmas 3.13 and 3.15 and induction. □

4 Appendix

In this appendix, we provide details about im-small projective, im-closed projective and im-summand projective related to almost projective and generalized projective, and rings whose factor square full modules are closed under essential extensions or essential submodules.

4.1 Im-small coinvariant modules and im-summand coinvariant modules

The content of this section is described in [20].

A module M is said to be *im-small* (*im-coclosed*, *im-summand*, resp.) *N-projective* for a module N if, for any module X, any homomorphism $f : M \to X$ with $f(M) \ll X$ ($f(M)$ is coclosed in X, $f(M) \leq_{\oplus} X$, resp.) and any epimorphism $g : N \to X$, there exists a homomorphism $h : M \to N$ such that $gh = f$.

Now we define notions of N-im-small coinvariant modules and N-im-summand coinvariant modules.

Definition. Let M and N be modules over a right perfect ring and let (P, φ) and (Q, ψ) be projective covers of M and N, respectively. M is called *N-im-small coinvariant* (*N-im-summand coinvariant*, resp.) if, for any homomorphism $\alpha : P \to Q$ with $\alpha(P) \ll Q$ ($\alpha(P) \leq_{\oplus} Q$, resp.), $\alpha(\operatorname{Ker}\varphi) \subseteq \operatorname{Ker}\psi$.

Proposition 4.1 *Let M, N, M_i ($i \in I$) and N_j ($j \in J$) be modules over a right perfect ring. Then*

(1) If M is N-im-small coinvariant, then M is N/X-im-small coinvariant and X-im-small coinvariant for any submodule X of N.

(2) If M is N-im-summand coinvariant, then M is N/X-im-summand coinvariant for any submodule X of N. Moreover, for any coclosed submodule X of N, M is X-im-summand coinvariant.

(3) If M/S is N-im-small (N-im-summand, resp.) coinvariant for some $S \ll M$, then M is N-im-small (N-im-summand, resp.) coinvariant.

(4) If M is N-im-small (N-im-summand, resp.) coinvariant, then M' is N-im-small (N-im-summand, resp.) coinvariant for any direct summand M' of M.

(5) If M_i is N_j-im-small coinvariant for any $i \in I$ and $j \in J$, then $\oplus_I M_i$ is $\oplus_J N_j$-im-small coinvariant.

Proof (1) Let X be a submodule of N, let (P, φ), (Q, ψ) and (Q', ψ') be the projective covers of M, N and N/X, respectively. Let $\alpha : P \to Q'$ be a homomorphism with $\alpha(P) \ll Q'$. Since Q is projective, there exists a homomorphism $f : Q \to Q'$ such that $\psi' f = \pi\psi$, where $\pi : N \to N/X$ is the natural epimorphism. As $\operatorname{Ker} \psi' \ll Q'$ and $\pi\psi$ is onto, f is an epimorphism. Therefore there exists a monomorphism $g : Q' \to Q$ such that $fg = 1_{Q'}$. Since M is N-im-small coinvariant, we see $g\alpha(\operatorname{Ker} \varphi) \subseteq \operatorname{Ker} \psi$. Then $\psi'\alpha(\operatorname{Ker} \varphi) = \psi' f g\alpha(\operatorname{Ker} \varphi) \subseteq \psi' f(\operatorname{Ker} \psi) = \pi\psi(\operatorname{Ker} \psi) = 0$, and hence $\alpha(\operatorname{Ker} \varphi) \subseteq \operatorname{Ker} \psi'$. Thus M is N/X-im-small coinvariant.

Next we show that M is X-im-small coinvariant. Let (Q'', ψ'') be the projective cover of X and let $\beta : P \to Q''$ be a homomorphism with $\beta(P) \ll Q''$. Since Q'' is projective, there exists a homomorphism $h : Q'' \to Q$ such that $\psi h = \psi''$. As M is N-im-small coinvariant, $h\beta(\operatorname{Ker} \varphi) \subseteq \operatorname{Ker} \psi$ and hence $\beta(\operatorname{Ker} \varphi) \subseteq \operatorname{Ker}(\psi h) = \operatorname{Ker} \psi''$. Thus M is X-im-small coinvariant.

(2) We can see that M is N/X-im-summand coinvariant for any submodule X of N, by similar proof of (1). Let X be a coclosed submodule of N and let (P, φ) and (Q, ψ) be projective covers of M and N, respectively. Since Q is lifting, by Lemma 0.7 (3), there exists a direct summand T of Q such that $X = q(T)$. Then, $(T, \psi|_T)$ is the projective cover of X. For any homomorphism $f : P \to T$ with $f(P) \leq_\oplus T$, since M is N-im-summand coinvariant, $f(\operatorname{Ker} \varphi) \subseteq \operatorname{Ker} \psi$. On the other hand, by $f(\operatorname{Ker} \varphi) \subseteq T$, we see $f(\operatorname{Ker} \varphi) \subseteq \operatorname{Ker} \psi \cap T = \operatorname{Ker} \psi|_T$. Thus M is X-im-summand coinvariant.

(3) We prove only the case for N-im-small coinvariant. Let S be a small submodule of M and let (P, φ) and (Q, ψ) be the projective covers of M and N, respectively.

By $S \ll M$, $(P, \pi\varphi)$ is the projective cover of M/S, where $\pi : M \to M/S$ is the natural epimorphism. Let $\alpha : P \to Q$ be a homomorphism with $\alpha(P) \ll Q$. Since M/S is N-im-small coinvariant, $\alpha(\operatorname{Ker}\varphi) \subseteq \alpha(\operatorname{Ker}\pi\varphi) \subseteq \operatorname{Ker}\psi$. Thus M is N-im-small coinvariant.

(4) We prove only the case for N-im-small coinvariant. Let $M = M' \oplus M''$ and let (P', φ'), (P'', φ'') and (Q, ψ) be the projective covers of M', M'' and N, respectively. Let $\alpha : P' \to Q$ be a homomorphism with $\alpha(P') \ll Q$. Put $P = P' \oplus P''$ and $\varphi = \varphi' \oplus \varphi''$. Then (P, φ) is the projective cover of M. Since M is N-im-small coinvariant, $\alpha(\operatorname{Ker}\varphi') = (\alpha \oplus 0)(\operatorname{Ker}\varphi) \subseteq \operatorname{Ker}\psi$. Thus M' is N-im-small coinvariant.

(5) First we show if each M_i is N-im-small coinvariant, then so is $\oplus_I M_i$. Let (P_i, φ_i) and (Q, ψ) be the projective covers of M_i $(i \in I)$ and N, respectively. Put $M = \oplus_I M_i$, $P = \oplus_I P_i$ and $\varphi = \oplus_I \varphi_i$. Then (P, φ) is the projective cover of M. Let $\alpha : P \to Q$ be a homomorphism with $\alpha(P) \ll Q$. Since each M_i is N-im-small coinvariant, $(\alpha|_{P_i})(\operatorname{Ker}\varphi_i) \subseteq \operatorname{Ker}\psi$. Hence $\alpha(\operatorname{Ker}\varphi) = \alpha(\oplus_I \operatorname{Ker}\varphi_i) = \sum_I \alpha(\operatorname{Ker}\varphi_i) \subseteq \operatorname{Ker}\psi$.

Next we show if M is N_j-im-small coinvariant for any $j \in J$, then M is $\oplus_J N_j$-im-small coinvariant. Let (P, φ) and (Q_j, ψ_j) be the projective covers of M and N_j $(j \in J)$, respectively. Put $N = \oplus_J N_j$, $Q = \oplus_J Q_j$ and $\psi = \oplus_J \psi_j$. Then (Q, ψ) is the projective cover of N. Let $p_k : Q = \oplus_J Q_j \to Q_k$ $(k \in J)$ be the projection and let $\beta : P \to Q$ be a homomorphism with $\beta(P) \ll Q$. Since M is N_j-im-small coinvariant, $p_j\beta(\operatorname{Ker}\varphi) \subseteq \operatorname{Ker}\psi_j$ for any $j \in J$. Hence $\beta(\operatorname{Ker}\varphi) \subseteq \oplus_J p_j\beta(\operatorname{Ker}\varphi) \subseteq \oplus_J \operatorname{Ker}\psi_j = \operatorname{Ker}\psi$. $\square$

Lemma 4.2 *Let M_1 and M_2 be modules and put $M = M_1 \oplus M_2$. If $M = X \oplus M_1'' \oplus M_2''$ for some $X \subseteq M$ and $M_i'' \subseteq M_i$ $(i = 1, 2)$, then there exist $M_i' \subseteq M_i$ $(i = 1, 2)$ and a homomorphism $\varepsilon_i : M_i' \to M_j''$ $(i \neq j)$ such that $M_i = M_i' \oplus M_i''$ and $X = \langle\varepsilon_1\rangle \oplus \langle\varepsilon_2\rangle$.*

Proof Let $M = X \oplus M_1'' \oplus M_2''$, where $M_i'' \subseteq M_i$ and put $M_i = A_i \oplus M_i''$ $(i = 1, 2)$. Let $p : M = A_1 \oplus A_2 \oplus M_1'' \oplus M_2'' \to A_1 \oplus A_2$ and $q : M = A_1 \oplus A_2 \oplus M_1'' \oplus M_2'' \to$

$M_1'' \oplus M_2''$ be the projections. Then we see $p(X) = A_1 \oplus A_2$ and we can define a homomorphism $f : p(X) \to q(X)$ by $f(p(x)) = q(x)$, where $x \in X$. So we see

$$X = \langle f \rangle = \langle f|_{A_1} \rangle \oplus \langle f|_{A_2} \rangle.$$

Let $\pi_i : M_1'' \oplus M_2'' \to M_i''$ be the projection and let $\beta_i : \langle \pi_i f|_{A_i} \rangle \to A_i$ be the natural isomorphism $(i = 1, 2)$. Then we obtain $\langle f|_{A_i} \rangle = \langle \pi_j f \beta_i \rangle$ $(i \neq j)$. Put $M_i' = \langle \pi_i f|_{A_i} \rangle$ and $\varepsilon_i = \pi_j f \beta_i$ $(i \neq j)$, then $M_i = M_i' \oplus M_i''$ $(i = 1, 2)$ and $X = \langle \varepsilon_1 \rangle \oplus \langle \varepsilon_2 \rangle$. □

Proposition 4.3 *Let $M_1, M_2, \ldots, M_m$, $N_1, N_2, \ldots, N_n$ be modules over a right perfect ring. If M_i is N_j-im-summand coinvariant and N_j-im-small coinvariant for any $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, n\}$, then $\oplus_{i=1}^m M_i$ is $\oplus_{j=1}^n N_j$-im-summand coinvariant.*

Proof It is enough to show the case of $m = n = 2$, by Proposition 4.1 (5).

First we show M is $N_1 \oplus N_2$-im-summand coinvariant if M is N_i-im-summand coinvariant and N_i-im-small coinvariant $(i = 1, 2)$. Let (P, φ) and (Q_i, ψ_i) be the projective covers of M and N_i $(i = 1, 2)$ respectively, put $Q = Q_1 \oplus Q_2$, $\psi = \psi_1 \oplus \psi_2$ and let $\alpha : P \to Q$ be a homomorphism with $\alpha(P) \leq_\oplus Q$. Since Q satisfies the finite internal exchange property, there exists $Q_i'' \subseteq Q_i$ $(i = 1, 2)$ such that $Q = \alpha(P) \oplus Q_1'' \oplus Q_2''$. By Lemma 4.2, there exist a direct summand Q_i' of Q_i and a homomorphism $\varepsilon_i : Q_i' \to Q_j''$ $(i \neq j)$ such that $Q_i = Q_i' \oplus Q_i''$ and $\alpha(P) = \langle \varepsilon_1 \rangle \oplus \langle \varepsilon_2 \rangle$. Let $p_i : \alpha(P) = \langle \varepsilon_1 \rangle \oplus \langle \varepsilon_2 \rangle \to \langle \varepsilon_i \rangle$, $q_i' : Q = Q_1' \oplus Q_2' \oplus Q_1'' \oplus Q_2'' \to Q_i'$ and $q_i'' : Q = Q_1' \oplus Q_2' \oplus Q_1'' \oplus Q_2'' \to Q_i''$ $(i = 1, 2)$ be the projections. Since M is N_i-im-summand coinvariant and $q_i' p_i \alpha(P) = Q_i' \leq_\oplus Q_i$, we see

$$q_i' p_i \alpha(\operatorname{Ker} \varphi) \subseteq \operatorname{Ker} \psi_i \ \cdots (i)$$

for $i = 1, 2$. As Q_j'' is lifting, there exists a decomposition $Q_j'' = Q_{j1}'' \oplus Q_{j2}''$ such that $Q_{j1}'' \subseteq_c^{Q_j''} q_j'' p_i \alpha(P)$ $(i \neq j)$. Let $s_{ij} : Q_i'' = Q_{i1}'' \oplus Q_{i2}'' \to Q_{ij}''$ be the projection $(i, j = 1, 2)$. By $q_j'' p_i \alpha(P) = Q_{j1}'' \oplus (q_j'' p_i \alpha(P) \cap Q_{j2}'')$, we see $s_{j1}(q_j'' p_i \alpha(P)) = Q_{j1}'' \leq_\oplus Q_j$

and $s_{j2}(q''_j p_i \alpha(P)) = q''_j p_i \alpha(P) \cap Q''_{i2} \ll Q''_{i2}$ $(i \neq j)$. Since M is N_j-im-summand coinvariant, we obtain $s_{j1} q''_j p_i \alpha(\operatorname{Ker}\varphi) \subseteq \operatorname{Ker}\psi_j$ $(i \neq j)$. On the other hand, since M is N_j-im-small coinvariant, we see $s_{j2} q''_j p_i \alpha(\operatorname{Ker}\varphi) \subseteq \operatorname{Ker}\psi_j$ $(i \neq j)$. Hence we obtain

$$q''_j p_i \alpha(\operatorname{Ker}\varphi) \subseteq s_{j1} q''_j p_i \alpha(\operatorname{Ker}\varphi) \oplus s_{j2} q''_j p_i \alpha(\operatorname{Ker}\varphi) \subseteq \operatorname{Ker}\psi_j \ \cdots (ii).$$

Since $p_i \alpha(\operatorname{Ker}\varphi) \subseteq Q'_1 \oplus Q''_2$, by (i) and (ii), we see $p_i \alpha(\operatorname{Ker}\varphi) \subseteq \operatorname{Ker}\psi$ $(i = 1, 2)$. Hence we obtain

$$\alpha(\operatorname{Ker}\varphi) \subseteq p_1 \alpha(\operatorname{Ker}\varphi) \oplus p_2 \alpha(\operatorname{Ker}\psi) \subseteq \operatorname{Ker}\psi.$$

Thus M is $N_1 \oplus N_2$-im-summand coinvariant.

Next we prove $M_1 \oplus M_2$ is N-im-summand coinvariant if M_i is N-im-summand coinvariant and N-im-small coinvariant $(i = 1, 2)$. Let (P_i, φ_i) and (Q, ψ) be the projective covers of M_i $(i = 1, 2)$ and N respectively, put $P = P_1 \oplus P_2$, $\varphi = \varphi_1 \oplus \varphi_2$ and let $\alpha : P \to Q$ be a homomorphism with $\alpha(P) \leq_\oplus Q$. Since $\alpha(P)$ is projective, $\operatorname{Ker}\alpha$ is a direct summand of P. Since P satisfies the finite internal exchange property, there exists $P''_i \subseteq P_i$ $(i = 1, 2)$ such that $P = \operatorname{Ker}\alpha \oplus P''_1 \oplus P''_2$. By Lemma 4.2, there exist a direct summand P'_i of P_i and a homomorphism $\varepsilon_i : P'_i \to P''_j$ $(i \neq j)$ such that $P_i = P'_i \oplus P''_i$ and $\operatorname{Ker}\alpha = \langle \varepsilon_1 \rangle \oplus \langle \varepsilon_2 \rangle$. Let $p'_i : P = P'_1 \oplus P'_2 \oplus P''_1 \oplus P''_2 \to P'_i$ be the projection $(i = 1, 2)$. Then $p'_i|_{\langle \varepsilon_i \rangle}$ is an isomorphism from $\langle \varepsilon_i \rangle$ to P'_i $(i \neq j)$. Put $\beta_i = (p'_i|_{\langle \varepsilon_i \rangle})^{-1} \oplus 1_{P''_i}$. Since M_i is N-im-summand coinvariant and $\alpha\beta_i(P_i) = \alpha(P''_i) \leq_\oplus Q$, we see

$$\alpha\beta_i(\operatorname{Ker}\varphi_i) \subseteq \operatorname{Ker}\psi \ \cdots (iii)$$

for $i = 1, 2$. As P''_j is lifting, there exists a decomposition $P''_j = P''_{j1} \oplus P''_{j2}$ such that $P''_{j1} \subseteq_c^{P''_j} \varepsilon_i(P'_i)$ $(i \neq j)$. Let $s_{ij} : P''_i = P''_{i1} \oplus P''_{i2} \to P''_{ij}$ be the projection $(i, j = 1, 2)$. Since M_i is N-im-summand coinvariant and $\alpha s_{j1} \varepsilon_i (p'_i|_{P_i})(P_i) = \alpha(P''_{j1}) \leq_\oplus Q$, we see $\alpha s_{j1} \varepsilon_i p'_i(\operatorname{Ker}\varphi_i) \subseteq \operatorname{Ker}\psi$. On the other hand, by $s_{j2}\varepsilon_i(P'_i) = \varepsilon_i(P'_i) \cap P''_{j2} \ll P''_{j2}$, $\alpha s_{j2} \varepsilon_i (p'_i|_{P_i})(P_i) = \alpha(\varepsilon_i(P'_i) \cap P''_{j2}) \ll Q$. Since M_i is N-im-small coinvariant, we see

$\alpha s_{j2}\varepsilon_i(p_i'|_{P_i})(\operatorname{Ker}\varphi_i) \subseteq \operatorname{Ker}\psi$. Hence

$$\alpha\varepsilon_i p_i'(\operatorname{Ker}\varphi_i) = \alpha(s_{j1}+s_{j2})\varepsilon_i p_i'(\operatorname{Ker}\varphi_i) \subseteq \operatorname{Ker}\psi \ \cdots (iv).$$

For any $k_i \in \operatorname{Ker}\varphi_i$, we express k_i in $P_i = P_i' \oplus P_i''$ as $k_i = k_i' + k_i''$, where $k_i' \in P_i'$ and $k_i'' \in P_i''$. By (iii) and (iv), $\alpha(k_i) = \alpha(k_i' + k_i'') = \alpha((k_i' + \varepsilon_i(k_i') + k_i'') - \varepsilon_i(k_i')) = \alpha\beta_i(k_i) - \alpha\varepsilon_i p_i'(k_i) \in \operatorname{Ker}\psi$ and therefore $\alpha(\operatorname{Ker}\varphi_i) \subseteq \operatorname{Ker}\psi$. Hence we obtain

$$\alpha(\operatorname{Ker}\varphi) = \alpha(\operatorname{Ker}\varphi_1 \oplus \operatorname{Ker}\varphi_2) \subseteq \operatorname{Ker}\psi.$$

Thus $M_1 \oplus M_2$ is N-im-summand coinvariant. □

Now we consider a connection between im-small coinvariance and im-small projectivity, over a right perfect ring.

Proposition 4.4 *Let M and N be modules over a right perfect ring. Then M is N-im-small coinvariant if and only if M is im-small N-projective.*

Proof Let (P,φ) and (Q,ψ) be projective covers of M and N, respectively.

($\Rightarrow$) Let $f : M \to X$ be a homomorphism with $f(M) \ll X$ and $g : N \to X$ an epimorphism. Since P is projective, there exists a homomorphism $\alpha : P \to Q$ such that $g\psi\alpha = f\varphi$. As Q is lifting, there exists a decomposition $Q = K \oplus Q'$ such that $K \subseteq_c^Q \operatorname{Ker} g\psi$. Let $q : Q = K \oplus Q' \to Q'$ be the projection. Suppose $Q' = q\alpha(P) + T$ for some $T \subseteq Q'$. By $g\psi(q\alpha(P)) = g\psi\alpha(P) = f\varphi(P) = f(M) \ll X$, $g\psi(Q') = g\psi(q\alpha(P)) + g\psi(T) = g\psi(T)$. So $Q' = \operatorname{Ker}(g\psi|_{Q'}) + T = T$ and hence $q\alpha(P) \ll Q'$. Since M is N-im-small coinvariant, $q\alpha(\operatorname{Ker}\varphi) \subseteq \operatorname{Ker}\psi$. Therefore we can define a homomorphism $h : M \to N$ by $h(\varphi(x)) = \psi q\alpha(x)$, where $x \in P$. Then for any $m = \varphi(x) \in M$, $gh(m) = gh(\varphi(x)) = g\psi q\alpha(x) = g\psi\alpha(x) = f\varphi(x) = f(m)$, where $x \in P$. Thus M is im-small N-projective.

($\Leftarrow$) Let $\alpha : P \to Q$ be a homomorphism with $\alpha(P) \ll Q$ and let $\pi : N \to N/\psi\alpha(\operatorname{Ker}\varphi)$ be the natural epimorphism. Then we can define a homomorphism $f : M \to N/\psi\alpha(\operatorname{Ker}\varphi)$ by $f(\varphi(x)) = \pi\psi\alpha(x)$, where $x \in P$. By $\alpha(P) \ll Q$, $f(M) \ll$

$N/\psi\alpha(\mathrm{Ker}\,\varphi)$. Since M is im-small N-projective, there exists a homomorphism $h : M \to N$ such that $\pi h = f$. By the same proof in (a) $\Rightarrow$ (b) of Theorem 3.3, we obtain $\alpha(\mathrm{Ker}\,\varphi) \subseteq \mathrm{Ker}\,\psi$. □

Next, we consider a connection between N-im-summand coinvariance and im-coclosed N-projectivity, over a right perfect ring.

Theorem 4.5 *Let M and N be modules over a right perfect ring. Then M is N-im-summand coinvariant if and only if M is im-coclosed N-projective.*

Proof Let (P, φ) and (Q, ψ) be projective covers of M and N, respectively.

($\Rightarrow$) Let X be a module, $f : M \to X$ a homomorphism such that $f(M)$ is coclosed in X and $g : N \to X$ an epimorphism. Since Q is lifting, there exists a decomposition $Q = K \oplus Q'$ such that $K \subseteq_c^Q \mathrm{Ker}\, g\psi$. Then $g\psi|_{Q'} : Q' \to X$ is a small epimorphism. Since Q' is also lifting, there exists a decomposition $Q' = Q_1 \oplus Q_2$ such that $Q_1 \subseteq_c^{Q'} (g\psi|_{Q'})^{-1}(f(M))$. By Lemma 0.7 (3), $g\psi(Q_1) \subseteq_c^X f(M)$. As $f(M)$ is coclosed in X, we see $g\psi(Q_1) = f(M)$. Since P is projective, there exists a homomorphism $\alpha : P \to Q_1$ such that $(g\psi|_{Q_1})\alpha = f\varphi$. By $\mathrm{Ker}\, g\psi|_{Q_1} \ll Q_1$, α is onto. So we see $\alpha(\mathrm{Ker}\,\varphi) \subseteq \mathrm{Ker}\,\psi$. Hence we can define a homomorphism $h : M \to N$ by $h(\varphi(x)) = \psi\alpha(x)$. Then $gh = f$.

($\Leftarrow$) Let $\alpha : P \to Q$ be a homomorphism with $\alpha(P) \leq_\oplus Q$. By $\psi\alpha(\mathrm{Ker}\,\varphi) \ll N$, the natural map $\pi : N \to N/\psi\alpha(\mathrm{Ker}\,\varphi)$ is a small epimorphism. Hence $\pi\psi\alpha(P)$ is coclosed in $N/\psi\alpha(\mathrm{Ker}\,\varphi)$ by Lemma 0.7 (4). Now we define a homomorphism $f : M \to N/\psi\alpha(\mathrm{Ker}\,\varphi)$ by $f(\varphi(x)) = \pi\psi\alpha(x)$, where $x \in P$. Since $f(M)$ is coclosed in $N/\psi\alpha(\mathrm{Ker}\,\varphi)$, there exists a homomorphism $h : M \to N$ such that $\pi h = f$. By the same proof in (a) $\Rightarrow$ (b) of Theorem 3.3, we obtain $\alpha(\mathrm{Ker}\,\varphi) \subseteq \mathrm{Ker}\,\psi$. □

Proposition 4.6 *Let M and N be modules over a right perfect ring. Then M is X-im-summand coinvariant for any submodule X of N if and only if M is N-projective.*

Proof ($\Leftarrow$) By Lemma 0.2 (1) and Theorem 4.5.

($\Rightarrow$) Let $f : M \to N/K$ be a homomorphism and let $\pi_N : N \to N/K$ be the natural epimorphism, where K is any submodule of N. Put $f(M) = A/K$

for some submodule A of N with $K \subseteq A$. Let (P, φ) and (Q, ψ) be projective covers of M and A, respectively. Since Q is lifting, there exists a decomposition $Q = T \oplus Q'$ such that $T \subseteq_c^Q \operatorname{Ker} \pi_A \psi$, where $\pi_A : A \to A/K$ is the natural epimorphism. Put $\psi' = \pi_A \psi|_{Q'}$, then (Q', ψ') is the projective cover of A/K. Since P is projective, there exists an epimorphism $\alpha : P \to Q'$ such that $f\varphi = \psi'\alpha$. Since M is A-im-summand coinvariant by the assumption, we see $\iota\alpha(\operatorname{Ker} \varphi) \subseteq \operatorname{Ker} \psi$, where $\iota : Q' \to Q = T \oplus Q'$ is the inclusion. Hence we can define a homomorphism $h : M \to N$ by $h(\varphi(x)) = \psi\iota\alpha(x)$, where $x \in P$. Then $\pi_N h = f$. $\square$

The following corollary is immediate from Propositions 4.1, 4.4 and Theorem 4.5.

Corollary 4.7 *Let M, N, M_i $(i \in I)$ and N_j $(j \in J)$ be modules over a right perfect ring. Then*

(1) If M is im-small N-projective, then M is im-small N/X-projective and im-small X-projective for any submodule X of N.

(2) If M is im-coclosed N-projective, then M is im-coclosed N/X-projective for any submodule X of N. Moreover, for any coclosed submodule X of N, M is im-coclosed X-projective.

(3) If M/S is im-small (im-coclosed, resp.) N-projective for some $S \ll M$, then M is im-small (im-coclosed, resp.) N-projective.

(4) If M is im-small (im-coclosed, resp.) N-projective, then M' is im-small (im-coclosed, resp.) N-projective for any direct summand M' of M.

(5) If M_i is im-small N_j-projective for any $i \in I$ and $j \in J$, then $\oplus_I M_i$ is im-small $\oplus_J N_j$-projective.

Example 4.8 *An N-im-summand coinvariant module is not always N-im-small coinvariant. Let $R = \begin{pmatrix} K & K & K & K \\ 0 & K & 0 & K \\ 0 & 0 & K & 0 \\ 0 & 0 & 0 & K \end{pmatrix}$, where K is any field. Then R is right* *perfect. Let $M = (0,K,K,K)/(0,0,K,K)$ and $N = (K,K,K,K)$. Since N is hollow, N/X is also hollow for any submodule X of N. A homomorphism $f : M \to N/X$ such that $f(M)$ is coclosed in N/X is only the zero map, because M is not isomorphic to N/X for any submodule X of N. Hence M is im-coclosed N-projective. On the other hand, the inclusion map $\iota : M \to N/(0,0,K,K)$ cannot be lifted to a homomorphism from M to N. Since $\operatorname{Im}\iota$ is small in $N/(0,0,K,K)$, M is not im-small N-projective. Thus M is N-im-summand coinvariant but not N-im-small coinvariant by Theorem 4.5 and Proposition 4.4.*

According to the above example, in general, an N-im-summand coinvariant module M need not be N-im-small coinvariant for a module N over a right perfect ring. However, if N is a small epimorphic image of M, the following holds.

Proposition 4.9 *Let M and N be modules over a right perfect ring. Suppose that there exists a small epimorphism from M to N. If M is N-im-summand coinvariant, then M is N-im-small coinvariant.*

Proof Let M be N-im-summand coinvariant and let $f : M \to N$ be a small epimorphism. Since f is a small epimorphism, we can take (P, φ) and $(P, f\varphi)$ as the projective covers of M and N respectively.

Let $\alpha : P \to P$ be an endomorphism with $\alpha(P) \ll P$. By $\alpha(P) \ll P$, $P = (1-\alpha)(P)+\alpha(P) = (1-\alpha)(P)$. So we see that $1-\alpha$ is onto. Since M is N-im-summand coinvariant, $(1-\alpha)(\operatorname{Ker}\varphi) \subseteq \operatorname{Ker} f\varphi$. By $\operatorname{Ker}\varphi \subseteq \operatorname{Ker} f\varphi$, $\alpha(\operatorname{Ker}\varphi) \subseteq \operatorname{Ker} f\varphi$. Therefore M is N-im-small coinvariant. □

Let M be a module over a right perfect ring and let (P, φ) be the projective cover of M. M is called *automorphism coinvariant* if $\alpha(\operatorname{Ker}\varphi) \subseteq \operatorname{Ker}\varphi$ for any automorphism α of P.

In the proof of Proposition 4.9, for any $k \in \operatorname{Ker}(1-\alpha)$, $k = \alpha(k) \in \alpha(P)$. So $\operatorname{Ker}(1-\alpha) \subseteq \alpha(P) \ll P$. On the other hand, $\operatorname{Ker}(1-\alpha)$ is a direct summand of P by $P/\operatorname{Ker}(1-\alpha) \cong (1-\alpha)(P) = P$. Hence $1-\alpha$ is a monomorphism. Thus by the similar proof of Proposition 4.9, we obtain the following:

Corollary 4.10 *Let M be a module over a right perfect ring and we consider the following conditions:*

(1) M is M-im-summand coinvariant,

(2) M is automorphism coinvariant,

(3) M is M-im-small coinvariant.

Then $(1) \Rightarrow (2) \Rightarrow (3)$ *holds.*

A module M is said to be *radical N-projective* for a module N if, for any module X, any homomorphism $f : M \to X$ and any epimorphism $g : N \to X$, there exists a homomorphism $h : M \to N$ such that $(f - gh)(M) \ll X$.

Lemma 4.11 ([25, Proposition 1.2]) *A module M is radical N-projective for a module N if and only if, for any module X, any homomorphism $f : M \to X$ and any epimorphism $g : N \to X$, there exist a module Y, a small epimorphism $\rho : X \to Y$ and a homomorphism $h : M \to N$ such that $\rho f = \rho gh$.*

Proposition 4.12 *Let M and N be modules over a right perfect ring. If M is im-summand N-projective, then M is radical N-projective.*

Proof Let X be a module, $f : M \to X$ a homomorphism and $g : N \to X$ an epimorphism. Let Y be a supplement of $f(M)$ in X. By $f(M) \cap Y$ is small in X, the natural epimorphism $\rho : X \to X/(f(M) \cap Y)$ is a small epimorphism. Then $X/(f(M) \cap Y) = f(M)/(f(M) \cap Y) \oplus Y/(f(M) \cap Y)$ and so $\rho f(M) = f(M)/(f(M) \cap Y)$ is a direct summand of $X/(f(M) \cap Y)$. Since M is im-summand N-projective, there exists a homomorphism $h : M \to N$ such that $\rho f = \rho gh$. Therefore M is radical N-projective by Lemma 4.11. □

Lemma 4.13 ([25, Proposition 1.3]) *Let M and N be modules. Then M is N-projective if and only if M is radical N-projective and im-small N-projective.*

Proposition 4.14 *Let M and N be modules over a right perfect ring. Then M is N-im-summand coinvariant and N-im-small coinvariant if and only if M is N-projective.*

Proof By Propositions 4.4, 4.12, Theorem 4.5 and Lemma 4.13. □

The following is obtained by Propositions 4.1 and 4.14.

Corollary 4.15 *Let M and N be modules over a right perfect ring and S a small submodule of M. If M/S is N-projective then M is N-projective.*

Theorem 4.16 *Let M be a module over a right perfect ring and let N be a small epimorphic image of M. Then the following conditions are equivalent:*

(a) M is N-projective,

(b) M is im-coclosed N-projective,

(c) M is N-im-summand coinvariant.

Proof By Theorem 4.5, Propositions 4.9 and 4.14. □

Next we show that M is M-projective if and only if M is im-summand M-projective, if and only if M is M-im-summand coinvariant, for any module M over a right perfect ring. We first need to give the following lemma:

Lemma 4.17 *Let M be an im-summand M-projective module with the projective cover (P, φ). Then for any decomposition $P = P_1 \oplus P_2$, $M = \varphi(P_1) \oplus \varphi(P_2)$.*

Proof Let $p_i : P = P_1 \oplus P_2 \to P_i$ $(i = 1, 2)$ be the projection. Given $\overline{\varphi(x_1)} = \overline{\varphi(x_2)} \in [\varphi(P_1)/\varphi p_1(\operatorname{Ker}\varphi)] \cap [(\varphi(P_2) + \varphi p_1(\operatorname{Ker}\varphi))/\varphi p_1(\operatorname{Ker}\varphi)]$. By $\varphi(x_1 - x_2) \in \varphi p_1(\operatorname{Ker}\varphi)$, there exists $k \in \operatorname{Ker}\varphi$ such that $\varphi(x_1 - x_2) = \varphi(p_1(k))$. Then $x_1 - x_2 - p_1(k) \in \operatorname{Ker}\varphi$ and so $x_1 - p_1(k) \in p_1(\operatorname{Ker}\varphi)$. By $x_1 \in p_1(\operatorname{Ker}\varphi)$, we see $[\varphi(P_1)/\varphi p_1(\operatorname{Ker}\varphi)] \cap [(\varphi(P_2) + \varphi p_1(\operatorname{Ker}\varphi))/\varphi p_1(\operatorname{Ker}\varphi)] = 0$. Now, we define a homomorphism $f : M \to M/\varphi p_1(\operatorname{Ker}\varphi)$ by $f(\varphi(x)) = \overline{\varphi p_1(x)}$, where $x \in P$.

Then $f(M) = \varphi(P_1)/\varphi p_1(\operatorname{Ker}\varphi) \leq_{\oplus} M/\varphi p_1(\operatorname{Ker}\varphi)$. Since M is im-summand M-projective, there exists an endomorphism h of M such that $\pi h = f$, where $\pi : M \to M/\varphi p_1(\operatorname{Ker}\varphi)$ is the natural epimorphism. By the same proof in (a) $\Rightarrow$ (b) of Theorem 3.3, we obtain $p_1(\operatorname{Ker}\varphi) \subseteq \operatorname{Ker}\varphi$.

By $\varphi(P_1) \cap \varphi(P_2) \subseteq \varphi p_1(\operatorname{Ker}\varphi) \subseteq \varphi(\operatorname{Ker}\varphi) = 0$, we obtain that $M = \varphi(P_1) \oplus \varphi(P_2)$. □

Theorem 4.18 *Let M be a module over a right perfect ring. Then M is im-summand M-projective if and only if M is M-im-summand coinvariant.*

Proof ($\Leftarrow$) By Theorem 4.5.

($\Rightarrow$) Let (P, φ) be the projective cover of M and let α be an endomorphism of P with $\alpha(P) \leq_{\oplus} P$. Put $P = \alpha(P) \oplus P'$. By Lemma 4.17, we see $M = \varphi(\alpha(P)) \oplus \varphi(P')$. Then $M/\varphi\alpha(\operatorname{Ker}\varphi) = (\varphi\alpha(P)/\varphi\alpha(\operatorname{Ker}\varphi)) \oplus ((\varphi(P') + \varphi\alpha(\operatorname{Ker}\varphi))/\varphi\alpha(\operatorname{Ker}\varphi))$. Now we define a homomorphism $f : M \to M/\varphi\alpha(\operatorname{Ker}\varphi)$ by $f(\varphi(x)) = \pi\varphi\alpha(x)$, where $x \in P$ and $\pi : M \to M/\varphi\alpha(\operatorname{Ker}\varphi)$ is the natural epimorphism. Since M is im-summand M-projective and $\varphi\alpha(P)/\varphi\alpha(\operatorname{Ker}\varphi) \leq_{\oplus} M/\varphi\alpha(\operatorname{Ker}\varphi)$, we obtain $\alpha(\operatorname{Ker}\varphi) \subseteq \operatorname{Ker}\varphi$ by the same proof in (a) $\Rightarrow$ (b) of Theorem 3.3. Thus M is M-im-summand coinvariant. □

A module M is called *pseudo-projective* if, for any module X, any epimorphisms f and g from M to X, there exists an endomorphism h of M such that $f = gh$.

Lemma 4.19 ([17, Corollary 2.6]) *A lifting module M is pseudo-projective if and only if it is M-projective.*

Lemma 4.20 ([11, Theorem 2.3]) *A module M over a right perfect ring is automorphism coinvariant if and only if it is pseudo-projective.*

Corollary 4.21 *Let M be a module over a right perfect ring. Then the following conditions are equivalent:*

(a) M is M-projective,

(b) M is im-coclosed M-projective,

(c) M is im-summand M-projective,

(d) M is M-im-summand coinvariant.

If M is lifting, then (a)-(d) are equivalent to :

(e) M is automorphism coinvariant.

Proof By Proposition 4.18, Theorem 4.16 and Lemmas 4.19 and 4.20. □

4.2 Rings whose factor square full modules are closed under essential extensions or essential submodules

According to Example 2.10, factor square full modules are not closed under essential extensions or essential submodules, in general. In this section, we study right perfect rings which satisfy the following conditions:

($*$) factor square full modules are closed under essential extensions,

($**$) factor square full modules are closed under essential submodules.

Throughout this section, we denote the Jacobson radical of R by J.

A module M is called *noetherian* (*artinian*, resp.) if it satisfies the ascending (descending, resp.) chain condition on submodules. A ring R is said to be *right noetherian* (*right artinian*, resp.) if the right R-module R is noetherian (artinian, resp.).

Lemma 4.22 ([1, Propositions 10.9 and 10.10])

(1) A module M is noetherian if and only if any submodule of M is finitely generated.

(2) A module M is artinian if and only if any factor module of M is finitely cogenerated.

Lemma 4.23 (cf. [1, Corollary 15.23]) *A ring is right artinian if and only if it is right noetherian and right (or left) perfect.*

Proposition 4.24 *(1) A uniserial module over a right perfect ring is noetherian.*

(2) A uniserial module over a left perfect ring is artinian.

Proof (1) Let U be a uniserial module over a right perfect ring. Then any submodule of U is uniserial with the maximum proper submodule by Lemmas 0.15 (6) and 0.18 (2). Therefore U is noetherian by Lemma 4.22 (1).

(2) Let U be a uniserial module over a left perfect ring. Then any nonzero factor module of U is uniserial with the minimum nonzero submodule by Lemmas 0.15 (6) and 0.18 (2). Therefore U is artinian by Lemma 4.22 (2). □

Lemma 4.25 *Let R be a right perfect ring with the condition $(*)$ or $(**)$, let e be a primitive idempotent of R and let S be a simple module. If S is embedded in eR/X for some submodule X of eR, then $S \cong eR/eJ$.*

Proof Let f be a primitive idempotent of R with $S \cong fR/fJ$. Since S is embedded in eR/X, there exists a complement submodule K/X of S in eR/X. By Lemma 0.6, $(S \oplus (K/X))/(K/X) \subseteq_e (eR/X)/(K/X)$.

In the case that R satisfies $(*)$, $(eR/X)/(K/X) \oplus fR/fJ$ is factor square full because it is an essential extension of a factor square full module $S \oplus S \cong (S \oplus (K/X))/(K/X) \oplus fR/fJ$. So its projective cover $eR \oplus fR$ is also factor square full by Proposition 2.9.

In the case that R satisfies $(**)$, since $eR \oplus eR$ is factor square full, we see $(eR/X)/(K/X) \oplus eR$ is factor square full, and so is its essential submodule $(S \oplus (K/X))/(K/X) \oplus eR \cong S \oplus eR \cong fR/fJ \oplus eR$. Hence $fR \oplus eR$ is also factor square full by Proposition 2.9.

In either case, $eR \cong fR$ by Proposition 2.12. Thus $S \cong fR/fJ \cong eR/eJ$. □

A ring R is said to be *right serial* if the right R-module R is a direct sum of uniserial modules.

Proposition 4.26 *Let R be a right perfect ring with the condition $(*)$ or $(**)$. Then R is right artinian right serial.*

Proof Let $\{e_1, e_2, \ldots, e_n\}$ be a complete set of primitive orthogonal idempotents of R. We show that any nonzero submodule of e_iR is local for each $i = 1, 2, \ldots, n$. Assume that there exists a nonzero submodule X of e_iR which has two distinct maximal submodules A and B. Put $T = A \cap B$. By $X = A + B$, we see $X/T = A/T \oplus B/T \cong X/B \oplus X/A$. Since $X/T \subseteq e_iR/T$, X/A and X/B are isomorphic to e_iR/e_iJ by Lemma 4.25. Now, there exists a homomorphism $g : e_iR/T \to E(X/T)$ such that $g|_{X/T}$ is the inclusion by injectivity of $E(X/T)$. Then $X/T \subseteq g(e_iR/T) \subseteq E(X/T)$, so $X/T \subseteq_e g(e_iR/T) \subseteq_e E(X/T)$. Since $X/T \cong X/A \oplus X/B \cong (e_iR/e_iJ)^2$ and $E(X/T) \cong E(e_iR/e_iJ)^2$ are both factor square full, $g(e_iR/T)$ is also factor square full. This contradicts that $g(e_iR/T)$ is hollow. Therefore any nonzero submodule of e_iR is local, and so e_iR is uniserial by Lemma 0.15 (5). By Lemma 4.24, each e_iR is noetherian. Hence $R_R = \oplus_{i=1}^n e_iR$ is also noetherian. Thus R is right artinian by Lemma 4.23. □

Proposition 4.27 *Let R be a basic right perfect ring with the condition $(*)$ or $(**)$ and $\{e_1, e_2, \ldots, e_n\}$ a basic set of primitive orthogonal idempotents of R. Then each e_i is central.*

Proof If there exists a nonzero homomorphism $f : e_kR \to e_lR$ for $k \neq l$, then $\operatorname{Ker} f \ll e_kR$ and $f(e_kR) \subseteq_e e_lR$ because each e_iR is uniserial by Proposition 4.26. Since R satisfies $(*)$ or $(**)$, $f(e_kR) \oplus e_lR$ is factor square full because it is an essential extension of $f(e_kR) \oplus f(e_kR)$ and an essential submodule of $e_lR \oplus e_lR$. Hence its projective cover $e_kR \oplus e_lR$ is also factor square full, a contradiction. So $e_lRe_k \cong \operatorname{Hom}_R(e_kR, e_lR) = 0$ for $k \neq l$. Thus for any element r of R, $e_ir = e_ir(e_1 + e_2 + \cdots + e_n) = e_ire_i = (e_1 + e_2 + \cdots + e_n)re_i = re_i$ for every $i = 1, 2, \ldots, n$. □

Theorem 4.28 *Let R be a basic right perfect ring and $\{e_1, e_2, \ldots, e_n\}$ a basic set of primitive orthogonal idempotents of R. Then the following conditions are equivalent:*

(a) *R satisfies the condition* $(*)$,

(b) *dual square free modules are closed under submodules,*

(c) $R \cong \Pi_{i=1}^{n} e_i R e_i$ *as rings and each* $e_i R e_i$ *is right artinian right uniserial.*

Proof $(b) \Leftrightarrow (c)$ is proved in [19, Proposition 4.6].

$(a) \Rightarrow (c)$: By Propositions 4.26 and 4.27.

$(c) \Rightarrow (a)$: Let $N \subseteq_e M$, let (P, φ) and (Q, ψ) be the projective covers of M and N respectively, and express $P = \oplus_{i=1}^{n} (e_i R)^{(I_i)}$ and $Q = \oplus_{i=1}^{n} (e_i R)^{(K_i)}$. Suppose that M is not factor square full, there exists $k \in \{1, 2, \dots, n\}$ such that $\# I_k = 1$. Since Q is projective, there exists a homomorphism $f : Q \to P$ such that $\varphi f = \psi$. As each e_i is central, $f((e_k R)^{(K_k)}) \subseteq e_k R$. Since $e_k R$ is uniserial and $\operatorname{Ker} f|_{(e_k R)^{(K_k)}} \ll (e_k R)^{(K_k)}$, $(e_k R)^{(K_k)}$ is hollow and so $\# K_k = 1$. Thus N is not factor square full. $\square$

Proposition 4.29 *Let R be a right perfect ring with the condition $(**)$ and let $\{e_1, e_2, \dots, e_n\}$ be a basic set of primitive orthogonal idempotents of R. Then each $e_k R$ is injective.*

Proof Let (P, φ) be the projective cover of $E(e_k R)$ and let $P = \oplus_{i=1}^{n} (e_i R)^{(\Lambda_i)}$. By Proposition 4.27, $\varphi((e_i R)^{(\Lambda_i)}) \cap e_k R = 0$ for every $i \neq k$, and hence $\varphi((e_i R)^{(\Lambda_i)}) = 0$ because $e_k R \subseteq_e E(e_k R)$. So $P = (e_k R)^{(\Lambda_k)}$. In addition, $\# \Lambda_k = 1$ by the condition $(**)$. Hence $|e_k R| \leq |E(e_k R)| \leq |e_k R|$, where $|X|$ means the length of a module X. Thus $E(e_k R) = e_k R$. $\square$

A ring R is called *quasi-Frobenius* if R is two-sided artinian and two-sided self-injective.

Lemma 4.30 ([1, Theorem 31.9]) *For a ring R, the following conditions are equivalent:*

(a) *R is quasi-Frobenius,*

(b) *any projective right R-module is injective,* *(c) any injective right R-module is projective.*

Lemma 4.31 ([1, Theorem 32.3]) *For a right artinian ring R, R is two-sided serial if and only if the injective hull and the projective cover of any simple right R-module is uniserial.*

Theorem 4.32 *Let R be a basic right perfect ring and $\{e_1, e_2, \ldots, e_n\}$ a basic set of primitive orthogonal idempotents of R. Then the following conditions are equivalent:*

(a) R satisfies the condition $(**)$,

(b) dual square free modules are closed under essential extensions,

(c) $R \cong \Pi_{i=1}^{n} e_i R e_i$ as rings and each $e_i R e_i$ is (quasi-)Frobenius uniserial.

Proof $(b) \Leftrightarrow (c)$ is proved in [19, Theorem 4.11].

$(a) \Rightarrow (c)$: By Propositions 4.26, 4.27, 4.29 and Lemma 4.31.

$(c) \Rightarrow (a)$: Let M be a factor square full module and N an essential submodule of M. Then $E(M)$ is factor square full by Theorem 4.28. By Lemma 4.30, $E(M)$ is projective and so we can express $E(M) = \oplus_{i=1}^{n}(e_iR)^{(\Lambda_i)}$ where $\#\Lambda_i \geq 2$ or $\#\Lambda_i = 0$ for $i = 1, 2, \ldots, n$. Let $\iota_{i\lambda} : e_iR \to (e_iR)^{(\Lambda_i)}$ be the λ-injection ($i = 1, 2, \ldots, n$ and $\lambda \in \Lambda_i$). For any $i = 1, 2, \ldots, n$ and any $\lambda \in \Lambda_i$, $\iota_{i\lambda}(e_iR) \cap N \subseteq_e \iota_{i\lambda}(e_iR)$ because N is essential in $E(N) = E(M)$ and each e_iR is uniserial. Hence $\oplus_{i=1}^{n} \oplus_{\Lambda_i} (\iota_{i\lambda}(e_iR) \cap N) \subseteq_e N \subseteq_e \oplus_{i=1}^{n}(e_iR)^{(\Lambda_i)}$. Since the projective cover of $\iota_{i\lambda}(e_iR) \cap N$ is e_iR ($i = 1, 2, \ldots, n$ and $\lambda \in \Lambda_i$), $\oplus_{i=1}^{n} \oplus_{\Lambda_i} (\iota_{i\lambda}(e_iR) \cap N)$ is factor square full by Proposition 2.9. Thus N is also factor square full by Theorem 4.28. □

Remark 4.33 *By Theorem 4.28 and Theorem 4.32, a ring R with $(**)$ satisfies $(*)$. However, the converse does not hold. Let $R = \mathbb{Q}(x) \oplus u\mathbb{Q}(x)$ and define its multiplication by $(f(x) + ug(x))(f'(x) + ug'(x)) = f(x)f'(x) + u(g(x)f'(x) + f(x^2)g'(x))$ for $f(x) + ug(x), f'(x) + ug'(x) \in R$. Then R is a right artinian right uniserial ring but not left uniserial (see [19, Example 4.12]). Hence R satisfies $(*)$ but it does not have $(**)$.*

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References

[1] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, Springer-Verlag, New York 1974.

[2] Y. Baba, *Note on almost M-injectives*, Osaka J. Math. **26** (1989) 687-698.

[3] Y. Baba and M. Harada, *On almost M-projectives and almost M-injectives*, Tsukuba J. Math. **14 (1)** (1990) 53-69.

[4] H. Bass, *Finitistic dimension and a homological generalization of semi-primary rings*, Trans. Amer. Math. Soc. **95** (1960) 466-488.

[5] V. Camillo, *Distributive modules*, J. Algebra **36** (1975) 16-25.

[6] J. Clark, C. Lomp, N. Vanaja and R. Wisbauer, *Lifting Modules. Supplements and Projectivity in Module Theory*, Frontiers in Math. Boston: Birkhäuser, 2006.

[7] N. Ding, Y. Ibrahim, M. Yousif, Y. Zhou, *D4-modules*, J. Algebra Appl. **16 (5)** (2017) 1750166-1-1750166-25.

[8] B. Eckmann and A. Schopf, *Über injektive Moduln*, Arch. Math. **4** (1953) 75-78.

[9] A. Facchini and L. Salce, *Uniserial modules; sums and isomorphisms of subquotients*, Comm. Algebra **18** (1990) 499-517.

[10] K. R. Goodearl, *Ring Theory. Nonsingular rings and modules*, Pure and Applied Mathematics, No. 33. Marcel Dekker, Inc., New York-Basel, 1976.

[11] P. A. Guil Asensio, D. Keskin Tütüncü, B. Kaleboğaz and A. K. Srivastava, *Modules which are coinvariant under automorphisms of their projective covers*, J. Algebra **466** (2016) 147-152.

[12] K. Hanada, Y. Kuratomi and K. Oshiro, *On direct sums of extending modules and internal exchange property*, J. Algebra **250** (2002) 115-133.

[13] M. Harada, *On lifting property on direct sums of hollow modules*, Osaka J. Math. **17** (1980) 783–791.

[14] M. Harada, *On almost relative projectives over perfect rings*, Osaka J. Math. **27** (1990) 655-665.

[15] M. Harada and T. Mabuchi, *On almost M-projectives*, Osaka J. Math. **26** (1989) 837-848.

[16] M. Harada and A. Tozaki, *Almost M-projectives and Nakayama rings*, J. Algebra **122** (1989) 447-474.

[17] D. Keskin Tütüncü and Y. Kuratomi, *On epi-projective modules*, East-West J. of Math. **10 (1)** (2008) 27-35.

[18] D. Keskin Tütüncü and Y. Kuratomi, *On generalize epi-projective modules*, Math. J. Okayama univ. **52 (1)** (2010) 111–122.

[19] D. Keskin Tütüncü, I. Kikumasa, Y. Kuratomi and Y. Shibata, *On dual of square free modules*, Comm. Algebra **46** (2018) 3365-3376.

[20] D. Keskin Tütüncü, Y. Kuratomi and Y. Shibata, *On image summand coinvariant modules and kernel summand invariant modules*, Turkish J. Math. **43** (2019) 1456-1473.

[21] I. Kikumasa and Y. Kuratomi, *On H-supplemented modules over a right perfect ring*, Comm. Algebra **46** (2018) 2063-2072.

[22] I. Kikumasa, Y. Kuratomi and Y. Shibata, *Factor square full modules*, Comm. Algebra **49** (2021) 2326-2336.

[23] I. Kikumasa, Y. Kuratomi and Y. Shibata, *Almost projective modules and generalized projective modules*, Preprint.

[24] Y. Kuratomi, *On direct sums of lifting modules and internal exchange property*, Comm. Algebra **33** (2005) 1795-1804.

[25] Y. Kuratomi, *Direct sums of H-supplemented modules*, J. Algebra Appl. **13** (2014) 1350075-1–1350075-12.

[26] Y. Kuratomi, *Lifting modules with finite internal exchange property and direct sums of hollow modules*, J. Algebra Appl. **20** (2021) 2150189-1–2150189-12.

[27] Y. Kuratomi and Ch. Chang, *Lifting modules over right perfect rings*, Comm. Algebra **35** (2007) 3103-3109.

[28] E. A. Mares, *Semi-perfect modules*, Math. Z. **82** (1963) 347-360.

[29] S. H. Mohamed and B. J. Müller, *Continuous and Discrete Modules*, London Mathematical Society Lecture Note Series **147**, Cambridge University Press, 1990.

[30] S. H. Mohamed and B. J. Müller, *Co-ojective modules*, J. Egypt. Math. Soc. **12 (2)** (2004) 83-96.

[31] K. Oshiro, *Semiperfect modules and quasi-semiperfect modules*, Osaka J. Math. **20** (1983) 337-372.

[32] Y. Shibata, *On lifting and extending properties on direct sums of hollow uniform modules*, to appear in Algebra and Discrete Mathematics.

[33] R. B. Warfield, *Krull-Shmidt theorem for infinite sums of modules*, Proc. Amer. Math. Soc. **22** (1969) 460–465.