Suppression of thermal noise in non-Markovian random velocity field

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Abstract. We study the diffusion of Brownian particles in a Gaussian random velocity field with short memory. By extending the derivation of an effective Fokker–Planck equation for the Lanvegin equation with weakly colored noise to a random velocity-field problem, we find that the effect of thermal noise on particles is suppressed by the existence of memory. We also find that the renormalization effect for the relative diffusion of two particles is stronger than that for single-particle diffusion. The results are compared with those of molecular dynamics simulations.

Keywords: Diffusion, Non-Markovian process, Fokker–Planck equation

1. Introduction

Diffusion in nonequilibrium environments has recently attracted considerable attention [1, 2, 3, 4, 5]. Particularly, the recent development of single-particle tracking [6, 7] enabled the observation of the trajectories of a tracer particle in biological systems, and many qualitatively new phenomena have been discovered. Although several anomalous diffusion phenomena have been explained in terms of the associated stochastic process [8], the origins of others are still unclear. In order to classify their origins, a deep understanding of theoretical models is necessary.

Compared with single-particle diffusion, the diffusive behavior of the relative distance between two noninteracting tracer particles in a common random velocity field is often diverse and complicated. Several counterintuitive phenomena have been theoretically discovered. One remarkable example is the aggregation of two independent particles in a common velocity field, which obeys Gaussian statistics with no memory [9, 10, 11]. Although normal diffusion is observed in the corresponding single-particle problem, two particles in a common velocity field aggregate with time, and therefore, relative diffusion is completely suppressed. It should be noted that while this behavior is not stable against the independent thermal noise that independently acts on the two particles and the two particles diffuse from each other in the large time limit in the existence of such independent noise, the relative diffusion is still slightly suppressed. Another example is diffusion in a time-independent random velocity field—Sinai diffusion [12]. In a single-particle problem, ultraslow diffusion is observed in the presence of thermal noise [13]. In contrast, it is known that the relative distance between two particles in a relative diffusion problem remains finite with probability one even in the infinite time limit, although they are subject to independent noise sources [14], which implies the complete suppression of relative diffusion. This phenomenon is understood as the existence of an infinitely deep well in a potential landscape [15].

The main purpose of this study is to investigate the effect of memory in random velocity fields on the thermal noise acting on the tracer particles. We study the case where the correlation time of a velocity field is small, and we perturbatively derive an effective time-evolution equation for the probability distribution of the particle positions by extending the derivation of an effective Fokker–Planck equation for the Lanvegin equation with weakly colored noise [16] to our random velocity-field problem. We then find that the effect of thermal noise on both single-particle and relative diffusion is suppressed by the existence of the memory of the velocity field, although suppression in the latter is stronger.

This paper is organized as follows. In section 2, the model is introduced. In section 3, an effective time-evolution equation for the probability distribution of two particles is perturbatively derived, and the properties of this equation are investigated. In section 4, the results in the previous section are compared with those obtained from molecular dynamics simulations. Section 5 is devoted to the concluding remarks.

2. Model

We study two Brownian particles on a common random velocity field:

$$\dot{x}^{(a)}(t) = F\left(x^{(a)}(t), t\right) + \xi^{(a)}(t), \tag{1}$$

where $a = 1, 2$. A zero-mean random velocity field $F(x, t)$ obeys Gaussian statistics as follows:

$$\langle F(x,t)F(x',t')\rangle = C(x - x', t - t') \tag{2}$$

with

$$C(x,t) = C_{\mathrm{s}}(x)\frac{1}{\tau}e^{-\frac{|t|}{\tau}}, \tag{3}$$

and the zero-mean Gaussian white thermal noise $\xi^{(a)}(t)$ obeys

$$\left\langle \xi^{(a)}(t)\xi^{(b)}(t')\right\rangle = 2T\delta_{a,b}\delta(t - t') \tag{4}$$

with $\left\langle F(x,t)\xi^{(a)}(t)\right\rangle = 0$, where $\langle\cdots\rangle$ represents the average with respect to both random variables. The parameter T is interpreted as the temperature of the thermal environment. We assume that (i) $C_{\mathrm{s}}(x) > 0$, (ii) $C_{\mathrm{s}}(-x) = C_{\mathrm{s}}(x)$, (iii) $C_{\mathrm{s}}(x)$ reaches a maximum at $x = 0$, and (iv) $C'_{\mathrm{s}}(0) = 0$. In the limit $\tau \to 0$, the Markovian case is recovered.

3. Fokker–Planck equation

3.1. Time-evolution equation for the probability distribution

We derive a time-evolution equation for the probability density function of two particles $P(\boldsymbol{x}, t) \equiv \langle\delta\left(\boldsymbol{x} - \boldsymbol{x}(t)\right)\rangle$ with $\boldsymbol{x} \equiv \left(x^{(1)}, x^{(2)}\right)$. From the Langevin equation in (1), the evolution equation for $P(\boldsymbol{x}, t)$ is written as

$$\frac{\partial}{\partial t}P(\boldsymbol{x},t) = -\sum_{a=1}^{2}\frac{\partial}{\partial x^{(a)}}\left\langle F\left(x^{(a)},t\right)\delta\left(\boldsymbol{x} - \boldsymbol{x}(t)\right)\right\rangle - \sum_{a=1}^{2}\frac{\partial}{\partial x^{(a)}}\left\langle \xi^{(a)}(t)\delta\left(\boldsymbol{x} - \boldsymbol{x}(t)\right)\right\rangle. \tag{5}$$

The second term leads to the standard diffusion term:

$$\left\langle \xi^{(a)}(t)\delta\left(\boldsymbol{x} - \boldsymbol{x}(t)\right)\right\rangle = -T\frac{\partial}{\partial x^{(a)}}P(\boldsymbol{x},t). \tag{6}$$

The first term is expressed by using the Furutsu–Novikov–Donsker formula [17, 18, 19]

$$\langle F(x,t)\mathcal{A}[F]\rangle = \int dx' \int dt' \,\langle F(x,t)F(x',t')\rangle \left\langle \frac{\delta\mathcal{A}[F]}{\delta F(x',t')}\right\rangle, \tag{7}$$

where $A[F]$ is an arbitrary functional of F. The result is

$$\begin{aligned}\frac{\partial}{\partial t}P(\boldsymbol{x},t) = &-\sum_{a=1}^{2}\frac{\partial}{\partial x^{(a)}}\int dx' \int dt' \left\langle F\left(x^{(a)},t\right)F\left(x',t'\right)\right\rangle \left\langle \frac{\delta}{\delta F(x',t')}\delta\left(\boldsymbol{x} - \boldsymbol{x}(t)\right)\right\rangle \\ &+ T\sum_{a=1}^{2}\frac{\partial^2}{\partial x^{(a)2}}P(\boldsymbol{x},t)\end{aligned}$$

$$= \sum_{a=1}^{2} \frac{\partial}{\partial x^{(a)}} \int dx' \int dt' C\left(x^{(a)} - x', t - t'\right) \sum_{b=1}^{2} \left\langle \frac{\delta x^{(b)}(t)}{\delta F(x', t')} \frac{\partial}{\partial x^{(b)}} \delta\left(\boldsymbol{x} - \boldsymbol{x}(t)\right) \right\rangle$$
$$+ T \sum_{a=1}^{2} \frac{\partial^2}{\partial x^{(a)2}} P(\boldsymbol{x}, t). \tag{8}$$

Furthermore, by differentiating the formal solution of (1),

$$x^{(a)}(t) = x^{(a)}(0) + \int_0^t ds \int dy \delta(y - x^{(a)}(s)) F(y, s) + \int_0^t ds \xi^{(a)}(s) \tag{9}$$

with respect to $F(x', t')$ $(t' < t)$, we obtain

$$\frac{\delta x^{(a)}(t)}{\delta F(x', t')} = \delta\left(x' - x^{(a)}(t')\right) - \int_{t'}^{t} ds \int dy F(y, s) \frac{\delta x^{(a)}(s)}{\delta F(x', t')} \frac{\partial}{\partial y} \delta\left(y - x^{(a)}(s)\right). \tag{10}$$

We note that the constraint $t' < s < t$ comes from causality. Differentiation of both sides with respect to t yields

$$\frac{\partial}{\partial t} \frac{\delta x^{(a)}(t)}{\delta F(x', t')} = \frac{\partial F}{\partial x}\left(x^{(a)}(t), t\right) \frac{\delta x^{(a)}(t)}{\delta F(x', t')}. \tag{11}$$

By integrating this with the initial condition $\delta x^{(a)}(t')/\delta F(x', t') = \delta\left(x' - x^{(a)}(t')\right)$, we obtain

$$\frac{\delta x^{(a)}(t)}{\delta F(x', t')} = \delta\left(x' - x^{(a)}(t')\right) e^{\int_{t'}^{t} ds \frac{\partial F}{\partial x}\left(x^{(a)}(s), s\right)}. \tag{12}$$

Therefore, (8) becomes

$$\frac{\partial}{\partial t} P(\boldsymbol{x}, t) = \sum_{a=1}^{2} \sum_{b=1}^{2} \frac{\partial}{\partial x^{(a)}} \int dx' \int_0^t dt' C\left(x^{(a)} - x', t - t'\right)$$
$$\times \frac{\partial}{\partial x^{(b)}} \left\langle \delta\left(x' - x^{(b)}(t')\right) e^{\int_{t'}^{t} ds \frac{\partial F}{\partial x}\left(x^{(b)}(s), s\right)} \delta\left(\boldsymbol{x} - \boldsymbol{x}(t)\right) \right\rangle$$
$$+ T \sum_{a=1}^{2} \frac{\partial^2}{\partial x^{(a)2}} P(\boldsymbol{x}, t). \tag{13}$$

It should be noted that for a general $C(x, t)$, this is not a closed equation for $P(\boldsymbol{x}, t)$.

3.2. Perturbation analysis for small τ

We assume that τ is small and derive a perturbative expression for the time-evolution equation of $P(\boldsymbol{x}, t)$ by extending the derivation of an effective Fokker–Planck equation for weakly colored noise [16] to our random velocity-field problem. The explicit meaning of "small" τ is explained later. First, (13) is rewritten as

$$\frac{\partial}{\partial t} P(\boldsymbol{x}, t) = \sum_{a=1}^{2} \sum_{b=1}^{2} \frac{\partial}{\partial x^{(a)}} \int dx' \int_0^t dt' C_{\mathrm{s}}\left(x^{(a)} - x'\right) \frac{1}{\tau} e^{-\frac{t-t'}{\tau}}$$
$$\times \frac{\partial}{\partial x^{(b)}} \left\langle \delta\left(x' - x^{(b)}(t')\right) e^{\int_{t'}^{t} ds \frac{\partial F}{\partial x}\left(x^{(b)}(s), s\right)} \delta\left(\boldsymbol{x} - \boldsymbol{x}(t)\right) \right\rangle$$

$$
\begin{aligned}
&+T\sum_{a=1}^{2}\frac{\partial^2}{\partial x^{(a)2}}P(\boldsymbol{x},t)\\
&=\sum_{a=1}^{2}\sum_{b=1}^{2}\frac{\partial}{\partial x^{(a)}}\int dx'\int_0^{\frac{t}{\tau}}duC_{\rm s}\left(x^{(a)}-x'\right)e^{-u}\frac{\partial}{\partial x^{(b)}}\\
&\qquad\times\left\langle\delta\left(x'-x^{(b)}(t-\tau u)\right)e^{\int_{t-\tau u}^{t}ds\frac{\partial F}{\partial x}\left(x^{(b)}(s),s\right)}\delta\left(\boldsymbol{x}-\boldsymbol{x}(t)\right)\right\rangle\\
&+T\sum_{a=1}^{2}\frac{\partial^2}{\partial x^{(a)2}}P(\boldsymbol{x},t).
\end{aligned}
\tag{14}
$$

By expanding it with respect to τ, we obtain the following up to the first order of τ:

$$
\begin{aligned}
\frac{\partial}{\partial t}P(\boldsymbol{x},t)\simeq&\sum_{a=1}^{2}\sum_{b=1}^{2}\frac{\partial}{\partial x^{(a)}}\int dx'C_{\rm s}\left(x^{(a)}-x'\right)\frac{\partial}{\partial x^{(b)}}\left[\left\langle\delta\left(x'-x^{(b)}\right)\delta\left(\boldsymbol{x}-\boldsymbol{x}(t)\right)\right\rangle\right.\\
&\qquad+\tau\left\langle\dot{x}^{(b)}(t)\frac{\partial}{\partial x'}\delta\left(x'-x^{(b)}\right)\delta\left(\boldsymbol{x}-\boldsymbol{x}(t)\right)\right\rangle\\
&\qquad\left.+\tau\left\langle\delta\left(x'-x^{(b)}\right)\frac{\partial F}{\partial x}\left(x^{(b)},t\right)\delta\left(\boldsymbol{x}-\boldsymbol{x}(t)\right)\right\rangle\right]\\
&+T\sum_{a=1}^{2}\frac{\partial^2}{\partial x^{(a)2}}P(\boldsymbol{x},t)\\
\equiv&\ J_1+J_2+J_3+T\sum_{a=1}^{2}\frac{\partial^2}{\partial x^{(a)2}}P(\boldsymbol{x},t),
\end{aligned}
\tag{15}
$$

where we have defined J_i ($i=1,2,3$) as each term on the right hand side. The first term J_1 is calculated by using $C_{\rm s}'(0)=0$ as

$$
\begin{aligned}
J_1=&\sum_{a=1}^{2}\sum_{b=1}^{2}\frac{\partial}{\partial x^{(a)}}\frac{\partial}{\partial x^{(b)}}\int dx'C_{\rm s}\left(x^{(a)}-x'\right)\left\langle\delta\left(x'-x^{(b)}\right)\delta\left(\boldsymbol{x}-\boldsymbol{x}(t)\right)\right\rangle\\
&-\sum_{a=1}^{2}\sum_{b=1}^{2}\frac{\partial}{\partial x^{(a)}}\int dx'\left\{\frac{\partial}{\partial x^{(b)}}C_{\rm s}\left(x^{(a)}-x'\right)\right\}\left\langle\delta\left(x'-x^{(b)}\right)\delta\left(\boldsymbol{x}-\boldsymbol{x}(t)\right)\right\rangle\\
=&\sum_{a=1}^{2}\sum_{b=1}^{2}\frac{\partial^2}{\partial x^{(a)}\partial x^{(b)}}C_{\rm s}\left(x^{(a)}-x^{(b)}\right)P(\boldsymbol{x},t).
\end{aligned}
\tag{16}
$$

The second term J_2 is calculated as

$$
\begin{aligned}
J_2=&\ \tau\sum_{a=1}^{2}\sum_{b=1}^{2}\frac{\partial}{\partial x^{(a)}}\int dx'C_{\rm s}\left(x^{(a)}-x'\right)\frac{\partial}{\partial x^{(b)}}\frac{\partial}{\partial x'}\delta\left(x'-x^{(b)}\right)\left\langle\dot{x}^{(b)}(t)\delta\left(\boldsymbol{x}-\boldsymbol{x}(t)\right)\right\rangle\\
=&\ \tau\sum_{a=1}^{2}\sum_{b=1}^{2}\frac{\partial}{\partial x^{(a)}}\int dx'\frac{\partial C_{\rm s}}{\partial x}\left(x^{(a)}-x'\right)\frac{\partial}{\partial x^{(b)}}\delta\left(x'-x^{(b)}\right)\left\langle\dot{x}^{(b)}(t)\delta\left(\boldsymbol{x}-\boldsymbol{x}(t)\right)\right\rangle\\
=&\ \tau\sum_{a=1}^{2}\sum_{b=1}^{2}\frac{\partial}{\partial x^{(a)}}\frac{\partial}{\partial x^{(b)}}\frac{\partial C_{\rm s}}{\partial x}\left(x^{(a)}-x^{(b)}\right)\left\langle\dot{x}^{(b)}(t)\delta\left(\boldsymbol{x}-\boldsymbol{x}(t)\right)\right\rangle\\
&-\tau\sum_{a=1}^{2}\frac{\partial}{\partial x^{(a)}}\frac{\partial^2 C_{\rm s}}{\partial x^2}(0)\left\langle\dot{x}^{(a)}(t)\delta\left(\boldsymbol{x}-\boldsymbol{x}(t)\right)\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
&\simeq -\tau\sum_{a=1}^{2}\sum_{b=1}^{2}\frac{\partial^2}{\partial x^{(a)}\partial x^{(b)}}\frac{\partial C_{\mathrm{s}}}{\partial x}\left(x^{(a)}-x^{(b)}\right)\\
&\qquad\times\left[\sum_{c=1}^{2}\frac{\partial}{\partial x^{(c)}}C_{\mathrm{s}}\left(x^{(b)}-x^{(c)}\right)P(\boldsymbol{x},t)+T\frac{\partial}{\partial x^{(b)}}P(\boldsymbol{x},t)\right]\\
&\quad+\tau C_{\mathrm{s}}''(0)\frac{\partial}{\partial t}P(\boldsymbol{x},t).
\end{aligned}
\tag{17}
$$

In order to derive the first term in the last line, we have used the fact that J_2 is already $\mathcal{O}(\tau)$. Similarly, the third term J_3 is expressed as

$$
\begin{aligned}
J_3 &= \tau\sum_{a=1}^{2}\sum_{b=1}^{2}\frac{\partial^2}{\partial x^{(a)}\partial x^{(b)}}C_{\mathrm{s}}\left(x^{(a)}-x^{(b)}\right)\left\langle\frac{\partial F}{\partial x}\left(x^{(b)},t\right)\delta\left(\boldsymbol{x}-\boldsymbol{x}(t)\right)\right\rangle\\
&= \tau\sum_{a=1}^{2}\sum_{b=1}^{2}\frac{\partial^2}{\partial x^{(a)}\partial x^{(b)}}C_{\mathrm{s}}\left(x^{(a)}-x^{(b)}\right)\\
&\qquad\times\left[\frac{\partial}{\partial x^{(b)}}\left\langle F\left(x^{(b)},t\right)\delta\left(\boldsymbol{x}-\boldsymbol{x}(t)\right)\right\rangle-\left\langle F\left(x^{(b)},t\right)\frac{\partial}{\partial x^{(b)}}\delta\left(\boldsymbol{x}-\boldsymbol{x}(t)\right)\right\rangle\right]\\
&\simeq \tau\sum_{a=1}^{2}\sum_{b=1}^{2}\frac{\partial^2}{\partial x^{(a)}\partial x^{(b)}}C_{\mathrm{s}}\left(x^{(a)}-x^{(b)}\right)\\
&\qquad\times\left[-\frac{\partial}{\partial x^{(b)}}\int dx' C_{\mathrm{s}}\left(x^{(b)}-x'\right)\sum_{c=1}^{2}\frac{\partial}{\partial x^{(c)}}\left\langle\delta\left(x'-x^{(c)}\right)\delta\left(\boldsymbol{x}-\boldsymbol{x}(t)\right)\right\rangle\right.\\
&\qquad\left.+\int dx' C_{\mathrm{s}}\left(x^{(b)}-x'\right)\sum_{c=1}^{2}\frac{\partial^2}{\partial x^{(b)}\partial x^{(c)}}\left\langle\delta\left(x'-x^{(c)}\right)\delta\left(\boldsymbol{x}-\boldsymbol{x}(t)\right)\right\rangle\right]\\
&= -\tau\sum_{a=1}^{2}\sum_{b=1}^{2}\frac{\partial^2}{\partial x^{(a)}\partial x^{(b)}}C_{\mathrm{s}}\left(x^{(a)}-x^{(b)}\right)\int dx'\frac{\partial C_{\mathrm{s}}}{\partial x}\left(x^{(b)}-x'\right)\\
&\qquad\times\sum_{c=1}^{2}\frac{\partial}{\partial x^{(c)}}\left\langle\delta\left(x'-x^{(c)}\right)\delta\left(\boldsymbol{x}-\boldsymbol{x}(t)\right)\right\rangle\\
&= -\tau\sum_{a=1}^{2}\sum_{b=1}^{2}\frac{\partial^2}{\partial x^{(a)}\partial x^{(b)}}C_{\mathrm{s}}\left(x^{(a)}-x^{(b)}\right)\\
&\qquad\times\left[\sum_{c=1}^{2}\frac{\partial}{\partial x^{(c)}}\frac{\partial C_{\mathrm{s}}}{\partial x}\left(x^{(b)}-x^{(c)}\right)P(\boldsymbol{x},t)-\frac{\partial^2 C_{\mathrm{s}}}{\partial x^2}(0)P(\boldsymbol{x},t)\right].
\end{aligned}
\tag{18}
$$

Therefore, we finally obtain

$$
\begin{aligned}
\frac{\partial}{\partial t}P(\boldsymbol{x},t) = &\sum_{a=1}^{2}\sum_{b=1}^{2}\frac{\partial^2}{\partial x^{(a)}\partial x^{(b)}}C_{\mathrm{s}}\left(x^{(a)}-x^{(b)}\right)P(\boldsymbol{x},t)+\tau C_{\mathrm{s}}''(0)\frac{\partial}{\partial t}P(\boldsymbol{x},t)\\
&-\tau\sum_{a=1}^{2}\sum_{b=1}^{2}\frac{\partial^2}{\partial x^{(a)}\partial x^{(b)}}\frac{\partial C_{\mathrm{s}}}{\partial x}\left(x^{(a)}-x^{(b)}\right)\\
&\qquad\times\left[\sum_{c=1}^{2}\frac{\partial}{\partial x^{(c)}}C_{\mathrm{s}}\left(x^{(b)}-x^{(c)}\right)P(\boldsymbol{x},t)+T\frac{\partial}{\partial x^{(b)}}P(\boldsymbol{x},t)\right]
\end{aligned}
$$

$$
\begin{aligned}
&- \tau \sum_{a=1}^{2} \sum_{b=1}^{2} \frac{\partial^2}{\partial x^{(a)} \partial x^{(b)}} C_{\mathrm{s}} \left(x^{(a)} - x^{(b)} \right) \\
&\quad \times \left[\sum_{c=1}^{2} \frac{\partial}{\partial x^{(c)}} \frac{\partial C_{\mathrm{s}}}{\partial x} \left(x^{(b)} - x^{(c)} \right) P(\boldsymbol{x}, t) - C_{\mathrm{s}}''(0) P(\boldsymbol{x}, t) \right] \\
&+ T \sum_{a=1}^{2} \frac{\partial^2}{\partial x^{(a)2}} P(\boldsymbol{x}, t),
\end{aligned}
\tag{19}
$$

or

$$
\begin{aligned}
\frac{\partial}{\partial t} P(\boldsymbol{x}, t) \simeq\ & \left\{ 1 + \tau C_{\mathrm{s}}''(0) \right\} \sum_{a=1}^{2} \sum_{b=1}^{2} \frac{\partial^2}{\partial x^{(a)} \partial x^{(b)}} C_{\mathrm{s}} \left(x^{(a)} - x^{(b)} \right) P(\boldsymbol{x}, t) \\
&- \tau \sum_{a=1}^{2} \sum_{b=1}^{2} \frac{\partial^2}{\partial x^{(a)} \partial x^{(b)}} \frac{\partial C_{\mathrm{s}}}{\partial x} \left(x^{(a)} - x^{(b)} \right) \\
&\quad \times \left[\sum_{c=1}^{2} \frac{\partial}{\partial x^{(c)}} C_{\mathrm{s}} \left(x^{(b)} - x^{(c)} \right) P(\boldsymbol{x}, t) + T \frac{\partial}{\partial x^{(b)}} P(\boldsymbol{x}, t) \right] \\
&- \tau \sum_{a=1}^{2} \sum_{b=1}^{2} \frac{\partial^2}{\partial x^{(a)} \partial x^{(b)}} C_{\mathrm{s}} \left(x^{(a)} - x^{(b)} \right) \\
&\quad \times \left[\sum_{c=1}^{2} \frac{\partial}{\partial x^{(c)}} \frac{\partial C_{\mathrm{s}}}{\partial x} \left(x^{(b)} - x^{(c)} \right) P(\boldsymbol{x}, t) - C_{\mathrm{s}}''(0) P(\boldsymbol{x}, t) \right] \\
&+ \left\{ 1 + \tau C_{\mathrm{s}}''(0) \right\} T \sum_{a=1}^{2} \frac{\partial^2}{\partial x^{(a)2}} P(\boldsymbol{x}, t).
\end{aligned}
\tag{20}
$$

This is the time-evolution equation for $P(\boldsymbol{x}, t)$ up to the order τ.

First, we discuss the properties of single-particle diffusion. By integrating (20) with respect to $x^{(2)}$, we obtain the time-evolution equation for the single-particle probability distribution $P^{(1)}(x, t) \equiv \left\langle \delta \left(x - x^{(1)}(t) \right) \right\rangle$:

$$
\frac{\partial}{\partial t} P^{(1)}(x, t) = \left\{ C_{\mathrm{s}}(0) + T \right\} \frac{\partial^2}{\partial x^2} P^{(1)}(x, t) + \tau \left\{ 2 C_{\mathrm{s}}(0) + T \right\} C_{\mathrm{s}}''(0) \frac{\partial^2}{\partial x^2} P^{(1)}(x, t). \tag{21}
$$

This equation is of the diffusion type. Particularly, the result for the Markovian case [9] is recovered when $\tau = 0$. Furthermore, because $C_{\mathrm{s}}''(0) < 0$, we can find that the effect of thermal noise is suppressed by the existence of memory as $T \to T \left\{ 1 + \tau C_{\mathrm{s}}''(0) \right\}$.

In order to obtain a time-evolution equation for the probability distribution of the relative distance between two particles $P_{\mathrm{rel}}(r, t) \equiv \left\langle \delta \left(r - x^{(1)}(t) + x^{(2)}(t) \right) \right\rangle$, we rewrite (20) by using $R \equiv \left(x^{(1)} + x^{(2)} \right) / 2$ and $r \equiv x^{(1)} - x^{(2)}$ and then integrate it with respect to R:

$$
\begin{aligned}
\frac{\partial}{\partial t} P_{\mathrm{rel}}(r, t) = & \ 2 \frac{\partial^2}{\partial r^2} \left\{ C_{\mathrm{s}}(0) - C_{\mathrm{s}}(r) + T \right\} P_{\mathrm{rel}}(r, t) + 2\tau \frac{\partial^2}{\partial r^2} C_{\mathrm{s}}'(r)^2 P_{\mathrm{rel}}(r, t) \\
&+ 2\tau C_{\mathrm{s}}''(0) \frac{\partial^2}{\partial r^2} \left\{ 2 C_{\mathrm{s}}(0) - 2 C_{\mathrm{s}}(r) + T \right\} P_{\mathrm{rel}}(r, t) - 2\tau T \frac{\partial^3}{\partial r^3} C_{\mathrm{s}}'(r) P_{\mathrm{rel}}(r, t) \\
&+ 2\tau \frac{\partial^2}{\partial r^2} C_{\mathrm{s}}''(r) \left\{ C_{\mathrm{s}}(0) - C_{\mathrm{s}}(r) + T \right\} P_{\mathrm{rel}}(r, t).
\end{aligned}
\tag{22}
$$

When $\tau = 0$, the result for the Markovian case [9] is recovered, and particularly, when T is also zero, the local diffusion constant $2\{C_{\rm s}(0) - C_{\rm s}(r)\}$ is zero at $r = 0$, which implies that particles aggregate with time and the steady-state distribution is $P_{\rm rel}^{\rm (ss)}(r) = \delta(r)$. For finite τ, we can find that the effect of thermal noise at $r = 0$ is $2T\{1 + 2\tau C_{\rm s}''(0)\} < 2T$ and therefore is suppressed by the existence of memory. We remark that this renormalization effect for relative diffusion is stronger than that for single-particle diffusion since $\{1 + 2\tau C_{\rm s}''(0)\} < \{1 + \tau C_{\rm s}''(0)\}$. We also note that $r = 0$ becomes the absorbing point when T equals zero, similar to the case with $\tau = 0$, which means that two particles in a common velocity field aggregate with time, and $P_{\rm rel}^{\rm (ss)}(r) = \delta(r)$.

3.3. Convergence of the effective Fokker–Planck equations

We discuss the convergence of the effective Fokker–Planck equations obtained in the previous subsection and the meaning of "small" τ.

First, we consider (21). From the expression for $T = 0$, we can find that the solution of this equation converges for all x only when

$$1 + 2\tau C_{\rm s}''(0) > 0. \tag{23}$$

Therefore, for any T, the condition that τ should satisfy is $\tau < 1/\left(2\left|C_{\rm s}''(0)\right|\right)$.

Next, in order to discuss convergence of (22), we first consider the case $T = 0$:

$$\frac{\partial}{\partial t}P_{\rm rel}(r,t) = 2\frac{\partial^2}{\partial r^2}\left[\{1 + 2\tau C_{\rm s}''(0) + \tau C_{\rm s}''(r)\}\{C_{\rm s}(0) - C_{\rm s}(r)\} + \tau C_{\rm s}'(r)^2\right]P_{\rm rel}(r,t). \tag{24}$$

We find that the positivity of the local diffusion constant is satisfied for all r when

$$1 + 2\tau C_{\rm s}''(0) + \tau \min_r C_{\rm s}''(r) > 0. \tag{25}$$

We remark that this condition is more strict than (23).

For the case $T > 0$, (22) can be rewritten as

$$\begin{aligned}\frac{\partial}{\partial t}P_{\rm rel}(r,t) &= 2\frac{\partial^2}{\partial r^2}\left[\{1 + 2\tau C_{\rm s}''(0) + \tau C_{\rm s}''(r)\}\{C_{\rm s}(0) - C_{\rm s}(r)\} + \tau C_{\rm s}'(r)^2\right]P_{\rm rel}(r,t)\\ &\quad + 2\frac{\partial^2}{\partial r^2}T\{1 + \tau C_{\rm s}''(0) + \tau C_{\rm s}''(r)\}P_{\rm rel}(r,t) - 2\tau T\frac{\partial^3}{\partial r^3}C_{\rm s}'(r)P_{\rm rel}(r,t).\end{aligned} \tag{26}$$

The positivity of the local diffusion constant in the second term is satisfied when $1 + \tau C_{\rm s}''(0) + \tau \min_r C_{\rm s}''(r) > 0$, and this condition is weaker than (25). Thus, the condition that "small" τ should satisfy is (25). We remark that (26) contains a third derivative. In general, a truncated Kramers–Moyal expansion at a higher order than the second order yields a negative probability [20]. However, it is also known that the truncated equation often approximates the shape of the exact probability distribution very well. Therefore, we numerically investigate behavior of (22) for a concrete setup and check its usefulness. A comparison with molecular dynamics results is presented in the next section.

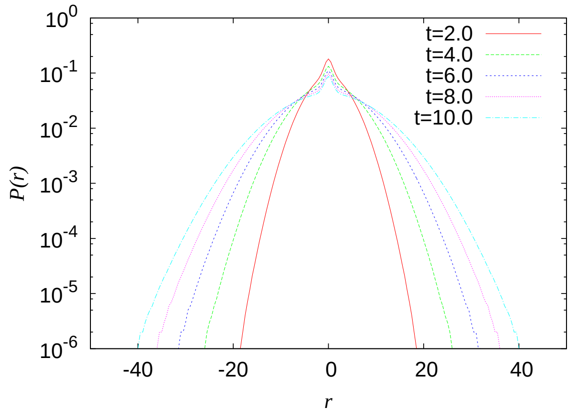

Figure 1. The probability distribution $P_{\rm rel}(r,t)$ for $\tau = 0.0$.

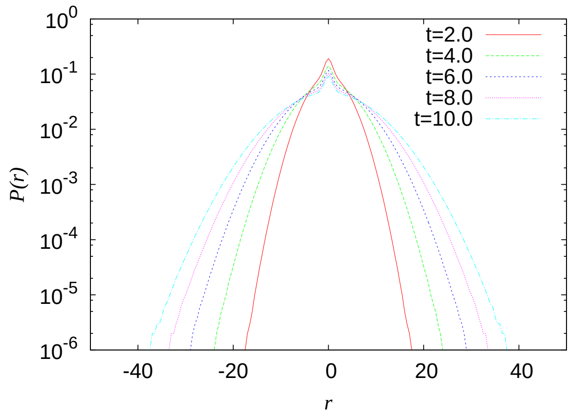

Figure 2. The probability distribution $P_{\rm rel}(r,t)$ for $\tau = 0.1$.

3.4. Numerical solution of (22)

We numerically solve the diffusion equation in (22) and investigate the properties of relative diffusion. We consider the case

$$C_{\rm s}(x) = Ae^{-\frac{x^2}{2\sigma^2}} \tag{27}$$

with $A = 1.0$ and $\sigma = 1.0$, and we set $T = 1.0$. The initial condition is given by

$$P_{\rm rel}(r,0) = \frac{1}{2}\delta(r) + \frac{1}{4}\left[\delta(r-0.5)+\delta(r+0.5)\right]. \tag{28}$$

In this setup, the condition in (25) becomes $\tau < 1/3$. The equation in (22) is numerically computed by differentiating it with step sizes $\Delta r = 0.5$ and $\Delta t = 0.01$.

We plot the numerical solution of (22) with $\tau = 0.0$ and $\tau = 0.1$ in Figure 1 and 2, respectively. We can see that there is no divergence, even for $\tau > 0$ and $T > 0$. We also find that $P_{\rm rel}(r,t)$ with $\tau = 0.1$ has a peak at $r = 0$, similar to that with $\tau = 0.0$. In order to focus on the behavior of the mean-squared relative distance $\langle r^2 \rangle_t$ and the peak value $P_{\rm rel}(0,t)$, we plot these two quantities in Figure 3 and 4, respectively. We observe the suppression of relative diffusion induced by the memory of a velocity field.

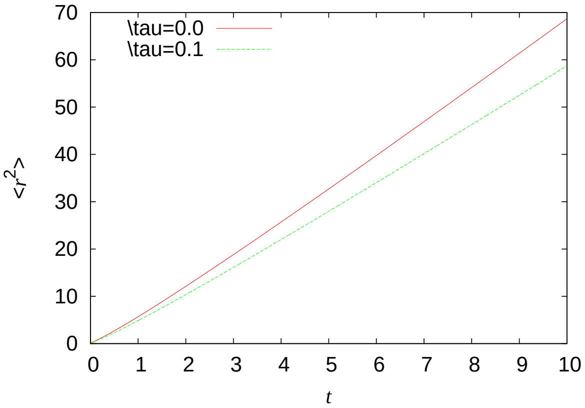

Figure 3. Time evolution of the mean-squared value $\langle r^2 \rangle_t$ for $\tau = 0.0$ and $\tau = 0.1$.

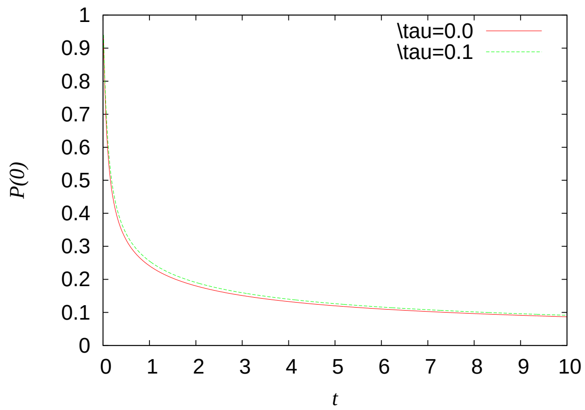

Figure 4. Time evolution of the peak value $P_{\rm rel}(0, t)$ for $\tau = 0.0$ and $\tau = 0.1$.

4. Comparison with molecular dynamics results

We check the validity of our perturbative treatment by comparing the bahavior of (22) with the results obtained from molecular dynamics simulation using (1). We can generate a random velocity field F that obeys (3) with (27) by

$$F(x,t) = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{t} \left(\frac{2}{\pi}\right)^{\frac{1}{4}} \sqrt{\frac{A}{\sigma}} \frac{1}{\tau} e^{-\frac{(x-x')^2}{\sigma^2}} e^{-\frac{t-t'}{\tau}} F_0(x',t'), \tag{29}$$

where F_0 is a zero-mean Gaussian random variable with

$$\langle F_0(x,t) F_0(x',t') \rangle = 2\delta(x-x')\delta(t-t'). \tag{30}$$

In the molecular dynamics simulation, F can also be prepared by numerically solving the equation

$$\frac{\partial}{\partial t} F(x,t) = -\frac{1}{\tau} F(x,t) + \int_{-\infty}^{\infty} dx' \left(\frac{2}{\pi}\right)^{\frac{1}{4}} \sqrt{\frac{A}{\sigma}} \frac{1}{\tau} e^{-\frac{(x-x')^2}{\sigma^2}} F_0(x',t') \tag{31}$$

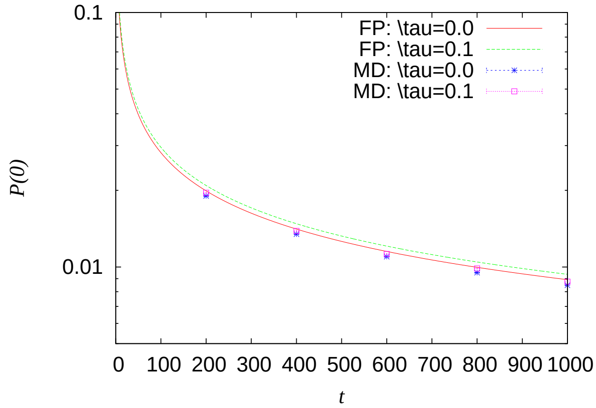

Figure 5. Molecular dynamics results for the peak value $P_{\rm rel}(0,t)$.

for the initial condition $F(x,-t_{\rm init})=0$ with a sufficiently large $t_{\rm init}>0$. The equation in (31) is numerically computed by differentiating it with step sizes $\Delta x=0.5$ and $\Delta t=0.01$. The equation in (1) is also computed by differentiating it with a step size $\Delta t=0.01$, and the force acting on the particles is evaluated by linear interpolation of F.

In Figure 5, we plot the behavior of $P_{\rm rel}(0,t)$ calculated by the molecular dynamics simulation together with the numerical solution of (22). The effective time-evolution equation in (22) is in good agreement with the molecular dynamics results. In particular, we observe that the suppression effect on relative diffusion becomes stronger as τ increases. Therefore, we conclude that (22) approximates the shape of the exact probability distribution very well, even though it contains a third derivative.

5. Concluding remarks

In this paper, we report the suppression of the effect of thermal noise induced by the memory of a random velocity field. The effective Fokker–Planck equation for the probability distribution of the positions of the two particles is perturbatively derived for a small corralation time τ. We find that the renormalized temperature of relative diffusion is different from that of single-particle diffusion. Furthermore, we check that the effective equation in (22) is in good agreement with the molecular dynamics results for small τ.

Before concluding the paper, we present two remarks. The first remark is related to the replica symmetry breaking in particle trajectories [21]. In reference [21], the diffusion of two Brownian particles in a velocity field obeying the noisy Burgers equation was studied, and it was found that the probability distribution of the overlap between two trajectories

$$q(t)\equiv\frac{1}{t}\int_0^t ds\delta\left(x^{(1)}(s)-x^{(2)}(s)\right) \tag{32}$$

takes a nontrivial form, even for a finite T. It should be noted that the trivial relation

$$\langle q(t)\rangle \equiv \frac{1}{t}\int_0^t ds P_{\mathrm{rel}}(0,s) \tag{33}$$

holds between the overlap and the relative diffusion. Therefore, a nonzero $P_{\mathrm{rel}}(0,\infty)$ is necessary for the nontrivial probability distribution of q. The results presented in this paper suggest that the existence of memory weakens relative diffusion, even for a finite T, although the effective temperature $T\{1+2\tau C_{\mathrm{s}}''(0)\}$ at $r=0$ cannot become zero under the condition in (25). This result is reasonable because memory plays a role in maintaining the shape of a velocity field. Although the mechanism of complete suppression $P_{\mathrm{rel}}(0,\infty)>0$ at a finite T is not yet clear, adding memory to the velocity fields may be significant for the replica symmetry breaking at a finite T.

The second remark is related to the common-noise-induced synchronization of two independent and identical phase oscillators [22]. In reference [23], a time-evolution equation was derived for the probability distribution of the phase difference between two oscillators subject to common white noise and was found to be equivalent to the Fokker–Planck equation for the relative distance between two particles in a Markovian random velocity field [9], i.e., (22) with $\tau=0$. In contrast, the relation between the synchronization of two oscillators subject to common colored noise [24] and our result for a non-Markovian random velocity field is rather unclear. This problem will be studied in the future.

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