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The N-soliton formulas for a multi-component modified nonlinear Schrödinger system with nonzero boundary conditions

Yoshimasa Matsuno
Division of Applied Mathematical Science
Graduate School of Sciences and Technology for Innovation
Yamaguchi University
E-mail address: matsuno@yamaguchi-u.ac.jp

Abstract

The present paper provides the dark N-soliton solution for a multi-component modified nonlinear Schrödinger (NLS) system with plane-wave boundary conditions, as well as the bright-dark N-soliton solution with mixed zero and plane-wave boundary conditions. The N-soliton formulas obtained here include as special cases the existing soliton solutions of the NLS and derivative NLS equations and their integrable multi-component analogs. The new features of soliton solutions are discussed. In particular, it is shown that the N constraints must be imposed on the amplitude parameters of solitons in constructing the dark N-soliton solution with plane-wave boundary conditions. For mixed type boundary conditions, the structure of soliton solutions is found to be more explicit than that of dark soliton solutions since no constraints are imposed on the soliton parameters. Last, we comment on an integrable multi-component system associated with the first negative flow of the multi-component derivative NLS hierarchy.

1. Introduction

The nonlinear Schrödinger (NLS) equation is a milestone in the theory of nonlinear waves. One remarkable feature is that it is a completely integrable equation, allowing us to solve it by means of several exact methods of solution such as the inverse scattering transform (IST) [1-3], Bäcklund-Darboux (BD) transformation [4-6] and Hirota's direct method [7-9]. In recent years, much attention has been paid to the multi-component generalizations of the NLS equation because of their wide applicability in many physical contexts such as nonlinear optics, nonlinear water waves and plasma physics and so on [10-12].

In this paper, we consider the following multi-component system of nonlinear partial differential equations (PDEs) which is a hybrid of the multi-component NLS system and multi-component derivative NLS system

$$\mathrm{i}\, q_{j,t} + q_{j,xx} + \mu \left(\sum_{s=1}^{n} \sigma_s |q_s|^2\right) q_j + \mathrm{i}\gamma \left[\left(\sum_{s=1}^{n} \sigma_s |q_s|^2\right) q_j\right]_x = 0, \quad j = 1, 2, ..., n, \tag{1.1}$$

where $q_j = q_j(x,t)$ $(j = 1, 2, ..., n)$ are complex-valued functions of x and t, μ and γ are real constants, n is an arbitrary positive integer, and subscripts x and t appended to q_j denote partial differentiations. The coefficients $\sigma_s = \pm 1$ $(s = 1, 2, ..., n)$ specify the sign of the nonlinearity. For the multi-component NLS system for which $\gamma = 0$, we can deal with the three types of cubic nonlinearities, i.e., focusing ($\sigma_s = 1,\ s = 1, 2, ..., n$), defocusing ($\sigma_s = -1,\ s = 1, 2, ..., n$) and mixed focusing-defocusing ($\sigma_s = 1, s = 1, 2, ..., m; \sigma_s = -1, s = m+1, m+2, ..., n$) nonlinearities with $\mu > 0$, where m is an arbitrary positive integer such that $1 \leq m < n$. The special systems reduced from (1.1) have been summarized in a previous paper [13]. The system (1.1) has been shown to be completely integrable [14] and hence the exact methods mentioned above can be applied to it to obtain various types of soliton solutions. The properties of solutions, however, depend essentially on the boundary conditions. We emphasize that most works have been concerned with the analysis of the system under vanishing boundary conditions at spatial infinity [13].

The main purpose of the present paper is to construct N-soliton solutions (N: arbitrary positive integer) of the system (1.1) with the following two types of boundary conditions:
1) Plane-wave boundary conditions.

$$q_j \sim \rho_j \exp \mathrm{i} \left(k_j x - \omega_j t + \phi_j^{(\pm)} \right), \quad x \to \pm\infty, \qquad j = 1, 2, ..., n, \tag{1.2a}$$

$$\omega_j = k_j^2 - \mu \sum_{s=1}^{n} \sigma_s \rho_s^2 + \gamma \left(\sum_{s=1}^{n} \sigma_s \rho_s^2 \right) k_j, \quad j = 1, 2, ..., n. \tag{1.2b}$$

2) Mixed type boundary conditions.

$$q_j \sim 0, \quad x \to \pm\infty, \quad j = 1, 2, ..., m, \tag{1.3a}$$

$$q_{m+j} \sim \rho_j \exp \mathrm{i} \left(k_j x - \omega_j t + \phi_j^{(\pm)} \right), \quad x \to \pm\infty, \quad j = 1, 2, ..., n-m, \tag{1.3b}$$

$$\omega_j = k_j^2 - \mu \sum_{s=1}^{n-m} \sigma_s \rho_s^2 + \gamma \left(\sum_{s=1}^{n-m} \sigma_s \rho_s^2 \right) k_j, \quad j = 1, 2, ..., n-m. \tag{1.3c}$$

Here, $\rho_j,\ k_j$ and $\phi_j^{(\pm)}$ are arbitrary real constants and ω_j from (1.2b) and (1.3c) represent the linear dispersion relations for components with the plane-wave boundary conditions. For case 1), all the components approach the specified plane waves at infinity whereas for case 2), the first m components vanish at infinity and the remaining $n-m$ components have the specified plane waves. It is important that for the nonzero background fields like (1.2a) and (1.3b), each component has different wavenumber and frequency. This causes difficulties in constructing soliton solutions. In particular, in the case of 1) above, one must impose certain constraints between the real and imaginary parts of the complex parameters characterizing the amplitude (or the velocity) of solitons. This peculiar feature has never been encountered in the case of zero boundary conditions [13].

The approach which will be employed in our analysis is the direct method [7-9]. While both the IST and BD transformation are based on the Lax representations of completely integrable PDEs, the direct method does not need the knowledge of the IST, and it can also be applicable to nonintegrable equations. It provides a powerful tool to obtain special solutions such as soliton and periodic solutions irrespective of the boundary conditions. The N-soliton formulas presented in this paper include as special cases the existing N-soliton solutions of the NLS and derivative NLS equations as well as the Manakov system with nonzero boundary conditions. The compact expressions of the soliton solutions are particularly useful for investigating their structures, asymptotic behaviors and dynamics. On the other hand, the IST is technically involved for nonzero boundary conditions, and hence it has been applied mainly to the multi-component system such as (1.1) under the restricted class of boundary conditions in which all the components have the same plane-wave boundary condition at infinity, for instance [15, 16]. The development of the IST for the multi-component systems with the general nonzero boundary conditions is still in progress. In particular, the derivation of the N-soliton solutions by means of the IST is an important issue left to the future. In the following, we describe only the main results, and the details will be reported in a separate paper.

2. Notation and basic formulas for determinants

First of all, we introduce for convenience the notation and some basic formulas for determinants associated with the N-soliton solutions.

2.1. Notation

Let $D = (d_{jk})_{1\leq j,k\leq N}$ be an $N\times N$ matrix and $\mathbf{a} = (a_j)_{1\leq j\leq N}, \mathbf{b} = (b_j)_{1\leq j\leq N}, \mathbf{c} = (c_j)_{1\leq j\leq N}, \mathbf{d} = (d_j)_{1\leq j\leq N}$ be the N-component row vectors, where all the entries in the matrices and vectors are complex-valued functions of x and t. The bordered marices associated with the matrix D are defined by

$$D(\mathbf{a};\mathbf{b}) = \begin{pmatrix} D & \mathbf{b}^T \\ \mathbf{a} & 0 \end{pmatrix}, \quad D(\mathbf{a},\mathbf{b};\mathbf{c},\mathbf{d}) = \begin{pmatrix} D & \mathbf{c}^T & \mathbf{d}^T \\ \mathbf{a} & 0 & 0 \\ \mathbf{b} & 0 & 0 \end{pmatrix}, \tag{2.1}$$

where the symbol T denotes transpose. The first cofactor of d_{ij} is defined by $D_{ij} = \partial|D|/\partial d_{ij}$, ($|D| = \det D$) and the second cofactor by $D_{ij,pq} = \partial^2|D|/\partial d_{ip}\partial d_{jq}$ $(i < j, p < q)$.

2.2. Basic formulas for determinants

The following formulas are well-known in the theory of determinants and will be used frequently in our analysis [17]:

$$\frac{\partial|D|}{\partial x} = \sum_{i,j=1}^{N} \frac{\partial d_{ij}}{\partial x} D_{ij}, \tag{2.2}$$

$$\begin{vmatrix} D & \mathbf{a}^T \\ \mathbf{b} & z \end{vmatrix} = |D|z - \sum_{i,j=1}^{N} D_{ij} a_i b_j, \tag{2.3}$$

$$|D(\mathbf{a},\mathbf{b};\mathbf{c},\mathbf{d})||D| = |D(\mathbf{a};\mathbf{c})||D(\mathbf{b};\mathbf{d})| - |D(\mathbf{a};\mathbf{d})||D(\mathbf{b};\mathbf{c})|, \tag{2.4}$$

$$\delta_{ij}|D| = \sum_{k=1}^{N} d_{ik} D_{jk} = \sum_{k=1}^{N} d_{ki} D_{kj}, \tag{2.5}$$

$$D_{ij} = \sum_{p=1}^{N} d_{pq}\, D_{ip,jq} \quad (j \neq q) \tag{2.6a}$$

$$= \sum_{q=1}^{N} d_{pq}\, D_{ip,jq} \quad (i \neq p), \tag{2.6b}$$

$$|D(\mathbf{a},\mathbf{b};\mathbf{c},\mathbf{d})| = \sum_{\substack{i,j,p,q=1\\(i\neq j,p\neq q)}}^{N} a_p b_q c_i d_j D_{ij,pq}. \tag{2.7}$$

3. The modified NLS system with plane-wave boundary conditions

3.1. Gauge transformation

We consider the system of PDEs (1.1) subjected to the plane-wave boundary conditions (1.2). As in the case of zero boundary conditions considered in [13], we first apply the following gauge transformation

$$q_j = u_j \exp\left[-\frac{\mathrm{i}\gamma}{2}\int_{-\infty}^{x}\sum_{s=1}^{n}\sigma_s(|u_s|^2-\rho_s^2)dx\right], \quad j=1,2,...,n, \tag{3.1}$$

to the system, where $u_j = u_j(x,t)$ $(j=1,2,...,n)$ are complex-valued functions of x and t. The system (1.1) is then transformed to the system of nonlinear PDEs for u_j:

$$\mathrm{i}u_{j,t} + u_{j,xx} + \mathrm{i}\lambda\gamma u_{j,x} + \mathrm{i}\gamma\left(\sum_{s=1}^{n}\sigma_s u_s^* u_{s,x} - \mathrm{i}\sum_{s=1}^{n}\sigma_s k_s \rho_s^2\right)u_j$$

$$+\left[\left(\mu - \frac{\lambda\gamma^2}{2}\right)\sum_{s=1}^{n}\sigma_s|u_s|^2 + \frac{(\lambda\gamma)^2}{2}\right]u_j = 0, \quad j=1,2,...,n. \tag{3.2}$$

Here, we have put $\lambda = \sum_{s=1}^{n}\sigma_s\rho_s^2$ for simplicity and the asterisk appended to u_s denotes complex conjugate.

3.2. Bilinearization

The following proposition is the starting point in our analysis.

Proposition 3.1. *By means of the dependent variable transformations*

$$u_j = \rho_j e^{\mathrm{i}(k_j x - \omega_j t)}\frac{h_j}{f}, \quad j=1,2,...,n, \tag{3.3}$$

the system of nonlinear PDEs (3.2) can be decoupled into the following system of bilinear equations for f and h_j

$$\mathrm{i}D_t h_j\cdot f + \mathrm{i}(2k_j+\lambda\gamma)D_x h_j\cdot f + D_x^2 h_j\cdot f = 0, \quad j=1,2,...,n, \tag{3.4}$$

$$D_x f\cdot f^* - \frac{\mathrm{i}\gamma}{2}\sum_{s=1}^{n}\sigma_s\rho_s^2(h_s h_s^* - ff^*) = 0, \tag{3.5}$$

$$D_x^2 f\cdot f^* - \frac{\mathrm{i}\gamma}{2}\sum_{s=1}^{n}\sigma_s\rho_s^2 D_x h_s\cdot h_s^* + \gamma\sum_{s=1}^{n}\sigma_s k_s\rho_s^2(h_s h_s^* - ff^*)$$

$$+\left(\frac{\lambda\gamma^2}{4}-\mu\right)\sum_{s=1}^{n}\sigma_s\rho_s^2(h_sh_s^*-ff^*)=0. \tag{3.6}$$

Here, $f=f(x,t)$ and $h_j=h_j(x,t)$ $(j=1,2,...,n)$ are complex-valued functions of x and t, and the bilinear operators D_x and D_t are defined by

$$D_x^mD_t^nf\cdot g=\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^m\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^n f(x,t)g(x',t')\Big|_{x'=x,\,t'=t}, \tag{3.7}$$

where m and n are nonnegative integers.

The fundamental quantities f and h_j characterize completely solutions. They are sometimes called the tau functions. While the expressions (3.3) for u_j give the solutions of the system of PDEs (3.2) in terms of the tau functions f and h_j $(j=1,2,...,n)$, the original variables q_j are expressible by them as well. To show this, we use (3.3) and (3.5) to obtain the relation

$$\frac{\partial}{\partial x}\ln\frac{f}{f^*}=-\frac{\mathrm{i}\gamma}{2}\sum_{s=1}^{n}\sigma_s(|u_s|^2-\rho_s^2). \tag{3.8}$$

If we introduce (3.8) into (3.1), integrate with respect to x and note (3.3), we find the desired expressions

$$q_j=\rho_j\mathrm{e}^{\mathrm{i}(k_jx-\omega_jt)}\,\frac{h_jf^*}{f^2},\quad j=1,2,...,n. \tag{3.9}$$

Remark 3.1. If $\gamma=0$, then the bilinear equation (3.5) reduces to $D_xf\cdot f^*=0$, implying that $f^*=c(t)f$. An arbitrary function $c(t)$ can be set to 1 by imposing a boundary condition, $f=1,\ x\to-\infty$, for instance. Thus, q_j from (3.9) simplify to

$$q_j=\rho_j\mathrm{e}^{\mathrm{i}(k_jx-\omega_jt)}\,\frac{h_j}{f},\quad j=1,2,...,n, \tag{3.10}$$

and they satisfy the n-component NLS system with the plane-wave boundary conditions (1.2)

$$\mathrm{i}\,q_{j,t}+q_{j,xx}+\mu\left(\sum_{s=1}^{n}\sigma_s|q_s|^2\right)q_j=0,\quad j=1,2,...,n. \tag{3.11}$$

3.3. The dark N-soliton solution

Here, we establish the following theorem.

Theorem 3.1. *The N-soliton solution of the system of bilinear equations (3.4)-(3.6) is given in terms of the following determinants*

$$f=|D|,\quad h_s=|H_s|,\quad s=1,2,...,n, \tag{3.12a}$$

$$D=(d_{jk})_{1\le j,k\le N},\quad d_{jk}=\delta_{jk}-\frac{\mathrm{i}p_j+\frac{\mu}{\gamma}}{p_j+p_k^*}\,z_jz_k^*, \tag{3.12b}$$

$$H_s = (h_{jk}^{(s)})_{1\le j,k\le N}, \quad h_{jk}^{(s)} = \delta_{jk} + \frac{\left(\mathrm{i}p_j + \frac{\mu}{\gamma}\right)(p_j - \mathrm{i}k_s)}{(p_j + p_k^*)(p_k^* + \mathrm{i}k_s)} z_j z_k^*, \tag{3.12c}$$

$$z_j = \exp[p_j x + (\mathrm{i}p_j^2 - \lambda\gamma p_j)t + \zeta_{j0}], \quad j = 1, 2, ..., N, \quad \lambda = \sum_{s=1}^{n} \sigma_s \rho_s^2. \tag{3.12d}$$

Here, p_j and ζ_{j0} $(j = 1, 2, ..., N)$ are arbitrary complex parameters characterizing the amplitude and phase of the solitons, respectively, and the N constraints are imposed on the parameters p_j

$$\frac{\gamma}{2}\sum_{s=1}^{n} \sigma_s \rho_s^2 \frac{\mathrm{i}(p_j - p_j^*) + k_s + \frac{\mu}{\gamma}}{(p_j - \mathrm{i}k_s)(p_j^* + \mathrm{i}k_s)} = -1, \quad j = 1, 2, ..., N. \tag{3.13}$$

Remark 3.2. By means of the transformation of the variables, $f = \tilde{f}, h_s = \tilde{h}_s$ $(s = 1, 2, ..., n)$, $x = \tilde{x} + \frac{2\mu}{\gamma}\tilde{t}$ combined with the transformation of the parameters, $p_j = \tilde{p}_j + \mathrm{i}\,\frac{\mu}{\gamma}$ $(j = 1, 2, ..., N), k_s = \tilde{k}_s + \frac{\mu}{\gamma}$ $(s = 1, 2, ..., n)$, the form of the bilinear equations (3.4) and (3.5) is unchanged whereas the bilinear equation (3.6) reduces to a simplified form with $\mu = 0$. The N-soliton solution (3.12) and the constraints (3.13) remain the same form with $\mu = 0$. Thus, the proof of the N-soliton solution may be performed under the setting $\mu = 0$ without loss of generality.

Remark 3.3. Unlike the bright N-soliton solution with zero boundary conditions [13], the real part of p_j is related to its imaginary part by (3.13). In the general n-component system, one needs to solve the algebraic equation of order n for $(\mathrm{Re}\, p_j)^2$. As well-known, analytical solutions are not available for $n \ge 5$. If the wavenumbers k_j $(j = 1, 2, ..., n)$ take the same value which implies that all the components of the system have the same asymptotic form except the amplitudes ρ_j and the phase constants $\phi_j^{(\pm)}$ (see (1.2)), then the constraints (3.13) reduce simply to a single quadratic equation for $\mathrm{Re}\, p_j$. This special case has been dealt with by means of the IST for the multi-component NLS system [15, 16].

Remark 3.4. The multi-component NLS system (i.e., $\gamma = 0$ in (1.1)) is invariant under the transformation $t = -\tilde{t}, x = \mathrm{i}\tilde{x}, \mu = -\tilde{\mu}$. Suppose that $\sigma_s = 1$ $(s = 1, 2, ..., n)$ and $\tilde{\mu} > 0$. We change the parameters in (3.12) according to the rule $k_s = -\mathrm{i}\tilde{k}_s, \omega_s = -\tilde{\omega}_s$ $(s = 1, 2, ..., n)$, $p_j = -\mathrm{i}\tilde{p}_j, \zeta_{j0} = \tilde{\zeta}_{j0} + \ln\sqrt{\gamma}$ $(j = 1, 2, ..., N)$ and then take the limit $\gamma \to 0$. The resulting expression gives rise to the N-soliton solution of the focusing NLS system with plane-wave boundary conditions. The solutions thus constructed exhibit a rich mathematical structure. Specifically, a reduction procedure applied to the soliton solutions would produce the breather and rogue wave solutions, as already demonstrated for the soliton solutions of the focusing NLS equation [18].

The proof of theorem 3.1 will be performed by using a sequence of lemmas, which we shall summarize. In accordance with remark 3.2, we put $\mu = 0$ in formulas that follow.

Lemma 3.1. *The expression of h_s from* (3.12*a*) *is rewritten in the form*

$$h_s = |D| - \frac{\mathrm{i}}{k_s}|D(\mathbf{z}_s^*; \mathbf{z}_x)|, \quad s = 1, 2, ..., n, \tag{3.14a}$$

where

$$\mathbf{z} = (z_j)_{1\le j\le N}, \quad \mathbf{z}_s = \left(\frac{k_s}{p_j - \mathrm{i}k_s}\, z_j\right)_{1\le j\le N}, \quad s = 1,2,...,n, \tag{3.14b}$$

are N-component row vectors.

The following lemma provides the differentiation rules of f and h_s with respect to t and x.

Lemma 3.2.

$$f_t = \mathrm{i}|D(\mathbf{z}^*;\mathbf{z}_t)| + |D(\mathbf{z}_x^*;\mathbf{z}_x)|, \tag{3.15}$$

$$f_x = \mathrm{i}|D(\mathbf{z}^*;\mathbf{z}_x)|, \tag{3.16}$$

$$f_{xx} = \mathrm{i}|D(\mathbf{z}_x^*;\mathbf{z}_x)| + \mathrm{i}|D(\mathbf{z}^*;\mathbf{z}_{xx})|, \tag{3.17}$$

$$h_{s,t} = \mathrm{i}|D(\mathbf{z}^*;\mathbf{z}_t)| + |D(\mathbf{z}_x^*;\mathbf{z}_x)| - \frac{\mathrm{i}}{k_s}|D(\mathbf{z}_{s,t}^*;\mathbf{z}_x)| - \frac{\mathrm{i}}{k_s}|D(\mathbf{z}_s^*;\mathbf{z}_{xt})| + \frac{1}{k_s}|D(\mathbf{z}_s^*,\mathbf{z}^*;\mathbf{z}_x,\mathbf{z}_t)|, \tag{3.18}$$

$$h_{s,x} = -|D(\mathbf{z}_s^*;\mathbf{z}_x)| - \frac{\mathrm{i}}{k_s}|D(\mathbf{z}_s^*;\mathbf{z}_{xx})|, \tag{3.19}$$

$$\begin{aligned} h_{s,xx} = \mathrm{i}|D(\mathbf{z}_x^*;\mathbf{z}_x)| + \mathrm{i}|D(\mathbf{z}^*;\mathbf{z}_{xx})| - \frac{\mathrm{i}}{k_s}|D(\mathbf{z}_{s,xx}^*;\mathbf{z})| - \frac{2\mathrm{i}}{k_s}|D(\mathbf{z}_{s,x}^*;\mathbf{z}_{xx})| \\ -\frac{\mathrm{i}}{k_s}|D(\mathbf{z}_s^*;\mathbf{z}_{xxx})| + \frac{1}{k_s}|D(\mathbf{z}_s^*,\mathbf{z}^*;\mathbf{z}_{xx},\mathbf{z}_x)|, \end{aligned} \tag{3.20}$$

Lemma 3.3. *The complex conjugate expressions of f, f_x and h_s are given as follows.*

$$f^* = |D| - \mathrm{i}|D(\mathbf{z}^*;\mathbf{z})|, \tag{3.21}$$

$$f_x^* = -\mathrm{i}|D(\mathbf{z}_x^*;\mathbf{z})|, \tag{3.22}$$

$$h_s^* = |D| - \mathrm{i}|D(\mathbf{z}^*;\mathbf{z})| + \frac{\mathrm{i}}{k_s}\,|D(\mathbf{z}_x^*;\mathbf{z}_s)| + \frac{1}{k_s}|D(\mathbf{z}_x^*,\mathbf{z}^*;\mathbf{z}_s,\mathbf{z})|. \tag{3.23}$$

When one tries to show that the N-soliton solution (3.12) solves the bilinear equations (3.5) and (3.6), the following lemma plays the central role together with Jacobi's identity (2.4).

Lemma 3.4.

$$\sum_{s=1}^{n} \frac{\sigma_s\rho_s^2}{k_s^2}\,|D(\mathbf{z}_s^*;\mathbf{z}_s)| = |D(\mathbf{z}^*B;\mathbf{z})|, \tag{3.24}$$

$$\sum_{s=1}^{n} \frac{\sigma_s\rho_s^2}{k_s^2}\,|D(\mathbf{z}_s^*,\mathbf{a}^*;\mathbf{b},\mathbf{z}_s)| = |D(\mathbf{z}^*B,\mathbf{a}^*;\mathbf{b},\mathbf{z})| - \mathrm{i}(|D(\mathbf{a}^*;\mathbf{b}B)| - |D(\mathbf{a}^*B;\mathbf{b})|). \tag{3.25}$$

Here, B is a diagonal matrix given by

$$B = \mathrm{diag}\,(\beta_1,\beta_2,...,\beta_N), \quad \beta_j = \sum_{s=1}^{n} \frac{\sigma_s\rho_s^2}{(p_j - \mathrm{i}k_s)(p_j^* + \mathrm{i}k_s)}, \quad j = 1,2,...,N, \tag{3.26}$$

$\mathbf{z}^*B = (\beta_j z_j^*)_{1\leq j\leq N}$ *is the* N*-component row vector, and* $\mathbf{a} = (a_j)_{1\leq j\leq N}$ *and* $\mathbf{b} = (b_j)_{1\leq j\leq N}$ *are arbitrary* N*-component row vectors.*

3.4. Proof of theorem 3.1

We can show that the tau functions (3.12) associated with the dark N-soliton solution solve the bilinear equations (3.4)-(3.6). The proof will be performed by employing lemmas 3.1-3.4 and some basic formulas for determinants. In particular, Jacobi's identity (2.4) plays the central role, as in the case of the proof of the bright N-soliton solution of the modified NLS system (1.1) [13].

Remark 3.5. We remark that the constraints (3.13) has not been used for the proof of (3.4). On the other hand, in establishing (3.5) and (3.6), one has relied on lemma (3.4) which is essentially based on the constraints.

3.5. One-soliton solutions

The soliton solutions are characterized completely by the tau functions f and h_j given by (3.12). Here, we describe the feature of one-soliton solutions. The general N-soliton solutions will be considered elsewhere. To simplify the notation, we first put $p_1 = a + \mathrm{i}b$, $\zeta_{10} = a(x_0 + \mathrm{i}y_0)$ $(a, b, x_0, y_0 \in \mathbb{R})$, and

$$\frac{b - \frac{\mu}{\gamma} - \mathrm{i}a}{2a} = \beta\,\mathrm{e}^{2\mathrm{i}\phi}\ (\beta > 0), \quad \frac{a + \mathrm{i}(b - k_j)}{a - \mathrm{i}(b - k_j)} = \mathrm{e}^{2\mathrm{i}\theta_j},\ j = 1, 2, ..., n. \tag{3.27}$$

Then, the tau functions for the one-soliton solutions are written compactly in the form

$$f = 1 + \mathrm{e}^{2a(\xi+\xi_0)+2\mathrm{i}\phi}, \quad h_j = 1 - \mathrm{e}^{2a(\xi+\xi_0)+2\mathrm{i}(\theta_j+\phi)},\ j = 1, 2, ..., n, \tag{3.28}$$

with $\xi = x - (2b + \lambda\gamma)t + x_0$, and $\xi_0 = (1/2a)\ln\beta$. Formula (3.12) with (3.28) gives one-soliton solutions. The square modulas of the complex variable q_j is computed as

$$|q_j|^2 = \rho_j^2 - \frac{2a^2\,\mathrm{sgn}\,a}{\sqrt{a^2 + (b - \mu/\gamma)^2}}\,\frac{\rho_j^2(2b - \mu/\gamma - k_j)}{a^2 + (b - k_j)^2}\,\frac{1}{\cosh 2a(\xi + \xi_0) + \frac{(b-\mu/\gamma)\mathrm{sgn}\,a}{\sqrt{a^2+(b-\mu/\gamma)^2}}}. \tag{3.29}$$

In the simplest one-soliton case, the constraints (3.13) reduce simply to a single relation which connects a with b

$$\frac{\gamma}{2}\sum_{s=1}^{n}\sigma_s\rho_s^2\,\frac{-2b + k_s + \frac{\mu}{\gamma}}{a^2 + (b - k_s)^2} = -1. \tag{3.30}$$

For the n-component system, one must solve the algebraic equation of order n for a^2 to express a in terms of b. Note, however, that analytical solutions are obtainable up to $n = 4$. This is the main difficulty in constructing soliton solutions. Although one can deal with the degenerate case for which all the wave numbers k_s have the same (possibly zero) value, the resulting one soliton solutions coincide essentially with those of the one-component system.

Last, we consider the soliton solutions of the 1-component system. Using (3.30) with $n = 1$, the expression (3.29) simplifies to

$$|q_1|^2 = \rho_1^2 - \frac{4a^2 \operatorname{sgn} a}{\sigma_1\gamma\sqrt{a^2 + (b-\mu/\gamma)^2}} \frac{1}{\cosh 2a(\xi+\xi_0) + \frac{(b-\mu/\gamma)\operatorname{sgn} a}{\sqrt{a^2+(b-\mu/\gamma)^2}}}. \tag{3.31}$$

When the condition $\operatorname{sgn}(\sigma_1\gamma a) > 0$ is satisfied, then this represents the dark soliton solution with a constant background. Taking the limit $\gamma \to 0$ under the conditions $\operatorname{sgn}(\sigma_1\gamma a) > 0, \operatorname{sgn}(\gamma\mu a) < 0$, the expression (3.31) reduces to the dark soliton solution of the NLS equation

$$|q_1|^2 = \rho_1^2 - \frac{2a^2}{|\mu|}\operatorname{sech}^2 a(\xi+\xi_0), \tag{3.32}$$

with a constraint $a^2 + (b-k_1)^2 = -(\mu/2)\sigma_1\rho_1^2$ imposed on the parameters a and b.

4. The modified NLS system with mixed type boundary conditions

In this section, we present the N-soliton solution of the system of equations (1.1) with the mixed zero and plane-wave boundary conditions (1.3). We will see that solutions take the form of bright-dark type solitons.

4.1. Gauge transformation

We first apply the gauge transformation

$$q_j = u_j \exp\left[-\frac{\mathrm{i}\gamma}{2}\int_{-\infty}^{x}\left\{\sum_{s=1}^{n}\sigma_s|u_s|^2 - \sum_{s=1}^{n-m}\sigma_{m+s}\rho_s^2\right\}dx\right], \quad j = 1, 2, ..., n, \tag{4.1}$$

to the system (1.1) and obtain the system of nonlinear PDEs for u_j:

$$\begin{aligned} &\mathrm{i}u_{j,t} + u_{j,xx} + \mathrm{i}\hat{\lambda}\gamma u_{j,x} + \mathrm{i}\gamma\left(\sum_{s=1}^{n}\sigma_s u_s^* u_{s,x} - \mathrm{i}\sum_{s=1}^{n-m}\sigma_{m+s}k_s\rho_s^2\right)u_j \\ &+\left[\left(\mu - \frac{\hat{\lambda}\gamma^2}{2}\right)\sum_{s=1}^{n}\sigma_s|u_s|^2 + \frac{(\hat{\lambda}\gamma)^2}{2}\right]u_j = 0, \quad j = 1, 2, ..., n, \end{aligned} \tag{4.2}$$

where $\hat{\lambda} = \sum_{s=1}^{n-m}\sigma_{m+s}\rho_s^2$.

4.2. Bilinearization

The bilinearization of the system of nonlinear PDEs (4.2) is accomplished by the following proposition.

Proposition 4.1. *By means of the dependent variable transformations*

$$u_j = \mathrm{e}^{\mathrm{i}\hat{\lambda}\mu t}\frac{g_j}{f}, \quad j = 1, 2, ..., m, \tag{4.3a}$$

$$u_{m+j}=\rho_j \mathrm{e}^{\mathrm{i}(k_jx-\omega_jt)}\,\frac{h_j}{f},\quad j=1,2,...,n-m, \tag{4.3b}$$

the system of nonlinear PDEs (4.2) can be decoupled into the following system of bilinear equations for f, g_j and h_j

$$\mathrm{i}D_tg_j\cdot f+\mathrm{i}\hat{\lambda}\gamma D_xg_j\cdot f+D_x^2g_j\cdot f=0,\quad j=1,2,...,m, \tag{4.4}$$

$$\mathrm{i}D_th_j\cdot f+\mathrm{i}(2k_j+\hat{\lambda}\gamma)D_xh_j\cdot f+D_x^2h_j\cdot f=0,\quad j=1,2,...,n-m, \tag{4.5}$$

$$D_xf\cdot f^*-\frac{\mathrm{i}\gamma}{2}\left\{\sum_{s=1}^{m}\sigma_sg_sg_s^*+\sum_{s=1}^{n-m}\sigma_{m+s}\rho_s^2(h_sh_s^*-ff^*)\right\}=0, \tag{4.6}$$

$$D_x^2f\cdot f^*-\frac{\mathrm{i}\gamma}{2}\left(\sum_{s=1}^{m}\sigma_sD_xg_s\cdot g_s^*+\sum_{s=1}^{n-m}\sigma_{m+s}\rho_s^2D_xh_s\cdot h_s^*\right)+\gamma\sum_{s=1}^{n-m}\sigma_{m+s}k_s\rho_s^2(h_sh_s^*-ff^*)$$
$$+\left(\frac{\hat{\lambda}\gamma^2}{4}-\mu\right)\left\{\sum_{s=1}^{m}\sigma_sg_sg_s^*+\sum_{s=1}^{n-m}\sigma_{m+s}\rho_s^2(h_sh_s^*-ff^*)\right\}=0. \tag{4.7}$$

Here, $f=f(x,t), g_j(x,t)$ and $h_j=h_j(x,t)$ are complex-valued functions of x and t.

Using (4.1) and (4.6), we can express q_j in terms of the tau functions f, g_j and h_j as

$$q_j=\mathrm{e}^{\mathrm{i}\hat{\lambda}\mu t}\,\frac{g_jf^*}{f^2},\quad j=1,2,...,m, \tag{4.8a}$$

$$q_{m+j}=\rho_j\mathrm{e}^{\mathrm{i}(k_jx-\omega_jt)}\,\frac{h_jf^*}{f^2},\quad j=1,2,...,n-m. \tag{4.8b}$$

Remark 4.1. In the case of $\gamma=0$, the expressions (4.8) become

$$q_j=\mathrm{e}^{\mathrm{i}\hat{\lambda}\mu t}\,\frac{g_j}{f},\quad j=1,2,...,m, \tag{4.9a}$$

$$q_{m+j}=\rho_j\mathrm{e}^{\mathrm{i}(k_jx-\omega_jt)}\,\frac{h_j}{f},\quad j=1,2,...,n-m, \tag{4.9b}$$

and they satisfy the n-component NLS system (3.11).

4.3. The bright-dark N-soliton solution

The main result in this section is provided by the following theorem.

Theorem 4.1. *The N-soliton solution of the system of bilinear equations (4.4)-(4.7) is given in terms of the determinants*

$$f=|D|,\quad g_s=-|D(\mathbf{a}_s^*;\mathbf{z})|,\ s=1,2,...,m,\quad h_s=|D|+\frac{1}{k_s}|D(\mathbf{z}_s^*;\mathbf{z})|,\ s=1,2,...,n-m, \tag{4.10a}$$

$$D = (d_{jk})_{1\leq j,k\leq N}, \quad d_{jk} = \frac{z_j z_k^* + \frac{1}{2}(\mu - \mathrm{i}\gamma p_k^*)c_{jk}}{p_j + p_k^*}, \tag{4.10b}$$

$$z_j = \exp[p_j x + (\mathrm{i}p_j^2 - \hat{\lambda}\gamma p_j)t], \quad j = 1, 2, ..., N, \tag{4.10c}$$

$$c_{jk} = \frac{\sum_{s=1}^{m} \sigma_s \alpha_{sj}\alpha_{sk}^*}{1 + \frac{\gamma}{2}\sum_{s=1}^{n-m}\sigma_{m+s}\rho_s^2 \frac{\mathrm{i}(p_j - p_k^*) + k_s + \frac{\mu}{\gamma}}{(p_j - \mathrm{i}k_s)(p_k^* + \mathrm{i}k_s)}}, \quad j, k = 1, 2, ..., N, \tag{4.10d}$$

where $\mathbf{z}$ *and* $\mathbf{z}_s$ *are* N*-component row vectors defined by* (3.14b) *with* z_j *given by (4.10c) and*

$$\mathbf{a}_s = (\alpha_{s1}, \alpha_{s2}, ..., \alpha_{sN}), \quad s = 1, 2, ..., m, \tag{4.10e}$$

are row vectors with elements $\alpha_{sj} \in \mathbb{C}$ $(s = 1, 2, .., m; j = 1, 2, .., N)$.

The N-soliton solution is characterized by the N complex parameters p_j $(j = 1, 2, ..., N)$ and mN complex parameters α_{sj} $(s = 1, 2, ..., m; j = 1, 2, ..., N)$. The former parameters determine the amplitude of solitons and the latter ones determine the polarization and the envelope phases of solitons. Note that we have used the same symbol as that appears in theorem 3.1 for the tau functions f and h_s since no confusion would be likely to arise from this convention.

First, we summarize the differentiation rules corresponding to those given by lemma 3.2 and lemma 3.3. Remark 3.2 is applied to the current problem, so that we can put $\mu = 0$ without loss of generality.

Lemma 4.1.

$$f_t = -\mathrm{i}|D(\mathbf{z}^*; \mathbf{z}_x)| + \mathrm{i}|D(\mathbf{z}_x^*; \mathbf{z})| + \hat{\lambda}\gamma|D(\mathbf{z}^*; \mathbf{z})|, \tag{4.11}$$

$$f_x = -|D(\mathbf{z}^*; \mathbf{z})|, \tag{4.12}$$

$$f_{xx} = -|D(\mathbf{z}_x^*; \mathbf{z})| - |D(\mathbf{z}^*; \mathbf{z}_x)|, \tag{4.13}$$

$$g_{s,t} = -|D(\mathbf{a}_s^*; \mathbf{z}_t)| + \mathrm{i}|D(\mathbf{a}_s^*, \mathbf{z}^*; \mathbf{z}, \mathbf{z}_x)|, \tag{4.14}$$

$$g_{s,x} = -|D(\mathbf{a}_s^*; \mathbf{z}_x)|, \tag{4.15}$$

$$g_{s,xx} = -|D(\mathbf{a}_s^*; \mathbf{z}_{xx})| + |D(\mathbf{a}_s^*, \mathbf{z}^*; \mathbf{z}_x, \mathbf{z})|, \tag{4.16}$$

$$h_{s,t} = -\mathrm{i}|D(\mathbf{z}^*; \mathbf{z}_x)| + \mathrm{i}|D(\mathbf{z}_x^*; \mathbf{z})| + \hat{\lambda}\gamma|D(\mathbf{z}^*; \mathbf{z})| + \frac{1}{k_s}|D(\mathbf{z}_{s,t}^*; \mathbf{z})| + \frac{1}{k_s}|D(\mathbf{z}_s^*; \mathbf{z}_t)| - \frac{\mathrm{i}}{k_s}|D(\mathbf{z}_s^*, \mathbf{z}^*; \mathbf{z}, \mathbf{z}_x)|, \tag{4.17}$$

$$h_{s,x} = -\mathrm{i}|D(\mathbf{z}_s^*; \mathbf{z})| + \frac{1}{k_s}|D(\mathbf{z}_s^*; \mathbf{z}_x)|, \tag{4.18}$$

$$h_{s,xx} = -\mathrm{i}|D(\mathbf{z}_{s,x}^*; \mathbf{z})| - \mathrm{i}|D(\mathbf{z}_s^*; \mathbf{z}_x)| + \frac{1}{k_s}|D(\mathbf{z}_{s,x}^*; \mathbf{z}_x)| + \frac{1}{k_s}|D(\mathbf{z}_s^*; \mathbf{z}_{xx})| - \frac{1}{k_s}|D(\mathbf{z}_s^*, \mathbf{z}^*; \mathbf{z}_x, \mathbf{z})|. \tag{4.19}$$

The above formulas can be derived by using (2.2) and some basic properties of determinants.

The lemma 4.2 below provides the complex conjugate expressions of f, f_x, g_s and h_s.

Lemma 4.2.

$$f^* = |\bar{D}| = |(\bar{d}_{jk})_{1\le j,k\le N}|, \quad \bar{d}_{jk} = d_{jk} + \frac{\mathrm{i}\gamma}{2}\, c_{jk}, \tag{4.20}$$

$$f_x^* = -|\bar{D}(\mathbf{z}^*;\mathbf{z})|, \tag{4.21}$$

$$g_s^* = -|\bar{D}(\mathbf{z}^*;\mathbf{a}_s)|, \tag{4.22}$$

$$h_s^* = |\bar{D}| + \frac{1}{k_s}|\bar{D}(\mathbf{z}^*;\mathbf{z}_s)|. \tag{4.23}$$

The following two lemmas will be used effectively in the proof of (4.6) and (4.7).

Lemma 4.3.

$$\sum_{s=1}^{n-m}\frac{\sigma_{m+s}\rho_s^2}{k_s^2}\,|D(\mathbf{z}_s^*;\mathbf{z})||\bar{D}(\mathbf{z}^*;\mathbf{z}_s)| = -|D(\tilde{\mathbf{z}}^*;\mathbf{z})||\bar{D}| - |\bar{D}(\mathbf{z}^*;\tilde{\mathbf{z}})||D|$$
$$+\frac{2\mathrm{i}}{\gamma}(|D(\mathbf{z}^*;\mathbf{z})||\bar{D}| - |\bar{D}(\mathbf{z}^*;\mathbf{z})||D|) - \sum_{s=1}^{m}\sigma_s|D(\mathbf{a}_s^*;\mathbf{z})||\bar{D}(\mathbf{z}^*;\mathbf{a}_s)|, \tag{4.24}$$

$$\sum_{s=1}^{n-m}\frac{\sigma_{m+s}\rho_s^2}{k_s^2}\,|D(\mathbf{z}_s^*;\mathbf{z}_x)||\bar{D}(\mathbf{z}^*;\mathbf{z}_s)| = -|D(\tilde{\mathbf{z}}^*;\mathbf{z}_x)||\bar{D}| - |\bar{D}(\mathbf{z}^*;\tilde{\mathbf{z}}_x)||D|$$
$$+\frac{2\mathrm{i}}{\gamma}(|D(\mathbf{z}^*;\mathbf{z}_x)||\bar{D}| - |\bar{D}(\mathbf{z}^*;\mathbf{z}_x)||D|) - \sum_{s=1}^{m}\sigma_s|D(\mathbf{a}_s^*;\mathbf{z}_x)||\bar{D}(\mathbf{z}^*;\mathbf{a}_s)|. \tag{4.25}$$

Here, $\tilde{\mathbf{z}} = (\tilde{z}_j)_{1\le j\le N}$ is an N-component row vector with elements $\tilde{z}_j = \sum_{s=1}^{n-m}\sigma_{m+s}\rho_s^2\frac{z_j}{p_j - \mathrm{i}k_s}$.

Lemma 4.4.

$$|D(\mathbf{z}_x^*;\mathbf{z})||\bar{D}| + |D(\mathbf{z}^*;\mathbf{z})||\bar{D}(\mathbf{z}^*;\mathbf{z})| + |\bar{D}(\mathbf{z}^*;\mathbf{z}_x)||D| = 0. \tag{4.26}$$

4.4. Proof of theorem 4.1

The proof of theorem 4.1 can be performed by using the differentiation rules for the tau functions f, g_s and h_s given by lemmas 4.1 and 4.2 as well as lemmas 4.3 and 4.4.

4.5. One-soliton solutions

If we put

$$p_1 = a + \mathrm{i}b, \quad z = \mathrm{e}^{a\xi + \mathrm{i}\{bx + (a^2 - b^2 - \hat{\lambda}\gamma b)t\}}, \quad \xi = x - (2b + \hat{\lambda}\gamma)t, \tag{4.27a}$$

the tau functions (4.10) for the one-soliton solutions are written in the form

$$f = \frac{1}{2a}\left\{zz^* + \frac{1}{2}(\mu - \gamma b - \mathrm{i}\gamma a)c_{11}\right\}, \quad g_j = \alpha_{j1}^* z, \quad j = 1, 2, ..., m,$$

$$h_j = f - \frac{zz^*}{a - \mathrm{i}(b - k_j)}, \quad j = 1, 2, ..., n - m. \tag{4.27b}$$

The parameter c_{11} from (4.10d) is given by

$$c_{11} = \frac{\sum_{s=1}^{m} \sigma_s \alpha_{s1} \alpha_{s1}^*}{1 + \frac{\gamma}{2} \sum_{s=1}^{n-m} \sigma_{m+s} \rho_s^2 \frac{-2b + k_s + \mu/\gamma}{a^2 + (b - k_s)^2}}. \tag{4.27c}$$

Note that c_{11} is real. This also follows directly from the Hermitian nature of c_{jk}. If one introduces the real quantities β, ϕ and θ_j by (3.27) and puts $(a\beta\gamma)^2 = \mathrm{e}^{-4a\xi_0}$, then one can express the one-soliton solutions compactly in terms of these parameters. After some manipulations, one finds that

$$|q_j|^2 = \frac{2a^2 \alpha_{j1} \alpha_{j1}^* \mathrm{e}^{2a\xi_0}}{\cosh\, 2a(\xi + \xi_0) - \mathrm{sgn}(a\gamma c_{11})\, \cos\, 2\phi}, \quad j = 1, 2, ..., m, \tag{4.28}$$

$$|q_{j+m}|^2 = \rho_j^2 \left[1 + \frac{\mathrm{sgn}(a\gamma c_{11})\{\cos\, 2(\theta_j + \phi) + \cos\, 2\phi\}}{\cosh\, 2a(\xi + \xi_0) - \mathrm{sgn}(a\gamma c_{11})\, \cos\, 2\phi}\right], \quad j = 1, 2, ..., n - m. \tag{4.29}$$

The components q_j from (4.28) take the form of bright-solitons with zero background whereas those of (4.29) represent the dark- or bright-solitons with nonzero background. A striking feature of the soliton solutions is that the parameters a and b can be chosen independently unlike the soliton solutions with the pure plane-wave boundary conditions discussed in section 3. It turns out that in the case of the mixed type boundary conditions, the explicit form of the N-soliton solution is available without solving algebraic equations.

If one takes the limit $\gamma \to 0$ under the condition $\mu c_{11} > 0$, then the expressions (4.28) and (4.29) reduce to the one-soliton solutions of the n-component NLS system

$$|q_j|^2 = a^2 \alpha_{j1} \alpha_{j1}^* \mathrm{e}^{2a\xi_0}\, \mathrm{sech}^2 a(\xi + \xi_0), \quad j = 1, 2, ..., m, \tag{4.30}$$

$$|q_{j+m}|^2 = \rho_j^2 \left[1 - \frac{a^2}{a^2 + (b - k_j)^2}\, \mathrm{sech}^2 a(\xi + \xi_0)\right], \quad j = 1, 2, ..., n - m. \tag{4.31}$$

Note that the condition $\mu c_{11} > 0$ assures the regularity of the solutions. Actually, if $\mu c_{11} < 0$, then the solutions exhibit a singularity at $\xi = -\xi_0$.

Remark 4.2. The N-soliton formula presented in theorem 4.1 have an alternative expressions. Indeed, in accordance with the procedure developed in [13], we can show that the tau functions given below satisfy the system of bilinear equations (4.4)-(4.7)

$$f = \begin{vmatrix} \hat{A} & I \\ -I & \hat{B} \end{vmatrix}, \quad g_s = \begin{vmatrix} \hat{A} & I & \mathbf{z}^T \\ -I & \hat{B} & \mathbf{0}^T \\ \mathbf{0} & -\mathbf{a}_s^* & 0 \end{vmatrix}, \ s = 1, 2, ..., m,$$

$$h_s = \begin{vmatrix} \hat{A} & I & \mathbf{z}^T \\ -I & \hat{B} & \mathbf{0}^T \\ \mathbf{z}_s^*/k_s & \mathbf{0} & 1 \end{vmatrix}, \ s = 1, 2, ..., n - m, \tag{4.32a}$$

$$\hat{A} = (\hat{a}_{jk})_{1 \le j,k \le N}, \quad \hat{a}_{jk} = \frac{z_j z_k^*}{p_j + p_k^*}, \quad \hat{B} = (\hat{b}_{jk})_{1 \le j,k \le N}, \quad \hat{b}_{jk} = \frac{\frac{1}{2}(\mu + \mathrm{i}\gamma p_k) c_{jk}}{p_j + p_k^*}, \tag{4.32b}$$

where the N-component row vectors $\mathbf{z}, \mathbf{z}_s$ and $\mathbf{a}_s$ are defined in (4.10) and I is an $N \times N$ unit matrix. It is noteworthy that the tau functions f and g_s have the same forms as those of the bright N-soliton solution presented in [13]. Indeed, if one puts $\rho_s = 0$ $(s = 1, 2, ..., n - m)$, or equivalently $q_{m+s} = 0$ $(s = 1, 2, ..., n - m)$, then $q_s = g_s f^*/f^2$ $(s = 1, 2, ..., m)$ solve the system of PDEs (1.1) with the boundary conditions $q_s \to 0, |x| \to \infty$. Since m is an arbitrary positive integer, this gives another proof of the bright N-soliton solution.

5. Concluding remark

In conclusion, it will be worthwhile to comment on an integrable system associated with the system (1.1). The multi-component Fokas-Lenells (FL) system

$$u_{j,xt} = u_j - \mathrm{i}\left\{\left(\sum_{s=1}^{n}\sigma_s u_{s,x}u_s^*\right)u_j + \left(\sum_{s=1}^{n}\sigma_s u_s u_s^*\right)u_{j,x}\right\}, \quad u_j = u_j(x,t) \in \mathbb{C}, \quad j = 1, 2, ..., n, \tag{5.1}$$

is an integrable multi-component generalization of the FL equation which describes the nonlinear propagation of short pulses in a monomode fiber [19, 20]. It belongs to the first negative flow of the multi-component derivative NLS hierarchy [21, 22]. The FL equation is a special case of the system (5.1) with $n = 1$. Its N-soliton solutions have been obtained by employing the direct method for both zero and plane-wave boundary conditions [23, 24]. In view of the above observation on an integrable hierarchy, the structure of the N-soliton solution of the multi-component FL system is closely related to that of the multi-component derivative NLS system. This statement has been confirmed for the one-component system [24]. Thus, the bilinearization and the construction of the N-soliton solution of the system (5.1) with zero and nonzero boundary conditions will be performed in accordance with the procedure developed in the present paper. This interesting issue is currently under study.

References

[1] Dodd R K, Eilbeck J C, Gibbon J D and Morris H C 1982 *Solitons and Nonlinear Wave Equations* (London: Academic)

[2] Novikov S P, Manakov S V, Pitaevskii L P and Zakharov V E 1984 *Theory of Solitons: the Inverse Scattering Method* (New York: Plenum)

[3] Ablowitz M J and Clarkson P A 1992 *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge: Cambridge University Press)

[4] Lamb G L *Elements of Soliton Theory* 1980 (New York: John Wiley)

[5] Rogers C and Shadwick W F 1982 *Bäcklund Transformations and Their Applications* (New York: Academic)

[6] Matveev V B and Salle M A 1991 *Darboux Transformations and Solitons* (Berlin: Springer)

[7] Hirota R 1980 Direct methods in soliton theory *Solitons (Topics in Current Physics)* ed R K Bullough and P J Caudrey (Berlin: Springer) pp 157-75

[8] Matsuno Y 1984 Bilinear Transformation Method (New York: Academic)

[9] Hirota R 2004 The direct Method in Soliton Theory (Cambridge: Cambridge University Press)

[10] Hasegawa A and Kodama Y 1995 *Solitons in Optical Communications* (New York: Oxford)

[11] Kivshar Y S and Agrawal G P 2003 *Optical Solitons From Fibers to Photonic Crystals* (New York: Academic)

[12] Maimistov A I 2010 Solitons in nonlinear optics *Quantum Electron.* **40** 756-81

[13] Matsuno Y 2011 The bright N-soliton solution of a multi-component modified nonlinear Dchrödinger equation *J. Phys. A: Math. Theor.* **44** 495202

[14] Hisakado M and Wadati M 1995 Integrable multi-component hybrid nonlinear Schrödinger equations *J. Phys. Soc. Japan* **64** 408-13

[15] Ieda J, Uchiyama M and Wadati M 2007 Inverse scattering method for square matrix nonlinear Schrödinger equation under nonvanishing boundary conditions *J. Math. Phys.* **48** 013507

[16] Prinari B, Biondini G and Trubatch A D 2011 Inverse scattering transform for the multi-component nonlinear Schrödinger equation with nonzero boundary conditions *Stud. Appl. Math.* **126** 245-302

[17] Vein R and Dale P 1999 *Determinants and Their Applications in Mathematical Physics* (New York: Springer)

[18] Tajiri M and Watanabe Y 1998 Breather solutions to the focusing nonlinear Schrödinger equation *Phys. Rev. E* **57** 3510-19

[19] Fokas A S 1995 On a class of physically important integrable equations *Physica* D **87** 145-50

[20] Lenells J 2009 Exactly solvable model for nonlinear pulse propagation in optical fibers *Stud. Appl. Math.* **123** 215-32

[21] Fordy A P 1984 Derivative nonlinear Schrödinger equations and Hermitian symmetric spaces *J. Phys. A: Math. Gen.* **17** 1235-45

[22] Tsuchida T and Wadati M 1999 New integrable systems of derivative nonlinear Schrödinger equations with multiple components *Phys. Lett.* A **257** 53-6

[23] Matsuno Y 2011 A direct method of solution for the Fokas-Lenells derivative nonlinear Schrödinger equation: II. Bright soliton solutions *J. Phys. A: Math. Theor.* **44** 235202

[24] Matsuno Y 2012 A direct method of solution for the Fokas-Lenells derivative nonlinear Schrödinger equation: II. Dark soliton solutions *J. Phys. A: Math. Theor.* **45** 475202