The multi-component modified nonlinear Schrödinger system with nonzero boundary conditions

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Abstract

In a previous paper (Matsuno 2011 *J. Phys. A: Math. Theore.* **44** 495202), we have presented a determinantal expression of the bright N-soliton solution for a multi-component modified nonlinear Schrödinger (NLS) system with zero boundary conditions. The present paper provides the dark N-soliton solution for the same system of equations with plane-wave boundary conditions, as well as the bright-dark N-soliton solution with mixed zero and plane-wave boundary conditions. The proof of the N-soliton solutions is performed by means of an elementary theory of determinants in which Jacobi's identity plays the central role. The N-soliton formulas obtained here include as special cases the existing soliton solutions of the NLS and derivative NLS equations and their integrable multi-component analogs. The new features of soliton solutions are discussed. In particular, it is shown that the N constraints must be imposed on the amplitude parameters of solitons in constructing the dark N-soliton solution with plane-wave boundary conditions. This makes it difficult to analyze the solutions as the number of components increases. For mixed type boundary conditions, the structure of soliton solutions is found to be more explicit than that of dark soliton solutions since no constraints are imposed on the soliton parameters. Last, we comment on an integrable multi-component system associated with the first negative flow of the multi-component derivative NLS hierarchy.

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1. Introduction

The nonlinear Schrödinger (NLS) equation is a milestone in the theory of nonlinear waves. One remarkable feature is that it is a completely integrable equation, allowing us to solve it by means of several exact methods of solution such as the inverse scattering transform (IST) [1-3], Bäcklund-Darboux (BD) transformation [4-6] and Hirota's direct method [7-9]. In recent years, much attention has been paid to the multi-component generalizations of the NLS equation because of their wide applicability in many physical contexts such as nonlinear optics, nonlinear water waves and plasma physics and so on [10-12].

In this paper, we consider the following multi-component system of nonlinear partial differential equations (PDEs) which is a hybrid of the multi-component NLS system and multi-component derivative NLS system

$$\mathrm{i}\, q_{j,t} + q_{j,xx} + \mu \left(\sum_{s=1}^{n} \sigma_s |q_s|^2 \right) q_j + \mathrm{i}\gamma \left[\left(\sum_{s=1}^{n} \sigma_s |q_s|^2 \right) q_j \right]_x = 0, \quad j = 1, 2, ..., n, \tag{1.1}$$

where $q_j = q_j(x,t)$ $(j = 1, 2, ..., n)$ are complex-valued functions of x and t, μ and γ are real constants, n is an arbitrary positive integer, and subscripts x and t appended to q_j denote partial differentiations. The coefficients $\sigma_s = \pm 1$ $(s = 1, 2, ..., n)$ specify the sign of the nonlinearity. For the multi-component NLS system for which $\gamma = 0$, we can deal with the three types of cubic nonlinearities, i.e., focusing $(\sigma_s = 1,\ s = 1, 2, ..., n)$, defocusing $(\sigma_s = -1,\ s = 1, 2, ..., n)$ and mixed focusing-defocusing $(\sigma_s = 1, s = 1, 2, ..., m; \sigma_s = -1, s = m+1, m+2, ..., n)$ nonlinearities with $\mu > 0$, where m is an arbitrary positive integer such that $1 \le m < n$. The special systems reduced from (1.1) have been summarized in a previous paper [13]. The system (1.1) has been shown to be completely integrable [14] and hence the exact methods mentioned above can be applied to it to obtain various types of soliton solutions. The properties of solutions, however, depend essentially on the boundary conditions. We emphasize that most works have been concerned with the analysis of the system under vanishing boundary conditions at spatial infinity.

The main purpose of the present paper is to construct N-soliton solutions (N: arbitrary positive integer) of the system (1.1) with the following two types of boundary conditions:

1) Plane-wave boundary conditions.

$$q_j \sim \rho_j \exp \mathrm{i} \left(k_j x - \omega_j t + \phi_j^{(\pm)} \right), \quad x \to \pm\infty, \qquad j = 1, 2, ..., n, \tag{1.2a}$$

$$\omega_j = k_j^2 - \mu \sum_{s=1}^{n} \sigma_s \rho_s^2 + \gamma \left(\sum_{s=1}^{n} \sigma_s \rho_s^2 \right) k_j, \quad j = 1, 2, ..., n. \tag{1.2b}$$

2) Mixed type boundary conditions.

$$q_j \sim 0, \quad x \to \pm\infty, \quad j = 1, 2, ..., m, \tag{1.3a}$$

$$q_{m+j} \sim \rho_j \exp \mathrm{i}\left(k_j x - \omega_j t + \phi_j^{(\pm)}\right), \quad x \to \pm\infty, \quad j = 1, 2, ..., n-m, \tag{1.3b}$$

$$\omega_j = k_j^2 - \mu \sum_{s=1}^{n-m} \sigma_s \rho_s^2 + \gamma \left(\sum_{s=1}^{n-m} \sigma_s \rho_s^2\right) k_j, \quad j = 1, 2, ..., n-m. \tag{1.3c}$$

Here, $\rho_j,\ k_j$ and $\phi_j^{(\pm)}$ are arbitrary real constants and ω_j from (1.2b) and (1.3c) represent the linear dispersion relations for components with the plane-wave boundary conditions. For case 1), all the components approach the specified plane waves at infinity whereas for case 2), the first m components vanish at infinity and the remaining $n-m$ components have the specified plane waves. It is important that for the nonzero background fields like (1.2a) and (1.3b), each component has different wavenumber and frequency. This causes difficulties in constructing soliton solutions. In particular, in the case of 1) above, one must impose certain constraints between the real and imaginary parts of the complex parameters characterizing the amplitude (or the velocity) of solitons. This peculiar feature has never been encountered in the case of zero boundary conditions [13].

The several exact methods are now available for the nonlinear PDEs. Among them, the IST provides a general scheme for solving the initial value problems of a wide class of completely integrable nonlinear PDEs [1-3]. For instance, the defocusing NLS equation has been solved by the IST [15, 16] while the analysis of the focusing NLS equation with nonzero boundary conditions was made quite recently in [17, 18] and used to explore the nonlinear stage of modulational instability. The IST has been developed as well for the derivative NLS equation subjected to nonzero boundary conditions [19]. The situation changes drastically for the multi-component systems with nonzero boundary conditions. At present, the applicability of the IST is restricted to the specific boundary conditions in which all the components have the same wavenumber and frequency at infinity. See [20, 21], for instance. We recall that there exists a unified approach for constructing multisoliton solutions of integrable PDEs which is based on a method of algebraic geometry [22].

The BD transformation is a useful tool to construct exact solutions of completely integrable PDEs [4-6]. In this framework, one needs to solve the linear eigenvalue problem associated with the Lax pair with a given seed solution of the nonlinear PDE. Once this procedure has been completed, the construction of solutions can be performed by purely algebraic mean. Although the successive application of the BD transformation to seed solutions allows us to obtain multisoliton solutions, the algebra involved becomes formidable as the number of components increases. A systematic method has been developed for constructing soliton solutions of the n-component NLS system with plane-wave boundary conditions [23-26]. See also [27] and references theirin. A difficulty of the method is that one must solve the characteristic equation of the Lax pair with a given seed solution. Consequently, the problem reduces to find eigenvalues of the $(n+1)\times(n+1)$ constant matrices whose entries are complex numbers. It turns out that even in the simplest 2-component system, the computation of eigenvalues is really troublesome, making the derivation of one-solutions difficult to address [23].

The third approach which will be employed in our analysis is the direct method [7-9]. While both the IST and BD transformation are based on the Lax representations of completely integrable PDEs, the direct method does not need the knowledge of the IST, and it can also be applicable to nonintegrable equations. It provides a powerful tool to obtain special solutions such as soliton and periodic solutions, even though it is not amenable to the analysis of the general initial value problems. The starting point of the method is to find appropriate dependent variable transformations to bilinearize a given system of PDEs. The most of the existing integrable PDEs can be bilinearized and the multisoliton solutions of the corresponding bilinear equations are constructed in systematic ways. Now, the application of the direct method to the system (1.1) has been done for various settings. The most results are concerned with the multi-component NLS system. A number of works have been devoted to studying the physically important two-component system which is sometimes called the Manakov system [28]. An article [29] reviews various types of solutions for the system with focusing, defocusing and mixed type nonlinearities. In particular, it addresses the properties of the one- and two soliton solutions in detail. Quite recently, the n-component NLS system subjected to the boundary conditions (1.2) and (1.3) was analyzed based on the reduction method of the multi-component KP hierarchy, and the determinantal expressions of the N-soliton solutions were presented [30]. See also [31] for the 2-component NLS system with the boundary conditions (1.2) where the dark-dark solitons and their dynamics are explored in quite some detail. In the most general case with $\mu \neq 0$ and $\gamma \neq 0$, we have constructed the bright N-soliton solution by imposing zero boundary conditions at infinity i.e., $q_j \to 0,\ |x| \to \infty\ (j = 1, 2, ..., n)$. Specifically, we have presented the two different expressions of the N-soliton solution both of which have a simple structure in terms of determinants. See theorem 2.1 and theorem 5.1 in [13]. The specific two-component system has been considered in [32, 33]. To the best of our knowledge, the N-soliton formulas for the general n-component system (1.1) with the boundary conditions 1) and 2) are presented here for the first time. They unify most of the existing soliton solutions of physically important equations of the NLS type.

The outline of this paper is the following. In section 2, we introduce a few notations and some basic formulas for determinants. In section 3, we bilinearize the system (1.1) with the boundary conditions (1.2) through appropriate dependent variable transformations. Subsequently, the construction of the N-soliton solution is done following the standard recipe of the direct method. It is shown that the tau functions associated with the N-soliton solution have a simple structure expressed in terms of determinants, and the proof of the N-soliton solution is performed by means of an elementary theory of determinants. Unlike the bright soliton solutions with zero boundary conditions presented in [13, 33], certain constraints must be imposed on soliton parameters, or in the terminology of the IST, the eigenvalues of the Lax operator. This leads to a difficulty in investigating soliton solutions of the n-component system for large n. Actually, one must solve the algebraic equation of order n with which the real and imaginary parts of the amplitude parameter of each soliton are connected. We exemplify this statement for the one-soliton solutions. It is important that the algebraic equations arising from constraints have real coefficients, and hence they are tractable when compapred with the characteristic equations with complex coefficients in the BD formulation. In section 4, the similar bilinearization is performed for the system (1.1) under the boundary conditions (1.3) and the explicit N-soliton solution is presented. The structure of the N-soliton solution is found to be quite different from that given in section 3 for the pure plane-wave background fields. To be more specific, no constraints are imposed on the soliton parameters. This important feature facilitates the analysis of soliton solutions. As an example, the one-soliton solutions are given explicitly and their limiting profiles as γ tends to zero are discussed. Section 5 is devoted to concluding remarks. In appendices A-C, the three lemmas are proved which play the important role in establishing the N-soliton formulas.

2. Notation and basic formulas for determinants

First of all, we introduce for convenience the notation and some basic formulas for determinants associated with the N-soliton solutions.

2.1. Notation

Let $D = (d_{jk})_{1\leq j,k\leq N}$ be an $N \times N$ matrix and $\mathbf{a} = (a_j)_{1\leq j\leq N}, \mathbf{b} = (b_j)_{1\leq j\leq N}, \mathbf{c} = (c_j)_{1\leq j\leq N}, \mathbf{d} = (d_j)_{1\leq j\leq N}$ be the N-component row vectors, where all the entries in the matrices and vectors are complex-valued functions of x and t. The bordered marices associated with the matrix D are defined by

$$D(\mathbf{a};\mathbf{b}) = \begin{pmatrix} D & \mathbf{b}^T \\ \mathbf{a} & 0 \end{pmatrix}, \quad D(\mathbf{a},\mathbf{b};\mathbf{c},\mathbf{d}) = \begin{pmatrix} D & \mathbf{c}^T & \mathbf{d}^T \\ \mathbf{a} & 0 & 0 \\ \mathbf{b} & 0 & 0 \end{pmatrix}, \tag{2.1}$$

where the symbol T denotes transpose. The first cofactor of d_{ij} is defined by $D_{ij} = \partial|D|/\partial d_{ij}$, ($|D| = \det D$) and the second cofactor by $D_{ij,pq} = \partial^2|D|/\partial d_{ip}\partial d_{jq}$ ($i < j, p < q$).

2.2. Basic formulas for determinants

The following formulas are well-known in the theory of determinants and will be used frequently in our analysis [34]:

$$\frac{\partial |D|}{\partial x} = \sum_{i,j=1}^{N} \frac{\partial d_{ij}}{\partial x} D_{ij}, \tag{2.2}$$

$$\begin{vmatrix} D & \mathbf{a}^T \\ \mathbf{b} & z \end{vmatrix} = |D|z - \sum_{i,j=1}^{N} D_{ij} a_i b_j, \tag{2.3}$$

$$|D(\mathbf{a}, \mathbf{b}; \mathbf{c}, \mathbf{d})||D| = |D(\mathbf{a}; \mathbf{c})||D(\mathbf{b}; \mathbf{d})| - |D(\mathbf{a}; \mathbf{d})||D(\mathbf{b}; \mathbf{c})|, \tag{2.4}$$

$$\delta_{ij}|D| = \sum_{k=1}^{N} d_{ik} D_{jk} = \sum_{k=1}^{N} d_{ki} D_{kj}, \tag{2.5}$$

$$D_{ij} = \sum_{p=1}^{N} d_{pq}\, D_{ip,jq} \quad (j \neq q) \tag{2.6a}$$

$$= \sum_{q=1}^{N} d_{pq}\, D_{ip,jq} \quad (i \neq p), \tag{2.6b}$$

$$|D(\mathbf{a}, \mathbf{b}; \mathbf{c}, \mathbf{d})| = \sum_{\substack{i,j,p,q=1 \\ (i \neq j, p \neq q)}}^{N} a_p b_q c_i d_j D_{ij,pq}. \tag{2.7}$$

Formula (2.2) is the differentiation rule of the determinant, and (2.3) is the expansion formula for a bordered determinant with respect to the last row and last column. Formula (2.4) is Jacobi's identity. Formula (2.5) is the expansion rule of the determinant whereas formula (2.6) is the expansion rule of the first cofactor of d_{ij} in terms of the second cofactors, where δ_{jk} is Kronecker's delta. Formula (2.7) is the expansion rule of the determinant with double borders in terms of the second cofactors.

3. The modified NLS system with plane-wave boundary conditions

3.1. Gauge transformation

We consider the system of PDEs (1.1) subjected to the plane-wave boundary conditions (1.2). As in the case of zero boundary conditions considered in [13], we first apply the following gauge transformation

$$q_j = u_j \exp\left[-\frac{\mathrm{i}\gamma}{2}\int_{-\infty}^{x} \sum_{s=1}^{n} \sigma_s(|u_s|^2 - \rho_s^2)dx\right], \quad j = 1, 2, ..., n, \tag{3.1}$$

to the system, where $u_j = u_j(x,t)$ $(j=1,2,...,n)$ are complex-valued functions of x and t. The system (1.1) is then transformed to the system of nonlinear PDEs for u_j:

$$\mathrm{i}u_{j,t} + u_{j,xx} + \mathrm{i}\lambda\gamma u_{j,x} + \mathrm{i}\gamma\left(\sum_{s=1}^{n}\sigma_s u_s^* u_{s,x} - \mathrm{i}\sum_{s=1}^{n}\sigma_s k_s \rho_s^2\right)u_j$$
$$+\left[\left(\mu - \frac{\lambda\gamma^2}{2}\right)\sum_{s=1}^{n}\sigma_s|u_s|^2 + \frac{(\lambda\gamma)^2}{2}\right]u_j = 0, \quad j=1,2,...,n. \tag{3.2}$$

Here, we have put $\lambda = \sum_{s=1}^{n}\sigma_s\rho_s^2$ for simplicity and the asterisk appended to u_s denotes complex conjugate.

3.2. Bilinearization

The following proposition is the starting point in our analysis.

Proposition 3.1. *By means of the dependent variable transformations*

$$u_j = \rho_j \mathrm{e}^{\mathrm{i}(k_j x - \omega_j t)}\,\frac{h_j}{f}, \quad j=1,2,...,n, \tag{3.3}$$

the system of nonlinear PDEs (3.2) can be decoupled into the following system of bilinear equations for f and h_j

$$\mathrm{i}D_t h_j\cdot f + \mathrm{i}(2k_j + \lambda\gamma)D_x h_j\cdot f + D_x^2 h_j\cdot f = 0, \quad j=1,2,...,n, \tag{3.4}$$

$$D_x f\cdot f^* - \frac{\mathrm{i}\gamma}{2}\sum_{s=1}^{n}\sigma_s\rho_s^2(h_s h_s^* - ff^*) = 0, \tag{3.5}$$

$$D_x^2 f\cdot f^* - \frac{\mathrm{i}\gamma}{2}\sum_{s=1}^{n}\sigma_s\rho_s^2 D_x h_s\cdot h_s^* + \gamma\sum_{s=1}^{n}\sigma_s k_s\rho_s^2(h_s h_s^* - ff^*)$$
$$+\left(\frac{\lambda\gamma^2}{4} - \mu\right)\sum_{s=1}^{n}\sigma_s\rho_s^2(h_s h_s^* - ff^*) = 0. \tag{3.6}$$

Here, $f = f(x,t)$ and $h_j = h_j(x,t)$ $(j=1,2,...,n)$ are complex-valued functions of x and t, and the bilinear operators D_x and D_t are defined by

$$D_x^m D_t^n f\cdot g = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n f(x,t)g(x',t')\Big|_{x'=x,\,t'=t}, \tag{3.7}$$

where m and n are nonnegative integers.

Proof. Substituting (3.3) into (3.2) and using the definition of the bilinear operators, the sysytem (3.2) can be rewritten in the form

$$\frac{1}{f^2}\left\{\mathrm{i}D_t h_j\cdot f+\mathrm{i}(2k_j+\lambda\gamma)D_x h_j\cdot f+D_x^2 h_j\cdot f\right\}$$

$$+\frac{h_j}{f^3f^*}\Bigg[-f^*D_x^2f\cdot f+\mathrm{i}\gamma\Big\{\sum_{s=1}^n\sigma_s\rho_s^2h_s^*D_xh_s\cdot f+\mathrm{i}f\sum_{s=1}^n\sigma_sk_s\rho_s^2(h_sh_s^*-ff^*)\Big\}$$

$$+\left(-\frac{\lambda\gamma}{2}+\mu\right)f\sum_{s=1}^n\sigma_s\rho_s^2(h_sh_s^*-ff^*)\Bigg]=0. \tag{3.8}$$

Inserting the identity

$$f^*D_x^2f\cdot f=fD_x^2f\cdot f^*-2f_xD_xf\cdot f^*+f(D_xf\cdot f^*)_x, \tag{3.9}$$

into the corresponding term in (3.8), we can put it into the form

$$\frac{1}{f^2}\left\{\mathrm{i}D_t h_j\cdot f+\mathrm{i}(2k_j+\lambda\gamma)D_x h_j\cdot f+D_x^2 h_j\cdot f\right\}$$

$$+\frac{h_j}{f^3f^*}\Bigg[f\Bigg\{-D_x^2f\cdot f^*-(D_xf\cdot f^*)_x+\mathrm{i}\gamma\Big\{\sum_{s=1}^n\sigma_s\rho_s^2h_s^*h_{s,x}+\mathrm{i}\sum_{s=1}^n\sigma_sk_s\rho_s^2(h_sh_s^*-ff^*)\Big\}$$

$$+\left(-\frac{\lambda\gamma^2}{2}+\mu\right)\sum_{s=1}^n\sigma_s\rho_s^2(h_sh_s^*-ff^*)\Bigg\}+f_x\left(2D_xf\cdot f^*-\mathrm{i}\gamma\sum_{s=1}^n\sigma_s\rho_s^2h_sh_s^*\right)\Bigg]=0. \tag{3.10}$$

One can confirm that the left-hand side of (3.10) becomes zero by using the bilinear equations (3.4)-(3.6). □

The fundamental quantities f and h_j characterize completely solutions. They are sometimes called the tau functions. While the expressions (3.3) for u_j give the solutions of the system of PDEs (3.2) in terms of the tau functions f and h_j ($j=1,2,...,n$), the original variables q_j are expressible by them as well. To show this, we use (3.3) and (3.5) to obtain the relation

$$\frac{\partial}{\partial x}\ln\frac{f}{f^*}=-\frac{\mathrm{i}\gamma}{2}\sum_{s=1}^n\sigma_s(|u_s|^2-\rho_s^2). \tag{3.11}$$

If we introduce (3.11) into (3.1), integrate with respect to x and note (3.3), we find the desired expressions

$$q_j=\rho_j\mathrm{e}^{\mathrm{i}(k_jx-\omega_jt)}\,\frac{h_jf^*}{f^2},\quad j=1,2,...,n. \tag{3.12}$$

Remark 3.1. If $\gamma = 0$, then the bilinear equation (3.5) reduces to $D_x f \cdot f^* = 0$, implying that $f^* = c(t) f$. An arbitrary function $c(t)$ can be set to 1 by imposing a boundary condition, $f = 1,\ x \to -\infty$, for instance. Thus, q_j from (3.12) simplify to

$$q_j = \rho_j \mathrm{e}^{\mathrm{i}(k_j x - \omega_j t)} \frac{h_j}{f}, \quad j = 1, 2, ..., n, \tag{3.13}$$

and they satisfy the n-component NLS system with the plane-wave boundary conditions (1.2)

$$\mathrm{i}\, q_{j,t} + q_{j,xx} + \mu \left(\sum_{s=1}^{n} \sigma_s |q_s|^2 \right) q_j = 0, \quad j = 1, 2, ..., n. \tag{3.14}$$

3.3. The dark N-soliton solution

Here, we establish the following theorem.

Theorem 3.1. *The N-soliton solution of the system of bilinear equations (3.4)-(3.6) is given in terms of the following determinants*

$$f = |D|, \quad h_s = |H_s|, \quad s = 1, 2, ..., n, \tag{3.15a}$$

$$D = (d_{jk})_{1 \le j,k \le N}, \quad d_{jk} = \delta_{jk} - \frac{\mathrm{i}p_j + \frac{\mu}{\gamma}}{p_j + p_k^*} z_j z_k^*, \tag{3.15b}$$

$$H_s = (h_{jk}^{(s)})_{1 \le j,k \le N}, \quad h_{jk}^{(s)} = \delta_{jk} + \frac{\left(\mathrm{i}p_j + \frac{\mu}{\gamma}\right)(p_j - \mathrm{i}k_s)}{(p_j + p_k^*)(p_k^* + \mathrm{i}k_s)} z_j z_k^*, \tag{3.15c}$$

$$z_j = \exp[p_j x + (\mathrm{i}p_j^2 - \lambda\gamma p_j)t + \zeta_{j0}], \quad j = 1, 2, ..., N, \quad \lambda = \sum_{s=1}^{n} \sigma_s \rho_s^2. \tag{3.15d}$$

Here, p_j and ζ_{j0} ($j = 1, 2, ..., N$) are arbitrary complex parameters characterizing the amplitude and phase of the solitons, respectively, and the N constraints are imposed on the parameters p_j

$$\frac{\gamma}{2} \sum_{s=1}^{n} \sigma_s \rho_s^2 \frac{\mathrm{i}(p_j - p_j^*) + k_s + \frac{\mu}{\gamma}}{(p_j - \mathrm{i}k_s)(p_j^* + \mathrm{i}k_s)} = -1, \quad j = 1, 2, ..., N. \tag{3.16}$$

Remark 3.2. By means of the transformation of the variables, $f = \tilde{f}, h_s = \tilde{h}_s$ ($s = 1, 2, ..., n$), $x = \tilde{x} + \frac{2\mu}{\gamma}\tilde{t}$ combined with the transformation of the parameters, $p_j = \tilde{p}_j + \mathrm{i}\frac{\mu}{\gamma}$ ($j = 1, 2, ..., N$), $k_s = \tilde{k}_s + \frac{\mu}{\gamma}$ ($s = 1, 2, ..., n$), the form of the bilinear equations (3.4) and (3.5) is unchanged whereas the bilinear equation (3.6) reduces to a simplified form with $\mu = 0$. The N-soliton solution (3.15) and the constraints (3.16) remain the same form with $\mu = 0$. Thus, the proof of the N-soliton solution may be performed under the setting $\mu = 0$ without loss of generality.

Remark 3.3. Unlike the bright N-soliton solution with zero boundary conditions [13], the real part of p_j is related to its imaginary part by (3.16). In the general n-component system, one needs to solve the algebraic equation of order n for $(\mathrm{Re}\, p_j)^2$. As well-known, analytical solutions are not available for $n \geq 5$. If the wavenumbers k_j $(j = 1, 2, ..., n)$ take the same value which implies that all the components of the system have the same asymptotic form except the amplitudes ρ_j and the phase constants $\phi_j^{(\pm)}$ (see (1.2)), then the constraints (3.16) reduce simply to a single quadratic equation for $\mathrm{Re}\, p_j$. This special case has been dealt with by means of the IST for the multi-component NLS system [20, 21].

Remark 3.4. The multi-component NLS system (i.e., $\gamma = 0$ in (1.1)) is invariant under the transformation $t = -\tilde{t}, x = \mathrm{i}\tilde{x}, \mu = -\tilde{\mu}$. Suppose that $\sigma_s = 1$ $(s = 1, 2, ..., n)$ and $\tilde{\mu} > 0$. We change the parameters in (3.15) according to the rule $k_s = -\mathrm{i}\tilde{k}_s, \omega_s = -\tilde{\omega}_s$ $(s = 1, 2, ..., n)$, $p_j = -\mathrm{i}\tilde{p}_j, \zeta_{j0} = \tilde{\zeta}_{j0} + \ln\sqrt{\gamma}$ $(j = 1, 2, ..., N)$ and then take the limit $\gamma \to 0$. The resulting expression gives rise to the N-soliton solution of the focusing NLS system with plane-wave boundary conditions. The solutions thus constructed exhibit a rich mathematical structure. Specifically, a reduction procedure applied to the soliton solutions would produce the breather and rogue wave solutions, as already demonstrated for the soliton solutions of the focusing NLS equation [35].

The proof of theorem 3.1 will be performed by using a sequence of lemmas, which we shall summarize. In accordance with remark 3.2, we put $\mu = 0$ in formulas that follow.

Lemma 3.1. *The expression of h_s from (3.15a) is rewritten in the form*

$$h_s = |D| - \frac{\mathrm{i}}{k_s}\,|D(\mathbf{z}_s^*; \mathbf{z}_x)|, \quad s = 1, 2, ..., n, \tag{3.17a}$$

where

$$\mathbf{z} = (z_j)_{1\leq j\leq N}, \quad \mathbf{z}_s = \left(\frac{k_s}{p_j - \mathrm{i}k_s}\, z_j\right)_{1\leq j\leq N}, \quad s = 1, 2, ..., n, \tag{3.17b}$$

are N-component row vectors.

Proof. Using an identity

$$\frac{\mathrm{i}p_j}{p_j + p_k^*}\,\frac{p_j - \mathrm{i}k_s}{p_k^* + \mathrm{i}k_s} = -\frac{\mathrm{i}p_j}{p_j + p_k^*} + \frac{\mathrm{i}p_j}{p_k^* + \mathrm{i}k_s},$$

the determinant $|H_s|$ from (3.15c) is modified in the form

$$|H_s| = \left|\left(d_{jk} + \frac{\mathrm{i}p_j}{p_k^* + \mathrm{i}k_s}\, z_j z_k^*\right)_{1\le j,k\le N}\right|$$
$$= \begin{vmatrix} D & \mathrm{i}\mathbf{z}_x^T \\ -\frac{1}{k_s}\,\mathbf{z}_s^* & 1 \end{vmatrix}.$$

Applying formula (2.3) to the last expression gives (3.17). □

The following lemma provides the differentiation rules of f and h_s with respect to t and x.

Lemma 3.2.

$$f_t = \mathrm{i}|D(\mathbf{z}^*;\mathbf{z}_t)| + |D(\mathbf{z}_x^*;\mathbf{z}_x)|, \tag{3.18}$$

$$f_x = \mathrm{i}|D(\mathbf{z}^*;\mathbf{z}_x)|, \tag{3.19}$$

$$f_{xx} = \mathrm{i}|D(\mathbf{z}_x^*;\mathbf{z}_x)| + \mathrm{i}|D(\mathbf{z}^*;\mathbf{z}_{xx})|, \tag{3.20}$$

$$h_{s,t} = \mathrm{i}|D(\mathbf{z}^*;\mathbf{z}_t)| + |D(\mathbf{z}_x^*;\mathbf{z}_x)| - \frac{\mathrm{i}}{k_s}|D(\mathbf{z}_{s,t}^*;\mathbf{z}_x)| - \frac{\mathrm{i}}{k_s}|D(\mathbf{z}_s^*;\mathbf{z}_{xt})| + \frac{1}{k_s}|D(\mathbf{z}_s^*,\mathbf{z}^*;\mathbf{z}_x,\mathbf{z}_t)|, \tag{3.21}$$

$$h_{s,x} = -|D(\mathbf{z}_s^*;\mathbf{z}_x)| - \frac{\mathrm{i}}{k_s}|D(\mathbf{z}_s^*;\mathbf{z}_{xx})|, \tag{3.22}$$

$$h_{s,xx} = \mathrm{i}|D(\mathbf{z}_x^*;\mathbf{z}_x)| + \mathrm{i}|D(\mathbf{z}^*;\mathbf{z}_{xx})| - \frac{\mathrm{i}}{k_s}|D(\mathbf{z}_{s,xx}^*;\mathbf{z})| - \frac{2\mathrm{i}}{k_s}|D(\mathbf{z}_{s,x}^*;\mathbf{z}_{xx})|$$
$$-\frac{\mathrm{i}}{k_s}|D(\mathbf{z}_s^*;\mathbf{z}_{xxx})| + \frac{1}{k_s}|D(\mathbf{z}_s^*,\mathbf{z}^*;\mathbf{z}_{xx},\mathbf{z}_x)|, \tag{3.23}$$

Proof. We prove (3.18). Applying formula (2.2) to f from (3.15) gives

$$f_t = \sum_{j,k=1}^{N} D_{jk}\{\mathrm{i}(p_j^2 - p_k^{*2}) - \lambda\gamma(p_j + p_k^*)\}\frac{-\mathrm{i}p_j}{p_j + p_k^*}\, z_j z_k^*$$
$$= \sum_{j,k=1}^{N} D_{jk}\{-\mathrm{i}(\mathrm{i}p_j^2 - \lambda\gamma p_j) - p_j p_k^*\} z_j z_k^*.$$

In view of formula (2.3) and the realtions $z_{j,t} = (\mathrm{i}p_j^2 - \lambda\gamma p_j)z_j, z_{j,x} = p_j z_j$, the last expression coincides with (3.18). A key feature in the above computation is that the factor $(p_j + p_k^*)^{-1}$ in the element a_{jk} has been canceled after differentiation with respect to t. All the other formulas can be proved in the same way by using formulas (2.2) and (2.3) as well as some basic propeerties of determinants. □

Lemma 3.3. *The complex conjugate expressions of f, f_x and h_s are given as follows.*

$$f^* = |D| - \mathrm{i}|D(\mathbf{z}^*; \mathbf{z})|, \tag{3.24}$$

$$f_x^* = -\mathrm{i}|D(\mathbf{z}_x^*; \mathbf{z})|, \tag{3.25}$$

$$h_s^* = |D| - \mathrm{i}|D(\mathbf{z}^*; \mathbf{z})| + \frac{\mathrm{i}}{k_s}|D(\mathbf{z}_x^*; \mathbf{z}_s)| + \frac{1}{k_s}|D(\mathbf{z}_x^*, \mathbf{z}^*; \mathbf{z}_s, \mathbf{z})|. \tag{3.26}$$

Proof. Let $D^\dagger$ be the Hermitian conjugate of the matrix D. Since $|D| = |D^T|$, one can see that $f^* = |D^*| = |D^\dagger|$. It follows from (3.15$b$) that $D^\dagger = D + \mathrm{i}\mathbf{z}^T\mathbf{z}^*$. These relations lead to formula (3.24) with the help of formula (2.3). Formulas (3.25) and (3.26) can be proved in the same way by taking the complex conjugate expression of (3.19) and (3.17), respectively. □

When one tries to show that the N-soliton solution (3.15) solves the bilinear equations (3.5) and (3.6), the following lemma plays the central role together with Jacobi's identity (2.4).

Lemma 3.4.

$$\sum_{s=1}^{n} \frac{\sigma_s\rho_s^2}{k_s^2}|D(\mathbf{z}_s^*; \mathbf{z}_s)| = |D(\mathbf{z}^*B; \mathbf{z})|, \tag{3.27}$$

$$\sum_{s=1}^{n} \frac{\sigma_s\rho_s^2}{k_s^2}|D(\mathbf{z}_s^*, \mathbf{a}^*; \mathbf{b}, \mathbf{z}_s)| = |D(\mathbf{z}^*B, \mathbf{a}^*; \mathbf{b}, \mathbf{z})| - \mathrm{i}(|D(\mathbf{a}^*; \mathbf{b}B)| - |D(\mathbf{a}^*B; \mathbf{b})|). \tag{3.28}$$

Here, B is a diagonal matrix given by

$$B = \mathrm{diag}\,(\beta_1, \beta_2, ..., \beta_N), \quad \beta_j = \sum_{s=1}^{n} \frac{\sigma_s\rho_s^2}{(p_j - \mathrm{i}k_s)(p_j^* + \mathrm{i}k_s)}, \quad j = 1, 2, ..., N, \tag{3.29}$$

$\mathbf{z}^*B = (\beta_j z_j^*)_{1\le j\le N}$ *is the N-component row vector, and* $\mathbf{a} = (a_j)_{1\le j\le N}$ *and* $\mathbf{b} = (b_j)_{1\le j\le N}$ *are arbitrary N-component row vectors.*

The proof of lemma 3.4 will be given in appendix A.

3.4. Proof of theorem 3.1

Here, we show that the tau functions (3.15) associated with the dark N-soliton solution solve the bilinear equations (3.4)-(3.6). The proof will be performed by employing lemmas 3.1-3.4 and some basic formulas for determinants. In particular, Jacobi's identity (2.4) plays the central role, as in the case of the proof of the bright N-soliton solution of the modified NLS system (1.1) [13].

3.4.1. **Proof of (3.4).** Let P be the left-hand side of (3.4). We write it in the form $P = P_1 f - P_2 h_s - 2h_{s,x} f_x$ with

$$P_1 = \mathrm{i}h_{s,t} + \mathrm{i}(2k_s + \lambda\gamma)h_{s,x} + h_{s,xx}, \quad P_2 = \mathrm{i}f_t + \mathrm{i}(2k_s + \lambda\gamma)f_x - f_{xx}.$$

First, we compute P_1 by using (3.21)-(3.23) to obtain

$$P_1 = -|D(\mathbf{z}^*; \mathbf{z}_t)| + \frac{1}{k_s}|D(\mathbf{z}^*_{s,t} - \mathrm{i}(2k_s + \lambda\gamma)k_s\mathbf{z}^*_s - k_s\mathbf{z}^*_{s,x}; \mathbf{z}_x)|$$

$$+\frac{1}{k_s}|D(\mathbf{z}^*_s; \mathbf{z}_{xt} + (k_s + \lambda\gamma)\mathbf{z}_{s,xx} - \mathrm{i}\mathbf{z}_{s,xxx})| + \mathrm{i}|D(\mathbf{z}^*_x; \mathbf{z}_x)| - \frac{\mathrm{i}}{k_s}|D(\mathbf{z}^*_{s,x}; \mathbf{z}_{xx})|$$

$$+\frac{\mathrm{i}}{k_s}|D(\mathbf{z}^*_s, \mathbf{z}^*; \mathbf{z}_x, \mathbf{z}_t)| + \frac{1}{k_s}|D(\mathbf{z}^*_s, \mathbf{z}^*; \mathbf{z}_{xx}, \mathbf{z}_x)|.$$

It follows from (3.15d) and (3.17b) that

$$\mathbf{z}^*_{s,t} - \mathrm{i}(2k_s + \lambda\gamma)k_s\mathbf{z}^*_s - k_s\mathbf{z}^*_{s,x} = -\mathrm{i}k_s\mathbf{z}^*_x - (2k_s + \lambda\gamma)k_s\mathbf{z}^*, \quad \mathbf{z}_{xt} + (k_s + \lambda\gamma)\mathbf{z}_{s,xx} - \mathrm{i}\mathbf{z}_{s,xxx} = k_s\mathbf{z}_{xx},$$

which reduce P_1 to

$$P_1 = -|D(\mathbf{z}^*; \mathbf{z}_t + (2k_s + \lambda\gamma)\mathbf{z}_x + \mathrm{i}\mathbf{z}_{xx})| + \frac{\mathrm{i}}{k_s}|D(\mathbf{z}^*_s, \mathbf{z}^*; \mathbf{z}_x, \mathbf{z}_t)| + \frac{1}{k_s}|D(\mathbf{z}^*_s, \mathbf{z}^*; \mathbf{z}_{xx}, \mathbf{z}_x)|.$$

To compute P_2, we use (3.18)-(3.20), giving

$$P_2 = -|D(\mathbf{z}^*; \mathbf{z}_t + (2k_s + \lambda\gamma)\mathbf{z}_x + \mathrm{i}\mathbf{z}_{xx})|.$$

Last, substituting above expressions for P_1 and P_2 into P and using (3.15) and (3.22) as well as the relations

$$\mathbf{z}_t = \mathrm{i}\mathbf{z}_{xx} - \lambda\gamma\mathbf{z}_x, \quad \mathbf{z}_t + (2k_s + \lambda\gamma)\mathbf{z}_x + \mathrm{i}\mathbf{z}_{xx} = 2\mathrm{i}\mathbf{z}_{xx} + 2k_s\mathbf{z}_x,$$

one arrives at the expression

$$P = -\frac{2}{k_s}\Big[|D(\mathbf{z}^*_s, \mathbf{z}^*; \mathbf{z}_x, \mathbf{z}_{xx})||D| - |D(\mathbf{z}^*_s; \mathbf{z}_x)||D(\mathbf{z}^*; \mathbf{z}_{xx})| + |D(\mathbf{z}^*_s; \mathbf{z}_{xx})||D(\mathbf{z}^*; \mathbf{z}_x)|\Big].$$

In view of Jacobi's identity (2.4), P becomes zero. □

3.4.2. **Proof of (3.5).** Let Q be the left-hand side of (3.5). Substituting (3.17), (3.19), (3.24), (3.25) and (3.26) into Q and using the relation $\sum_{s=1}^{n}(\sigma_s\rho_s^2/k_s)\,\mathbf{z}_s = -(2\mathrm{i}/\gamma)\,\mathbf{z} + \mathbf{z}_x B$ which can be derived from (3.16), (3.17b) and (3.29), we obtain

$$Q = -\frac{\mathrm{i}\gamma}{2}\Big[\mathrm{i}|D|\{-|D(\mathbf{z}^*_x B; \mathbf{z}_x)| + |D(\mathbf{z}^*_x; \mathbf{z}_x B)| - \mathrm{i}|D(\mathbf{z}^*_x, \mathbf{z}^*; \mathbf{z}_x B, \mathbf{z})|\} - |D(\mathbf{z}^*_x B; \mathbf{z}_x)||D(\mathbf{z}^*; \mathbf{z})|\Big]$$

$$-\frac{\mathrm{i}\gamma}{2}\sum_{s=1}^{n}\frac{\sigma_s\rho_s^2}{k_s^2}\Big[|D(\mathbf{z}_s^*;\mathbf{z}_x)||D(\mathbf{z}_x^*;\mathbf{z}_s)|-\mathrm{i}|D(\mathbf{z}_s^*;\mathbf{z}_x)||D(\mathbf{z}_x^*,\mathbf{z}^*;\mathbf{z}_s,\mathbf{z})|\Big].$$

It follows from Jacobi's identity (2.4) that

$$|D(\mathbf{z}_x^*,\mathbf{z}^*;\mathbf{z}_xB,\mathbf{z})||D|=|D(\mathbf{z}_x^*;\mathbf{z}_xB)||D(\mathbf{z}^*;\mathbf{z})|-|D(\mathbf{z}_x^*;\mathbf{z})||D(\mathbf{z}^*;\mathbf{z}_xB)|,$$

$$|D(\mathbf{z}_s^*;\mathbf{z}_x)||D(\mathbf{z}_x^*;\mathbf{z}_s)|=|D(\mathbf{z}_s^*,\mathbf{z}_x^*;\mathbf{z}_x,\mathbf{z}_s)||D|+|D(\mathbf{z}_s^*;\mathbf{z}_s)||D(\mathbf{z}_x^*;\mathbf{z}_x)|,$$

$$|D(\mathbf{z}_s^*;\mathbf{z}_x)||D(\mathbf{z}_x^*,\mathbf{z}^*;\mathbf{z}_s,\mathbf{z})|$$

$$=|D(\mathbf{z}_s^*,\mathbf{z}_x^*;\mathbf{z}_x,\mathbf{z}_s)||D(\mathbf{z}^*;\mathbf{z})|-|D(\mathbf{z}_s^*,\mathbf{z}^*;\mathbf{z}_x,\mathbf{z}_s)||D(\mathbf{z}_x^*;\mathbf{z})|+|D(\mathbf{z}_x^*,\mathbf{z}^*;\mathbf{z}_x,\mathbf{z})||D(\mathbf{z}_s^*;\mathbf{z}_s)|.$$

The third formula may be proved by multiplying $|D|$ on both sides and using Jacobi's identity (2.4). If we identify $\mathbf{a}=\mathbf{z}_x,\mathbf{b}=\mathbf{z}_x$ and $\mathbf{a}=\mathbf{z},\mathbf{b}=\mathbf{z}_x$, respectively in (3.28), then we obtain

$$\sum_{s=1}^{n}\frac{\sigma_s\rho_s^2}{k_s^2}|D(\mathbf{z}_s^*,\mathbf{z}_x^*;\mathbf{z}_x,\mathbf{z}_s)|=|D(\mathbf{z}^*B,\mathbf{z}_x^*;\mathbf{z}_x,\mathbf{z})|-\mathrm{i}(|D(\mathbf{z}_x^*;\mathbf{z}_xB)|-|D(\mathbf{z}_x^*B;\mathbf{z}_x)|),$$

$$\sum_{s=1}^{n}\frac{\sigma_s\rho_s^2}{k_s^2}|D(\mathbf{z}_s^*,\mathbf{z}^*;\mathbf{z}_x,\mathbf{z}_s)|=|D(\mathbf{z}^*B,\mathbf{z}^*;\mathbf{z}_x,\mathbf{z})|-\mathrm{i}(|D(\mathbf{z}^*;\mathbf{z}_xB)|-|D(\mathbf{z}^*B;\mathbf{z}_x)|).$$

The last step is to substitute (3.27) and the above five relations into Q, which recasts it to

$$Q=\frac{\gamma}{2}\Big[-|D(\mathbf{z}^*;\mathbf{z})||D(\mathbf{z}^*B,\mathbf{z}_x^*;\mathbf{z}_x,\mathbf{z})|+|D(\mathbf{z}_x^*;\mathbf{z})||D(\mathbf{z}^*B,\mathbf{z}^*;\mathbf{z}_x,\mathbf{z})|$$
$$-|D(\mathbf{z}^*B;\mathbf{z})||D(\mathbf{z}_x^*,\mathbf{z}^*;\mathbf{z}_x,\mathbf{z})|\Big].$$

This expression becomes zero by Jacobi's identity (2.4). □

3.4.3. **Proof of (3.6).** We replace the last term of the left-hand side of (3.6) by $D_xf\cdot f$ from (3.5) and then add $Q_x(=0))$ to the resulting expression, and show that $R\equiv 2R_1-\mathrm{i}\gamma R_2+\gamma R_3=0$, where

$$R_1=f_{xx}f^*-f_xf_x^*,\quad R_2=\sum_{n=1}^{n}\sigma_s\rho_s^2(h_{s,x}h_s^*-ff_x^*),\quad R_3=\sum_{n=1}^{n}\sigma_sk_s\rho_s^2(h_sh_s^*-ff^*).$$

Now, substituting (3.17), (3.19), (3.20), (3.24) and (3.25) into R_1, we obtain

$$R_1=\mathrm{i}(|D(\mathbf{z}_x^*;\mathbf{z}_x)|+|D(\mathbf{z}_x^*;\mathbf{z}_{xx})|)f^*-|D(\mathbf{z}^*;\mathbf{z}_x)||D(\mathbf{z}_x^*;\mathbf{z})|.$$

The expression of R_2 can be recast, after using (3.22), (3.25) and (3.26) as well as the relation $\sum_{s=1}^{n}(\sigma_s\rho_s^2/k_s)\,\mathbf{z}_s^* = (2\mathrm{i}/\gamma)\,\mathbf{z}^* + \mathbf{z}_x^*B$, to

$$R_2 = -\left\{\sum_{s=1}^{n}\sigma_s\rho_s^2|D(\mathbf{z}_s^*;\mathbf{z}_x)| - \frac{2}{\gamma}|D(\mathbf{z}^*;\mathbf{z}_{xx})| + \mathrm{i}|D(\mathbf{z}_x^*B;\mathbf{z}_{xx})|\right\} f^* + \mathrm{i}\lambda|D||D(\mathbf{z}_x^*;\mathbf{z})|$$

$$-\sum_{s=1}^{n}\frac{\sigma_s\rho_s^2}{k_s}\Big[\mathrm{i}|D(\mathbf{z}_s^*;\mathbf{z}_x)||D(\mathbf{z}_x^*;\mathbf{z}_s)| - \frac{1}{k_s}|D(\mathbf{z}_s^*;\mathbf{z}_{xx})||D(\mathbf{z}_x^*;\mathbf{z}_s)|$$

$$+|D(\mathbf{z}_s^*;\mathbf{z}_x)||D(\mathbf{z}_x^*,\mathbf{z}^*;\mathbf{z}_s,\mathbf{z})| + \frac{\mathrm{i}}{k_s}|D(\mathbf{z}_s^*;\mathbf{z}_{xx})||D(\mathbf{z}_x^*,\mathbf{z}^*;\mathbf{z}_s,\mathbf{z})|\Big].$$

It follows from (3.17), (3.24) and (3.26) that

$$R_3 = |D|\sum_{s=1}^{n}\sigma_s\rho_s^2(\mathrm{i}|D(\mathbf{z}_x^*;\mathbf{z}_s)| + |D(\mathbf{z}_x^*,\mathbf{z}^*;\mathbf{z}_s,\mathbf{z})|) - \mathrm{i}\lambda|D(\mathbf{z}_s^*;\mathbf{z}_x)|f^*$$

$$-\mathrm{i}\sum_{s=1}^{n}\frac{\sigma_s\rho_s^2}{k_s}|D(\mathbf{z}_s^*;\mathbf{z}_x)|(\mathrm{i}|D(\mathbf{z}_x^*;\mathbf{z}_s)| + |D(\mathbf{z}_x^*,\mathbf{z}^*;\mathbf{z}_s,\mathbf{z})|).$$

Referring to the above expressions of R_1, R_2 and R_3, R becomes

$$R = 2\mathrm{i}|D(\mathbf{z}_x^*;\mathbf{z}_x)|f^* - 2|D(\mathbf{z}^*;\mathbf{z}_x)||D(\mathbf{z}_x^*;\mathbf{z})| - \mathrm{i}\gamma\Big[-\mathrm{i}|D(\mathbf{z}_x^*B;\mathbf{z}_{xx})|f^* + \mathrm{i}\lambda|D||D(\mathbf{z}_x^*;\mathbf{z})|$$

$$-\sum_{s=1}^{n}\frac{\sigma_s\rho_s^2}{k_s^2}|D(\mathbf{z}_s^*;\mathbf{z}_x)|\{-|D(\mathbf{z}_x^*;\mathbf{z}_s)| + \mathrm{i}|D(\mathbf{z}_x^*,\mathbf{z}^*;\mathbf{z}_s,\mathbf{z})|\}\Big]$$

$$+\gamma|D|\sum_{s=1}^{n}\sigma_s\rho_s^2(\mathrm{i}|D(\mathbf{z}_x^*;\mathbf{z}_s)| + |D(\mathbf{z}_x^*,\mathbf{z}^*;\mathbf{z}_s,\mathbf{z})|).$$

To simplify R further, we use the relation $\sum_{s=1}^{n}\sigma_s\rho_s^2\,\mathbf{z}_s = \mathrm{i}\lambda\mathbf{z} - (2/\gamma)\,\mathbf{z}_x - \mathrm{i}\mathbf{z}_{xx}B$ in the last term and (3.24) for f^*, and then apply Jacobi's identity (2.4), resulting in

$$R = -\mathrm{i}\gamma\Big[-\mathrm{i}|D(\mathbf{z}_x^*B;\mathbf{z}_{xx})|(|D| - \mathrm{i}|D(\mathbf{z}_x^*;\mathbf{z}_s)|)$$

$$+\sum_{s=1}^{n}\frac{\sigma_s\rho_s^2}{k_s^2}|D(\mathbf{z}_s^*;\mathbf{z}_{xx})|\{|D(\mathbf{z}_x^*;\mathbf{z}_s)| - \mathrm{i}|D(\mathbf{z}_x^*,\mathbf{z}^*;\mathbf{z}_s,\mathbf{z})|\}\Big]$$

$$+\gamma|D|\Big[|D(\mathbf{z}_x^*;\mathbf{z}_{xx}B)| - \mathrm{i}|D(\mathbf{z}_x^*,\mathbf{z}^*;\mathbf{z}_{xx}B,\mathbf{z})|\Big].$$

An application of Jacobi's identity (2.4) gives

$$|D(\mathbf{z}_s^*;\mathbf{z}_{xx})||D(\mathbf{z}_x^*,\mathbf{z}^*;\mathbf{z}_s,\mathbf{z})|$$

$$= |D(\mathbf{z}_s^*, \mathbf{z}_x^*; \mathbf{z}_{xx}, \mathbf{z}_s)||D(\mathbf{z}^*; \mathbf{z})| - |D(\mathbf{z}_s^*, \mathbf{z}^*; \mathbf{z}_{xx}, \mathbf{z}_s)||D(\mathbf{z}_x^*; \mathbf{z})| + |D(\mathbf{z}_x^*, \mathbf{z}^*; \mathbf{z}_{xx}, \mathbf{z})||D(\mathbf{z}_s^*; \mathbf{z}_s)|,$$

and formula (3.28) leads to

$$\sum_{s=1}^{n} \frac{\sigma_s \rho_s^2}{k_s^2} |D(\mathbf{z}_s^*, \mathbf{z}_x^*; \mathbf{z}_{xx}, \mathbf{z}_s)| = |D(\mathbf{z}^* B, \mathbf{z}_x^*; \mathbf{z}_{xx}, \mathbf{z})| - \mathrm{i}(|D(\mathbf{z}_x^*; \mathbf{z}_x B)| - |D(\mathbf{z}_x^* B; \mathbf{z}_{xx})|),$$

$$\sum_{s=1}^{n} \frac{\sigma_s \rho_s^2}{k_s^2} |D(\mathbf{z}_s^*, \mathbf{z}^*; \mathbf{z}_{xx}, \mathbf{z}_s)| = |D(\mathbf{z}^* B, \mathbf{z}^*; \mathbf{z}_{xx}, \mathbf{z})| - \mathrm{i}(|D(\mathbf{z}^*; \mathbf{z}_{xx} B)| - |D(\mathbf{z}^* B; \mathbf{z}_{xx})|).$$

Using these relations together with (3.27), one obtains

$$\sum_{s=1}^{n} \frac{\sigma_s \rho_s^2}{k_s^2} |D(\mathbf{z}_s^*; \mathbf{z}_{xx})||D(\mathbf{z}_x^*, \mathbf{z}^*; \mathbf{z}_s, \mathbf{z})|$$

$$= -\mathrm{i}|D(\mathbf{z}^*; \mathbf{z})|(|D(\mathbf{z}_x^*; \mathbf{z}_{xx} B)| - |D(\mathbf{z}_x^* B; \mathbf{z}_{xx})|) + \mathrm{i}|D(\mathbf{z}_x^*; \mathbf{z})|(|D(\mathbf{z}^*; \mathbf{z}_{xx} B)| - |D(\mathbf{z}^* B; \mathbf{z}_{xx})|).$$

Similarly,

$$\sum_{s=1}^{n} \frac{\sigma_s \rho_s^2}{k_s^2} |D(\mathbf{z}_s^*; \mathbf{z}_{xx})|\{|D(\mathbf{z}_x^*; \mathbf{z}_s)|$$

$$= |D|\{|D(\mathbf{z}^* B, \mathbf{z}_x^*; \mathbf{z}_{xx}, \mathbf{z})| - \mathrm{i}(|D(\mathbf{z}_x^*; \mathbf{z}_{xx} B)| - |D(\mathbf{z}_x^* B; \mathbf{z}_{xx})|)\} + |D(\mathbf{z}^* B; \mathbf{z})||D(\mathbf{z}_x^*; \mathbf{z}_{xx}).$$

Last, substituting above two relations into R, one can see that R becomes zero. □

Remark 3.5. We remark that the constraints (3.16) has not been used for the proof of (3.4). On the other hand, in establishing (3.5) and (3.6), one has relied on lemma (3.4) which is essentially based on the constraints.

3.5. One-soliton solutions

The soliton solutions are characterized completely by the tau functions f and h_j given by (3.15). Here, we describe the feature of one-soliton solutions. The general N-soliton solutions will be considered elsewhere. To simplify the notation, we first put $p_1 = a + \mathrm{i}b,\ \zeta_{10} = a(x_0 + \mathrm{i}y_0)\ \ (a, b, x_0, y_0 \in \mathbb{R})$, and

$$\frac{b - \frac{\mu}{\gamma} - \mathrm{i}a}{2a} = \beta\, \mathrm{e}^{2\mathrm{i}\phi}\ (\beta > 0), \quad \frac{a + \mathrm{i}(b - k_j)}{a - \mathrm{i}(b - k_j)} = \mathrm{e}^{2\mathrm{i}\theta_j},\ j = 1, 2, ..., n. \tag{3.30}$$

Then, the tau functions for the one-soliton solutions are written compactly in the form

$$f = 1 + \mathrm{e}^{2a(\xi+\xi_0)+2\mathrm{i}\phi}, \quad h_j = 1 - \mathrm{e}^{2a(\xi+\xi_0)+2\mathrm{i}(\theta_j+\phi)},\ j = 1, 2, ..., n, \tag{3.31}$$

with $\xi = x - (2b + \lambda\gamma)t + x_0$, and $\xi_0 = (1/2a)\ln\beta$. Formula (3.12) with (3.31) gives one-soliton solutions. The square modulas of the complex variable q_j is computed as

$$|q_j|^2 = \rho_j^2 - \frac{2a^2\,\mathrm{sgn}\,a}{\sqrt{a^2+(b-\mu/\gamma)^2}}\,\frac{\rho_j^2(2b-\mu/\gamma-k_j)}{a^2+(b-k_j)^2}\,\frac{1}{\cosh 2a(\xi+\xi_0)+\frac{(b-\mu/\gamma)\mathrm{sgn}\,a}{\sqrt{a^2+(b-\mu/\gamma)^2}}}. \quad (3.32)$$

In the simplest one-soliton case, the constraints (3.16) reduce simply to a single relation which connects a with b

$$\frac{\gamma}{2}\sum_{s=1}^{n}\sigma_s\rho_s^2\,\frac{-2b+k_s+\frac{\mu}{\gamma}}{a^2+(b-k_s)^2} = -1. \quad (3.33)$$

For the n-component system, one must solve the algebraic equation of order n for a^2 to express a in terms of b. Note, however, that analytical solutions are obtainable up to $n = 4$. This is the main difficulty in constructing soliton solutions. Although one can deal with the degenerate case for which all the wave numbers k_s have the same (possibly zero) value, the resulting one soliton solutions coincide essentially with those of the one-component system.

Last, we consider the soliton solutions of the 1-component system. Using (3.33) with $n = 1$, the expression (3.32) simplifies to

$$|q_1|^2 = \rho_1^2 - \frac{4a^2\,\mathrm{sgn}\,a}{\sigma_1\gamma\sqrt{a^2+(b-\mu/\gamma)^2}}\,\frac{1}{\cosh 2a(\xi+\xi_0)+\frac{(b-\mu/\gamma)\mathrm{sgn}\,a}{\sqrt{a^2+(b-\mu/\gamma)^2}}}. \quad (3.34)$$

When the condition $\mathrm{sgn}(\sigma_1\gamma a) > 0$ is satisfied, then this represents the dark soliton solution with a constant background. Taking the limit $\gamma \to 0$ under the conditions $\mathrm{sgn}(\sigma_1\gamma a) > 0, \mathrm{sgn}(\gamma\mu a) < 0$, the expression (3.34) reduces to the dark soliton solution of the NLS equation

$$|q_1|^2 = \rho_1^2 - \frac{2a^2}{|\mu|}\mathrm{sech}^2 a(\xi+\xi_0), \quad (3.35)$$

with a constraint $a^2 + (b - k_1)^2 = -(\mu/2)\sigma_1\rho_1^2$ imposed on the parameters a and b.

4. The modified NLS system with mixed type boundary conditions

In this section, we present the N-soliton solution of the system of equations (1.1) with the mixed zero and plane-wave boundary conditions (1.3). We will see that solutions take the form of bright-dark type solitons. Since the procedure for constructing the solution almost parallels that developed for the plane-wave boundary conditions, we outline the results except for the proof of the N-soliton solution.

4.1. Gauge transformation

We first apply the gauge transformation

$$q_j = u_j \exp\left[-\frac{\mathrm{i}\gamma}{2}\int_{-\infty}^{x}\left\{\sum_{s=1}^{n}\sigma_s|u_s|^2 - \sum_{s=1}^{n-m}\sigma_{m+s}\rho_s^2\right\}dx\right], \quad j=1,2,...,n, \tag{4.1}$$

to the system (1.1) and obtain the system of nonlinear PDEs for u_j:

$$\mathrm{i}u_{j,t} + u_{j,xx} + \mathrm{i}\hat{\lambda}\gamma u_{j,x} + \mathrm{i}\gamma\left(\sum_{s=1}^{n}\sigma_s u_s^* u_{s,x} - \mathrm{i}\sum_{s=1}^{n-m}\sigma_{m+s}k_s\rho_s^2\right)u_j$$
$$+\left[\left(\mu - \frac{\hat{\lambda}\gamma^2}{2}\right)\sum_{s=1}^{n}\sigma_s|u_s|^2 + \frac{(\hat{\lambda}\gamma)^2}{2}\right]u_j = 0, \quad j=1,2,...,n, \tag{4.2}$$

where $\hat{\lambda} = \sum_{s=1}^{n-m}\sigma_{m+s}\rho_s^2$.

4.2. Bilinearization

The bilinearization of the system of nonlinear PDEs (4.2) is accomplished by the following proposition.

Proposition 4.1. *By means of the dependent variable transformations*

$$u_j = \mathrm{e}^{\mathrm{i}\hat{\lambda}\mu t}\frac{g_j}{f}, \quad j=1,2,...,m, \tag{4.3a}$$

$$u_{m+j} = \rho_j \mathrm{e}^{\mathrm{i}(k_j x - \omega_j t)}\frac{h_j}{f}, \quad j=1,2,...,n-m, \tag{4.3b}$$

the system of nonlinear PDEs (4.2) can be decoupled into the following system of bilinear equations for f, g_j and h_j

$$\mathrm{i}D_t g_j\cdot f + \mathrm{i}\hat{\lambda}\gamma D_x g_j\cdot f + D_x^2 g_j\cdot f = 0, \quad j=1,2,...,m, \tag{4.4}$$

$$\mathrm{i}D_t h_j\cdot f + \mathrm{i}(2k_j+\hat{\lambda}\gamma)D_x h_j\cdot f + D_x^2 h_j\cdot f = 0, \quad j=1,2,...,n-m, \tag{4.5}$$

$$D_x f\cdot f^* - \frac{\mathrm{i}\gamma}{2}\left\{\sum_{s=1}^{m}\sigma_s g_s g_s^* + \sum_{s=1}^{n-m}\sigma_{m+s}\rho_s^2(h_s h_s^* - ff^*)\right\} = 0, \tag{4.6}$$

$$D_x^2 f\cdot f^* - \frac{\mathrm{i}\gamma}{2}\left(\sum_{s=1}^{m}\sigma_s D_x g_s\cdot g_s^* + \sum_{s=1}^{n-m}\sigma_{m+s}\rho_s^2 D_x h_s\cdot h_s^*\right) + \gamma\sum_{s=1}^{n-m}\sigma_{m+s}k_s\rho_s^2(h_s h_s^* - ff^*)$$
$$+\left(\frac{\hat{\lambda}\gamma^2}{4} - \mu\right)\left\{\sum_{s=1}^{m}\sigma_s g_s g_s^* + \sum_{s=1}^{n-m}\sigma_{m+s}\rho_s^2(h_s h_s^* - ff^*)\right\} = 0. \tag{4.7}$$

Here, $f=f(x,t), g_j(x,t)$ and $h_j=h_j(x,t)$ are complex-valued functions of x and t.

Using (4.1) and (4.6), we can express q_j in terms of the tau functions f, g_j and h_j as

$$q_j = \mathrm{e}^{\mathrm{i}\hat{\lambda}\mu t}\,\frac{g_j f^*}{f^2}, \quad j=1,2,...,m, \tag{4.8a}$$

$$q_{m+j} = \rho_j \mathrm{e}^{\mathrm{i}(k_j x-\omega_j t)}\,\frac{h_j f^*}{f^2}, \quad j=1,2,...,n-m. \tag{4.8b}$$

Remark 4.1. In the case of $\gamma=0$, the expressions (4.8) become

$$q_j = \mathrm{e}^{\mathrm{i}\hat{\lambda}\mu t}\,\frac{g_j}{f}, \quad j=1,2,...,m, \tag{4.9a}$$

$$q_{m+j} = \rho_j \mathrm{e}^{\mathrm{i}(k_j x-\omega_j t)}\,\frac{h_j}{f}, \quad j=1,2,...,n-m, \tag{4.9b}$$

and they satisfy the n-component NLS system (3.14).

4.3. The bright-dark N-soliton solution

The main result in this section is provided by the following theorem.

Theorem 4.1. *The N-soliton solution of the system of bilinear equations (4.4)-(4.7) is given in terms of the determinants*

$$f=|D|, \quad g_s = -|D(\mathbf{a}_s^*;\mathbf{z})|,\ s=1,2,...,m, \quad h_s = |D| + \frac{1}{k_s}|D(\mathbf{z}_s^*;\mathbf{z})|,\ s=1,2,...,n-m, \tag{4.10a}$$

$$D=(d_{jk})_{1\le j,k\le N}, \quad d_{jk} = \frac{z_j z_k^* + \frac{1}{2}(\mu - \mathrm{i}\gamma p_k^*)c_{jk}}{p_j+p_k^*}, \tag{4.10b}$$

$$z_j = \exp[p_j x + (\mathrm{i}p_j^2 - \hat{\lambda}\gamma p_j)t], \quad j=1,2,...,N, \tag{4.10c}$$

$$c_{jk} = \frac{\sum_{s=1}^m \sigma_s \alpha_{sj}\alpha_{sk}^*}{1+\frac{\gamma}{2}\sum_{s=1}^{n-m}\sigma_{m+s}\rho_s^2\,\frac{\mathrm{i}(p_j-p_k^*)+k_s+\frac{\mu}{\gamma}}{(p_j-\mathrm{i}k_s)(p_k^*+\mathrm{i}k_s)}}, \quad j,k=1,2,...,N, \tag{4.10d}$$

where $\mathbf{z}$ and $\mathbf{z}_s$ are N-component row vectors defined by (3.17b) *with z_j given by (4.10c) and*

$$\mathbf{a}_s = (\alpha_{s1}, \alpha_{s2}, ..., \alpha_{sN}), \quad s=1,2,...,m, \tag{4.10e}$$

are row vectors with elements $\alpha_{sj}\in\mathbb{C}$ $(s=1,2,..,m; j=1,2,..,N)$.

The N-soliton solution is characterized by the N complex parameters p_j $(j=1,2,...,N)$ and mN complex parameters α_{sj} $(s=1,2,...,m; j=1,2,...,N)$. The former parameters determine the amplitude of solitons and the latter ones determine the polarization and the envelope phases of solitons. Note that we have used the same symbol as that appears in theorem 3.1 for the tau functions f and h_s since no confusion would be likely to arise from this convention.

The proof of theorem 4.1 can be performed by using the differentiation rules for the tau functions f, g_s and h_s as well as the basic properties of determinants. Remark 3.2 is applied to the current problem, so that we can put $\mu = 0$ without loss of generality. First, we summarize the differentiation rules corresponding to those given by lemma 3.2 and lemma 3.3.

Lemma 4.1.

$$f_t = -\mathrm{i}|D(\mathbf{z}^*;\mathbf{z}_x)| + \mathrm{i}|D(\mathbf{z}_x^*;\mathbf{z})| + \hat{\lambda}\gamma|D(\mathbf{z}^*;\mathbf{z})|, \tag{4.11}$$

$$f_x = -|D(\mathbf{z}^*;\mathbf{z})|, \tag{4.12}$$

$$f_{xx} = -|D(\mathbf{z}_x^*;\mathbf{z})| - |D(\mathbf{z}^*;\mathbf{z}_x)|, \tag{4.13}$$

$$g_{s,t} = -|D(\mathbf{a}_s^*;\mathbf{z}_t)| + \mathrm{i}|D(\mathbf{a}_s^*,\mathbf{z}^*;\mathbf{z},\mathbf{z}_x)|, \tag{4.14}$$

$$g_{s,x} = -|D(\mathbf{a}_s^*;\mathbf{z}_x)|, \tag{4.15}$$

$$g_{s,xx} = -|D(\mathbf{a}_s^*;\mathbf{z}_{xx})| + |D(\mathbf{a}_s^*,\mathbf{z}^*;\mathbf{z}_x,\mathbf{z})|, \tag{4.16}$$

$$h_{s,t} = -\mathrm{i}|D(\mathbf{z}^*;\mathbf{z}_x)| + \mathrm{i}|D(\mathbf{z}_x^*;\mathbf{z})| + \hat{\lambda}\gamma|D(\mathbf{z}^*;\mathbf{z})| + \frac{1}{k_s}|D(\mathbf{z}_{s,t}^*;\mathbf{z})| + \frac{1}{k_s}|D(\mathbf{z}_s^*;\mathbf{z}_t)| - \frac{\mathrm{i}}{k_s}|D(\mathbf{z}_s^*,\mathbf{z}^*;\mathbf{z},\mathbf{z}_x)|, \tag{4.17}$$

$$h_{s,x} = -\mathrm{i}|D(\mathbf{z}_s^*;\mathbf{z})| + \frac{1}{k_s}|D(\mathbf{z}_s^*;\mathbf{z}_x)|, \tag{4.18}$$

$$h_{s,xx} = -\mathrm{i}|D(\mathbf{z}_{s,x}^*;\mathbf{z})| - \mathrm{i}|D(\mathbf{z}_s^*;\mathbf{z}_x)| + \frac{1}{k_s}|D(\mathbf{z}_{s,x}^*;\mathbf{z}_x)| + \frac{1}{k_s}|D(\mathbf{z}_s^*;\mathbf{z}_{xx})| - \frac{1}{k_s}|D(\mathbf{z}_s^*,\mathbf{z}^*;\mathbf{z}_x,\mathbf{z})|. \tag{4.19}$$

The above formulas can be derived by using (2.2) and some basic properties of determinants. See, for example, proof of (3.18).

The lemma 4.2 below provides the complex conjugate expressions of f, f_x, g_s and h_s.

Lemma 4.2.

$$f^* = |\bar{D}| = |(\bar{d}_{jk})_{1\leq j,k\leq N}|, \quad \bar{d}_{jk} = d_{jk} + \frac{\mathrm{i}\gamma}{2}\,c_{jk}, \tag{4.20}$$

$$f_x^* = -|\bar{D}(\mathbf{z}^*;\mathbf{z})|, \tag{4.21}$$

$$g_s^* = -|\bar{D}(\mathbf{z}^*; \mathbf{a}_s)|, \tag{4.22}$$

$$h_s^* = |\bar{D}| + \frac{1}{k_s}|\bar{D}(\mathbf{z}^*; \mathbf{z}_s)|. \tag{4.23}$$

Proof. It follows from a well-known property of the determinant that $f^* = |D^*| = |D^\dagger| = |(d_{kj}^*)_{1\leq j,k\leq N}|$. Referring to (4.10$b$), one obtains $d_{kj}^* = \frac{z_j z_k^* + \frac{1}{2}(\mu + \mathrm{i}\gamma p_j)c_{kj}^*}{p_j + p_k^*}$. Since $c_{kj}^* = c_{jk}$ by (4.10d), this expression is written as $d_{kj}^* = d_{jk} + (\mathrm{i}\gamma/2)c_{jk} = \bar{d}_{jk}$, which gives (4.20). The remaining relations can be proved in the same way. □

The following two lemmas will be used effectively in the proof of (4.6) and (4.7).

Lemma 4.3.

$$\sum_{s=1}^{n-m} \frac{\sigma_{m+s}\rho_s^2}{k_s^2}\,|D(\mathbf{z}_s^*; \mathbf{z})||\bar{D}(\mathbf{z}^*; \mathbf{z}_s)| = -|D(\tilde{\mathbf{z}}^*; \mathbf{z})||\bar{D}| - |\bar{D}(\mathbf{z}^*; \tilde{\mathbf{z}})||D|$$
$$+\frac{2\mathrm{i}}{\gamma}(|D(\mathbf{z}^*; \mathbf{z})||\bar{D}| - |\bar{D}(\mathbf{z}^*; \mathbf{z})||D|) - \sum_{s=1}^{m}\sigma_s|D(\mathbf{a}_s^*; \mathbf{z})||\bar{D}(\mathbf{z}^*; \mathbf{a}_s)|, \tag{4.24}$$

$$\sum_{s=1}^{n-m} \frac{\sigma_{m+s}\rho_s^2}{k_s^2}\,|D(\mathbf{z}_s^*; \mathbf{z}_x)||\bar{D}(\mathbf{z}^*; \mathbf{z}_s)| = -|D(\tilde{\mathbf{z}}^*; \mathbf{z}_x)||\bar{D}| - |\bar{D}(\mathbf{z}^*; \tilde{\mathbf{z}}_x)||D|$$
$$+\frac{2\mathrm{i}}{\gamma}(|D(\mathbf{z}^*; \mathbf{z}_x)||\bar{D}| - |\bar{D}(\mathbf{z}^*; \mathbf{z}_x)||D|) - \sum_{s=1}^{m}\sigma_s|D(\mathbf{a}_s^*; \mathbf{z}_x)||\bar{D}(\mathbf{z}^*; \mathbf{a}_s)|. \tag{4.25}$$

Here, $\tilde{\mathbf{z}} = (\tilde{z}_j)_{1\leq j\leq N}$ is an N-component row vector with elements $\tilde{z}_j = \sum_{s=1}^{n-m}\sigma_{m+s}\rho_s^2\frac{z_j}{p_j - \mathrm{i}k_s}$.

Lemma 4.4.

$$|D(\mathbf{z}_x^*; \mathbf{z})||\bar{D}| + |D(\mathbf{z}^*; \mathbf{z})||\bar{D}(\mathbf{z}^*; \mathbf{z})| + |\bar{D}(\mathbf{z}^*; \mathbf{z}_x)||D| = 0. \tag{4.26}$$

The proof of lemma 4.3 and lemma 4.4 will be given in appendix B and appendix C, respectively.

4.4. Proof of theorem 4.1

Here, we show that the tau functions (4.10) associated with the N-soliton solution solve the bilinear equations (4.4)-(4.7). A transparent proof will be presented with the aid of lemmas 4.3 and 4.4.

4.4.1. **Proof of (4.4)** Let $\hat{P}$ be the left-hand side of (4.4). Substituting (4.10a) and(4.11)-(4.16) into $\hat{P}$ and rearranging terms, one obtains

$$\hat{P} = -|D(\mathbf{a}_j^*; \mathrm{i}\mathbf{z}_t + \mathrm{i}\hat{\lambda}\gamma\mathbf{z}_x + \mathbf{z}_{xx})||D| - 2|D(\mathbf{a}_j^*, \mathbf{z}^*; \mathbf{z}, \mathbf{z}_x)||D|$$

$$+2(|D(\mathbf{a}_j^*; \mathbf{z})||D(\mathbf{z}^*; \mathbf{z}_x)| - |D(\mathbf{a}_j^*; \mathbf{z}_x)||D(\mathbf{z}^*; \mathbf{z})|).$$

The first term vanishes by virtue of the relation $\mathrm{i}\mathbf{z}_t + \mathrm{i}\hat{\lambda}\gamma\mathbf{z}_x + \mathbf{z}_{xx} = \mathbf{0}$ which comes from (4.10c). The sum of the second and third terms becomes zero by Jacobi's identity (2.4). □

4.4.2. **Proof of (4.5)** Let $\hat{Q}$ be the left-hand side of (4.5) and write it in the form $\hat{Q} = \hat{Q}_1 f - \hat{Q}_2 h_s - 2h_{s,x}f_x$ with

$$\hat{Q}_1 = \mathrm{i}h_{s,t} + \mathrm{i}(2k_s + \hat{\lambda}\gamma)h_{s.x} + h_{s,xx}, \quad \hat{Q}_2 = \mathrm{i}f_t + \mathrm{i}(2k_s + \hat{\lambda}\gamma)f_x - f_{xx}.$$

Using (4.17)-(4.19), $\hat{Q}_1$ recasts to

$$\hat{Q}_1 = |D(\mathbf{z}^* + (\mathrm{i}/k_s)(k_s + \hat{\lambda}\gamma)\mathbf{z}_s^* + (1/k_s)\mathbf{z}_{s,x}^*; \mathbf{z}_x)| + \frac{1}{k_s}|D(\mathbf{z}_s^*; \mathrm{i}\mathbf{z}_t + k_s(2k_s + \hat{\lambda}\gamma)\mathbf{z} + \mathbf{z}_{xx})|$$

$$+|D((\mathrm{i}/k_s)\mathbf{z}_{s,t}^* - \mathbf{z}_x^* - \mathrm{i}\mathbf{z}_{s,x}^*; \mathbf{z})| + \mathrm{i}\hat{\lambda}\gamma|D(\mathbf{z}^*; \mathbf{z})| + \frac{2}{k_s}|D(\mathbf{z}_s^*, \mathbf{z}^*; \mathbf{z}, \mathbf{z}_x)|.$$

It follows from the definition of $\mathbf{z}$ and $\mathbf{z}_s$ that

$$\mathrm{i}\mathbf{z}_t = -\mathbf{z}_{xx} + \mathrm{i}\hat{\lambda}\gamma\mathbf{z}_x, \quad \frac{\mathrm{i}}{k_s}\mathbf{z}_{s,t}^* - \mathbf{z}_x^* - \mathrm{i}\mathbf{z}_{s,x}^* = -\mathrm{i}(2k_s + \hat{\lambda}\gamma)\mathbf{z}^* - (2k_s + \hat{\lambda}\gamma)\mathbf{z}_s^*.$$

Taking into account the above relations, $\hat{Q}_1$ simplifies to

$$\hat{Q}_1 = 2\left\{|D(\mathbf{z}^*; \mathbf{z}_x)| - \mathrm{i}k_s|D(\mathbf{z}^*; \mathbf{z})| + \frac{1}{k_s}|D(\mathbf{z}_s^*, \mathbf{z}^*; \mathbf{z}, \mathbf{z}_x)|\right\}.$$

On the other hand, introducing (4.11)-(4.13) into $\hat{Q}_2$, one obtains

$$\hat{Q}_2 = 2\left\{|D(\mathbf{z}^*; \mathbf{z}_x)| - \mathrm{i}k_s|D(\mathbf{z}^*; \mathbf{z})|\right\}.$$

Last, after a computation using $\hat{Q}_1$ and $\hat{Q}_2$ above as well as (4.10a), (4.12) and (4.18), it is found that

$$\hat{Q} = \frac{2}{k_s}\left\{|D(\mathbf{z}_s^*, \mathbf{z}^*; \mathbf{z}, \mathbf{z}_x)||D| - |D(\mathbf{z}^*; \mathbf{z}_x)||D(\mathbf{z}_s^*; \mathbf{z})| + |D(\mathbf{z}_s^*; \mathbf{z}_x)||D(\mathbf{z}^*; \mathbf{z})|\right\},$$

which becomes zero by Jacobi's identity (2.4). □

4.4.3. **Proof of (4.6)** Let $\hat{R}$ be the left-hand side of (4.6). Substituting (4.10a), (4.12), (4.20), (4.22) and (4.23) into $\hat{R}$, one obtains

$$\hat{R} = -\left\{|D(\mathbf{z}^*;\mathbf{z})| + \frac{\mathrm{i}\gamma}{2}\sum_{s=1}^{n-m}\frac{\sigma_{m+s}\rho_s^2}{k_s}|D(\mathbf{z}_s^*;\mathbf{z})|\right\}|\bar{D}|$$

$$+\left\{|\bar{D}(\mathbf{z}^*;\mathbf{z})| - \frac{\mathrm{i}\gamma}{2}\sum_{s=1}^{n-m}\frac{\sigma_{m+s}\rho_s^2}{k_s}|\bar{D}(\mathbf{z}^*;\mathbf{z}_s)|\right\}|D|$$

$$-\frac{\mathrm{i}\gamma}{2}\sum_{s=1}^{n-m}\frac{\sigma_{m+s}\rho_s^2}{k_s^2}|D(\mathbf{z}_s^*;\mathbf{z})||\bar{D}(\mathbf{z}^*;\mathbf{z}_s)| - \frac{\mathrm{i}\gamma}{2}\sum_{s=1}^{n-m}\sigma_s|D(\mathbf{a}_s^*;\mathbf{z})||\bar{D}(\mathbf{z}^*;\mathbf{a}_s)|.$$

Use (4.24) for the third term and introduce the definition of $\tilde{\mathbf{z}}$ for the first and second terms. Then, one can see that $\hat{R} = 0$. □

4.4.4. **Proof of (4.7)** We add $\hat{R}_x(=0)$ to (4.7), replace the last term of (4.7) by $D_x f\cdot f^*$ from (4.6) and prove that $\hat{S} = 0$, where

$$\hat{S} = f_{xx}f^* - f_x f_x^* - \frac{\mathrm{i}\gamma}{2}\left(\sum_{s=1}^{m}\sigma_s g_{s,x}g_s^* + \sum_{s=1}^{n-m}\sigma_{m+s}\rho_s^2 h_{s,x}h_s^*\right)$$

$$+\frac{\gamma}{2}\sum_{s=1}^{n-m}\sigma_{m+s}k_s\rho_s^2(h_s h_s^* - ff^*) + \frac{\mathrm{i}\hat{\lambda}\gamma}{2}ff_x^*.$$

Performing a straightforward computation using (4.10), (4.12), (4.13), (4.15), (4.18) and (4.20)-(4.23) as well as the relations

$$-\frac{\mathrm{i}}{k_s}\mathbf{z}_{s,x}^* + \mathbf{z}_s^* = -\mathrm{i}\mathbf{z}^*, \quad \sum_{s=1}^{n-m}\sigma_{m+s}\rho_s^2\mathbf{z}_s = \mathrm{i}\hat{\lambda}\mathbf{z} - \mathrm{i}\tilde{\mathbf{z}}_x,$$

$$\sum_{s=1}^{n-m}\frac{\sigma_{m+s}\rho_s^2}{k_s}|D(\mathbf{z}_{s,x}^*;\mathbf{z})| = \sum_{s=1}^{n-m}\sigma_{m+s}\rho_s^2(|D(\mathbf{z}^*;\mathbf{z})| - \mathrm{i}|D(\mathbf{z}_s^*;\mathbf{z})|),$$

which follow directly from the definition of the vectors $\mathbf{z}$ and $\tilde{\mathbf{z}}$, $\hat{S}$ can be put into the form

$$\hat{S} = -\{|D(\mathbf{z}_x^*;\mathbf{z})| + |D(\mathbf{z}^*;\mathbf{z}_x)|\}|\bar{D}| - |D(\mathbf{z}^*;\mathbf{z})||\bar{D}(\mathbf{z}^*;\mathbf{z})|$$

$$-\frac{\mathrm{i}\gamma}{2}\Big\{|D(\tilde{\mathbf{z}}^*;\mathbf{z}_x)||\bar{D}| + |\bar{D}(\mathbf{z}^*;\tilde{\mathbf{z}}_x)||D|$$

$$+\sum_{s=1}^{n-m}\frac{\sigma_{m+s}\rho_s^2}{k_s^2}|D(\mathbf{z}_s^*;\mathbf{z}_x)||\bar{D}(\mathbf{z}^*;\mathbf{z}_\mathbf{s})| + \sum_{s=1}^{m}\sigma_s|D(\mathbf{a}_s^*;\mathbf{z}_x)||\bar{D}(\mathbf{z}^*;\mathbf{a}_s)|\Big\}.$$

In view of the relation (4.25), this expression reduces to

$$\hat{S} = -(|D(\mathbf{z}_x^*;\mathbf{z})||\bar{D}| + |D(\mathbf{z}^*;\mathbf{z})||\bar{D}(\mathbf{z}^*;\mathbf{z})| + |\bar{D}(\mathbf{z}^*;\mathbf{z}_x)||D|),$$

which becomes zero by (4.26). □

4.5. One-soliton solutions

If we put

$$p_1 = a + \mathrm{i}b, \quad z = \mathrm{e}^{a\xi + \mathrm{i}\{bx + (a^2 - b^2 - \hat{\lambda}\gamma b)t\}}, \quad \xi = x - (2b + \hat{\lambda}\gamma)t, \tag{4.27a}$$

the tau functions (4.10) for the one-soliton solutions are written in the form

$$f = \frac{1}{2a}\left\{zz^* + \frac{1}{2}(\mu - \gamma b - \mathrm{i}\gamma a)c_{11}\right\}, \quad g_j = \alpha_{j1}^* z, \quad j = 1, 2, ..., m,$$

$$h_j = f - \frac{zz^*}{a - \mathrm{i}(b - k_j)}, \quad j = 1, 2, ..., n - m. \tag{4.27b}$$

The parameter c_{11} from (4.10d) is given by

$$c_{11} = \frac{\sum_{s=1}^{m} \sigma_s \alpha_{s1}\alpha_{s1}^*}{1 + \frac{\gamma}{2}\sum_{s=1}^{n-m} \sigma_{m+s}\rho_s^2 \frac{-2b + k_s + \mu/\gamma}{a^2 + (b - k_s)^2}}. \tag{4.27c}$$

Note that c_{11} is real. This also follows directly from the Hermitian nature of c_{jk}. If one introduces the real quantities β, ϕ and θ_j by (3.30) and puts $(a\beta\gamma)^2 = \mathrm{e}^{-4a\xi_0}$, then one can express the one-soliton solutions compactly in terms of these parameters. After some manipulations, one finds that

$$|q_j|^2 = \frac{2a^2 \alpha_{j1}\alpha_{j1}^* \mathrm{e}^{2a\xi_0}}{\cosh\, 2a(\xi + \xi_0) - \mathrm{sgn}(a\gamma c_{11})\, \cos\, 2\phi}, \quad j = 1, 2, ..., m, \tag{4.28}$$

$$|q_{j+m}|^2 = \rho_j^2 \left[1 + \frac{\mathrm{sgn}(a\gamma c_{11})\{\cos\, 2(\theta_j + \phi) + \cos\, 2\phi\}}{\cosh\, 2a(\xi + \xi_0) - \mathrm{sgn}(a\gamma c_{11})\, \cos\, 2\phi}\right], \quad j = 1, 2, ..., n - m. \tag{4.29}$$

The components q_j from (4.28) take the form of bright-solitons with zero background whereas those of (4.29) represent the dark- or bright-solitons with nonzero background. A striking feature of the soliton solutions is that the parameters a and b can be chosen independently unlike the soliton solutions with the pure plane-wave boundary conditions discussed in section 3. It turns out that in the case of the mixed type boundary conditions, the explicit form of the N-soliton solution is available without solving algebraic equations.

If one takes the limit $\gamma \to 0$ under the condition $\mu c_{11} > 0$, then the expressions (4.28) and (4.29) reduce to the one-soliton solutions of the n-component NLS system

$$|q_j|^2 = a^2 \alpha_{j1}\alpha_{j1}^* \mathrm{e}^{2a\xi_0}\, \mathrm{sech}^2 a(\xi + \xi_0), \quad j = 1, 2, ..., m, \tag{4.30}$$

$$|q_{j+m}|^2 = \rho_j^2\left[1 - \frac{a^2}{a^2+(b-k_j)^2}\,\mathrm{sech}^2 a(\xi+\xi_0)\right], \quad j = 1, 2, ..., n-m. \tag{4.31}$$

Note that the condition $\mu c_{11} > 0$ assures the regularity of the solutions. Actually, if $\mu c_{11} < 0$, then the solutions exhibit a singularity at $\xi = -\xi_0$.

Remark 4.2. The N-soliton formula presented in theorem 4.1 have an alternative expressions. Indeed, in accordance with the procedure developed in [13], we can show that the tau functions given below satisfy the system of bilinear equations (4.4)-(4.7)

$$f = \begin{vmatrix} \hat{A} & I \\ -I & \hat{B} \end{vmatrix}, \quad g_s = \begin{vmatrix} \hat{A} & I & \mathbf{z}^T \\ -I & \hat{B} & \mathbf{0}^T \\ \mathbf{0} & -\mathbf{a}_s^* & 0 \end{vmatrix}, \ s = 1, 2, ..., m,$$

$$h_s = \begin{vmatrix} \hat{A} & I & \mathbf{z}^T \\ -I & \hat{B} & \mathbf{0}^T \\ \mathbf{z}_s^*/k_s & \mathbf{0} & 1 \end{vmatrix}, \ s = 1, 2, ..., n-m, \tag{4.32a}$$

$$\hat{A} = (\hat{a}_{jk})_{1\le j,k\le N}, \quad \hat{a}_{jk} = \frac{z_j z_k^*}{p_j + p_k^*}, \quad \hat{B} = (\hat{b}_{jk})_{1\le j,k\le N}, \quad \hat{b}_{jk} = \frac{\frac{1}{2}(\mu + \mathrm{i}\gamma p_k)c_{jk}}{p_j + p_k^*}, \tag{4.32b}$$

where the N-component row vectors $\mathbf{z}, \mathbf{z}_s$ and $\mathbf{a}_s$ are defined in (4.10) and I is an $N \times N$ unit matrix. It is noteworthy that the tau functions f and g_s have the same forms as those of the bright N-soliton solution presented in [13]. Indeed, if one puts $\rho_s = 0$ ($s = 1, 2, ..., n-m$), or equivalently $q_{m+s} = 0$ ($s = 1, 2, ..., n-m$), then $q_s = g_s f^*/f^2$ ($s = 1, 2, ..., m$) solve the system of PDEs (1.1) with the boundary conditions $q_s \to 0, |x| \to \infty$. Since m is an arbitrary positive integer, this gives another proof of the bright N-soliton solution.

5. Concluding remarks

The primary advantage of the direct method is that it is capable of providing a simple mean to obtain soliton solutions irrespective of the boundary conditions. The N-soliton formulas presented in this paper include as special cases the existing N-soliton solutions of the NLS and derivative NLS equations as well as the Manakov system with nonzero boundary conditions. The compact expressions of the soliton solutions are particularly useful for investigating their structures, asymptotic behaviors and dynamics. On the other hand, the IST is technically involved for nonzero boundary conditions, and hence it has been applied mainly to the multi-component system such as (1.1) under the restricted class of boundary conditions in which all the components have the same plane-wave boundary condition at infinity, for instance. The development of the IST for the multi-component systems with the general nonzero boundary conditions is still in progress. In particular, the derivation of the N-soliton solutions by means of the IST is an important issue left to the future.

In conclusion, it will be worthwhile to comment on an integrable system associated with the system (1.1). The multi-component Fokas-Lenells (FL) system

$$u_{j,xt} = u_j - \mathrm{i}\left\{\left(\sum_{s=1}^{n}\sigma_s u_{s,x}u_s^*\right)u_j + \left(\sum_{s=1}^{n}\sigma_s u_s u_s^*\right)u_{j,x}\right\}, \quad u_j = u_j(x,t) \in \mathbb{C}, \quad j = 1,2,...,n, \tag{5.1}$$

is an integrable multi-component generalization of the FL equation which describes the nonlinear propagation of short pulses in a monomode fiber [36, 37]. It belongs to the first negative flow of the multi-component derivative NLS hierarchy [38, 39]. The FL equation is a special case of the system (5.1) with $n = 1$. Its N-soliton solutions have been obtained by employing the direct method for both zero and plane-wave boundary conditions [40, 41]. In view of the above observation on an integrable hierarchy, the structure of the N-soliton solution of the multi-component FL system is closely related to that of the multi-component derivative NLS system. This statement has been confirmed for the one-component system [41]. Thus, the bilinearization and the construction of the N-soliton solution of the system (5.1) with zero and nonzero boundary conditions will be performed in accordance with the procedure developed in the present paper. This interesting issue is currently under study, and the results will be reported elsewhere.

Appendix A. Proof of lemma 3.4

Proof of (3.27). Let L_1 be the left-hand side of (3.27). Applying formula (2.3) with $\mathbf{z}_s$ from (3.17b) and manipulating the resultant expression, one can show that

$$\begin{aligned} L_1 &= \sum_{s=1}^{n}\frac{\sigma_s\rho_s^2}{k_s^2}|D(\mathbf{z}_s^*;\mathbf{z}_s)| \\ &= -\sum_{q,r=1}^{N}\frac{z_q z_r^*}{p_q+p_r^*}D_{qr}\sum_{s=1}^{n}\sigma_s\rho_s^2\left[\frac{\mathrm{i}\{\mathrm{i}(p_q-p_q^*)+k_s\}+p_q}{(p_q-\mathrm{i}k_s)(p_q^*+\mathrm{i}k_s)} - \frac{\mathrm{i}\{\mathrm{i}(p_r-p_r^*)+k_s\}-p_r^*}{(p_r-\mathrm{i}k_s)(p_r^*+\mathrm{i}k_s)}\right]. \end{aligned} \tag{A.1}$$

In view of (3.16), this expression simplifies to

$$L_1 = -\sum_{q,r=1}^{N}D_{qr}\{\beta_r z_q z_r^* + \mathrm{i}(\beta_q-\beta_r)d_{qr}\}, \tag{A.2}$$

where the definition of d_{jk} from (3.15b) and that β_j from (3.29) have been used. Referring to formulas (2.3) and (2.5) , (A.2) becomes $L_1 = |D(\mathbf{z}^*B;\mathbf{z})|$. □

Proof of (3.28). Let L_2 be the left-hand side of (3.28). Use the expansion formula (2.7) to obtain

$$\begin{aligned} L_2 &= \sum_{s=1}^{n} \frac{\sigma_s \rho_s^2}{k_s^2} \, |D(\mathbf{z}_s^*, \mathbf{a}^*; \mathbf{b}, \mathbf{z}_s)| \\ &= \sum_{s=1}^{n} \frac{\sigma_s \rho_s^2}{k_s^2} \sum_{\substack{p,q,j,k=1 \\ (p\neq q, j\neq k)}}^{N} b_p (\mathbf{z}_s)_q (\mathbf{z}_s^*)_j a_k^* D_{pq,jk}. \end{aligned} \tag{A.3}$$

A similar computation to that leading to $(A.2)$ gives

$$\begin{aligned} L_2 &= \sum_{\substack{p,q,j,k=1 \\ (p\neq q, j\neq k)}}^{N} b_p a_k^* D_{pq,jk} \{\beta_j z_q z_j^* + \mathrm{i}(\beta_q - \beta_j) d_{qj}\} \\ &= \sum_{\substack{p,q,j,k=1 \\ (p\neq q, j\neq k)}}^{N} (\beta_j z_j^*) a_k^* b_p z_q D_{pq,jk} - \mathrm{i} \sum_{p,k=1}^{N} b_p a_k^* \left\{ \sum_{\substack{q=1 \\ (q\neq p)}}^{N} \beta_q D_{pk} - \sum_{\substack{j=1 \\ (j\neq k)}}^{N} \beta_j D_{pk} \right\} \\ &= \sum_{\substack{p,q,j,k=1 \\ (p\neq q, j\neq k)}}^{N} (\beta_j z_j^*) a_k^* b_p z_q D_{pq,jk} + \mathrm{i} \sum_{p,k=1}^{N} b_p a_k^* (\beta_p D_{pk} - \beta_k D_{pk}). \end{aligned} \tag{A.4}$$

Referring to formulas (2.3) and (2.7) with (2.1), $(A.4)$ can be written in the form

$$L_2 = |D(\mathbf{z}^* B, \mathbf{a}^*; \mathbf{b}, \mathbf{z})| - \mathrm{i}(|D(\mathbf{a}^*; \mathbf{b}B)| - |D(\mathbf{a}^* B; \mathbf{b})|), \tag{$A.5$}$$

which is just the right-hand side of (3.28). □

Appendix B. Proof of lemma 4.3

Let L_3 be the left-hand side of (4.24). In view of formula (2.3) with $\mathbf{z}_s$ from (3.17b), L_3 becomes

$$L_3 = \sum_{s=1}^{n-m} \sigma_{m+s} \rho_s^2 \sum_{j,l,q,r=1}^{N} \frac{z_j z_l^* z_q z_r^*}{p_q + p_l^*} D_{jl} \bar{D}_{qr} \left(\frac{1}{p_l^* + \mathrm{i}k_s} + \frac{1}{p_q - \mathrm{i}k_s} \right).$$

It follows from the definition of d_{jk} from (4.10b) and $\bar{d}_{jk}$ from (4.20) that

$$\frac{z_q z_l^*}{p_q + p_l^*} = d_{ql} + \frac{\mathrm{i}\gamma}{2} \frac{p_l^* c_{ql}}{p_q + p_l^*} = \bar{d}_{ql} - \frac{\mathrm{i}\gamma}{2} \frac{p_q c_{ql}}{p_q + p_l^*}.$$

Substituting this expression into L_3, using formula (2.5) and referring to formula (2.3), one obtains

$$L_3 = -|D(\tilde{\mathbf{z}}^*;\mathbf{z})||\bar{D}| - |\bar{D}(\mathbf{z}^*;\tilde{\mathbf{z}})||D| - \frac{\gamma}{2}\sum_{j,l,q,r=1}^{N}\sum_{s=1}^{n-m}\sigma_{m+s}\rho_s^2\,\frac{\mathrm{i}(p_q-p_l^*)+k_s}{(p_q-\mathrm{i}k_s)(p_l^*+\mathrm{i}k_s)}\,c_{ql}D_{jl}\bar{D}_{qr}z_jz_r^*.$$

The relation

$$\frac{\gamma}{2}\sum_{s=1}^{n-m}\sigma_{m+s}\rho_s^2\,\frac{\mathrm{i}(p_q-p_l^*)+k_s}{(p_q-\mathrm{i}k_s)(p_l^*+\mathrm{i}k_s)}\,c_{ql} = -c_{ql} + \sum_{s=1}^{m}\sigma_s\alpha_{sq}\alpha_{sl}^*,$$

follows directly from (4.10d), which simplifies the third term of L_3 as

$$\frac{\gamma}{2}\sum_{j,l,q,r=1}^{N}\sum_{s=1}^{n-m}\sigma_{m+s}\rho_s^2\,\frac{\mathrm{i}(p_q-p_l^*)+k_s}{(p_q-\mathrm{i}k_s)(p_l^*+\mathrm{i}k_s)}\,c_{ql}D_{jl}\bar{D}_{qr}z_jz_r^*$$

$$= -\sum_{j,l,q,r=1}^{N}c_{ql}D_{jl}\bar{D}_{qr}z_jz_r^* + \sum_{j,l,q,r=1}^{N}\sum_{s=1}^{m}\sigma_s\alpha_{sq}\alpha_{sl}^*D_{jl}\bar{D}_{qr}z_jz_r^*.$$

Using the relation $c_{ql} = -(2\mathrm{i}/\gamma)(\bar{d}_{ql}-d_{ql})$ from (4.20) and applying formulas (2.3) and (2.5), one can rewrite the above expression in terms of bordered determinans. Explicitly, it reads

$$\frac{\gamma}{2}\sum_{j,l,q,r=1}^{N}\sum_{s=1}^{n-m}\sigma_{m+s}\rho_s^2\,\frac{\mathrm{i}(p_q-p_l^*)+k_s}{(p_q-\mathrm{i}k_s)(p_l^*+\mathrm{i}k_s)}\,c_{ql}D_{jl}\bar{D}_{qr}z_jz_r^*$$

$$= -\frac{2\mathrm{i}}{\gamma}(|D(\mathbf{z}^*;\mathbf{z})||\bar{D}| - |\bar{D}(\mathbf{z}^*;\mathbf{z})||D|) + \sum_{s=1}^{m}\sigma_s|D(\mathbf{a}_s^*;\mathbf{z})||\bar{D}(\mathbf{z}^*;\mathbf{a}_s)|.$$

Last, on substituting this relation into L_3, one can see that it coinsides with the right-hand side of (4.24). The proof of (4.25) parallels completely to that of (4.24). □

Appendix C. Proof of lemma 4.4

Introduce an $N\times N$ matrix $\tilde{C} = (\tilde{c}_{jk})_{1\le j,k\le N}$ whose elements are defined by the relations

$$c_{jk} = \frac{2(p_j+p_k^*)z_jz_k^*}{\mathrm{i}\gamma p_jp_k^*}\left(\tilde{c}_{jk} - \frac{p_k^*}{p_j+p_k^*}\right), \quad j,k = 1,2,...,N.$$

Consequently, $d_{jk} = (1-\tilde{c}_{jk})z_jz_k^*/p_j$, $\bar{d}_{jk} = \tilde{c}_{jk}z_jz_k^*/p_k^*$. The determinants associated with the matrices D and $\bar{D}$ are expressed by the matrix $\tilde{C}$ as follows:

$$|D| = \kappa|U-\tilde{C}|, \quad |\bar{D}| = \kappa|\tilde{C}|, \quad |D(\mathbf{z}_x^*;\mathbf{z})| = \kappa\begin{vmatrix} U-\tilde{C} & \mathbf{p}^T \\ \mathbf{p}^* & 0\end{vmatrix}, \quad |\bar{D}(\mathbf{z}^*;\mathbf{z}_x)| = \kappa\begin{vmatrix}\tilde{C} & \mathbf{p}^T \\ \mathbf{p}^* & 0\end{vmatrix},$$

$$|D(\mathbf{z}^*;\mathbf{z})| = \kappa \begin{vmatrix} U-\tilde{C} & \mathbf{p}^T \\ \mathbf{1} & 0 \end{vmatrix}, \quad |\bar{D}(\mathbf{z}^*;\mathbf{z})| = \kappa \begin{vmatrix} \tilde{C} & \mathbf{1}^T \\ \mathbf{p}^* & 0 \end{vmatrix}, \quad \kappa = \prod_{j=1}^{N}(z_j z_j^*/p_j), \qquad (C.1)$$

where U is an $N \times N$ matrix whose elements are all unity, and $\mathbf{p}$ and $\mathbf{1}$ are N-component row vectors given by $\mathbf{p} = (p_1, p_2, ..., p_N)$, $\mathbf{1} = (1, 1, ..., 1)$. The above relations are verified by using a few basic properties of determinants. Substituting $(C.1)$ into the left-hand side of (4.26) which is denoted by L_4, one finds that

$$L_4 = \kappa\kappa^* \left\{ \begin{vmatrix} U-\tilde{C} & \mathbf{p}^T \\ \mathbf{p}^* & 0 \end{vmatrix} |\tilde{C}| + \begin{vmatrix} U-\tilde{C} & \mathbf{p}^T \\ \mathbf{1} & 0 \end{vmatrix} \begin{vmatrix} \tilde{C} & \mathbf{1}^T \\ \mathbf{p}^* & 0 \end{vmatrix} + \begin{vmatrix} \tilde{C} & \mathbf{p}^T \\ \mathbf{p}^* & 0 \end{vmatrix} |U-\tilde{C}| \right\}. \qquad (C.2)$$

By an elementary computation, one can show that all the expressions in $(C.2)$ can be written by the bordered deternminants associated with the matrix $\tilde{C}$. Explicitly,

$$\begin{vmatrix} U-\tilde{C} & \mathbf{p}^T \\ \mathbf{p}^* & 0 \end{vmatrix} = (-1)^{N-1}(|\tilde{C}(\mathbf{p}^*;\mathbf{p})| + |\tilde{C}(\mathbf{p}^*,\mathbf{1};\mathbf{p},\mathbf{1})|), \quad \begin{vmatrix} U-\tilde{C} & \mathbf{p}^T \\ \mathbf{1} & 0 \end{vmatrix} = (-1)^{N-1}|\tilde{C}(\mathbf{1};\mathbf{p})|,$$

$$|U-\tilde{C}| = (-1)^N(|\tilde{C}| + |\tilde{C}(\mathbf{1};\mathbf{1})|). \qquad (C.3)$$

Substitution of (C.3) into (C.2) gives

$$L_4 = (-1)^{N-1}\kappa\kappa^*(|\tilde{C}(\mathbf{p}^*,\mathbf{1};\mathbf{p},\mathbf{1})||\tilde{C}| + |\tilde{C}(\mathbf{1};\mathbf{p})||\tilde{C}(\mathbf{p}^*;\mathbf{1})| - |\tilde{C}(\mathbf{p}^*;\mathbf{p})||\tilde{C}(\mathbf{1};\mathbf{1})|).$$

This expression becomes zero by Jacobi's identity (2.4). □

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