

Mathematics and Sudoku VIII

KITAMOTO Takuya, WATANABE Tadashi *

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We discuss on the worldwide famous Sudoku by using mathematical approach. This paper is the 8th paper in our series, so we use the same notations and terminologies in [1], [2] without any descriptions.

13. A block coordinate system and a block structural system.

In this section we introduce a new useful coordinate system in Sudoku.

Ordinally we use the following notations and terminologies as follows:

We already use the basic notation $J = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. However, we change it as $J = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$. By this change, we does not change our previous arguments and proofs in our Propositions.

Let $Z = \{0, 1, 2\}$, $Z = Z_1 = Z_2$ and $U = Z_2 \times Z_1$. We define maps DEC, EXP as follows:

$$\begin{aligned} DEC: J &= \{0, 1, 2, 3, 4, 5, 6, 7, 8\} \rightarrow U = Z_2 \times Z_1 \\ DEC(n) &= (n_2, n_1) \text{ for each } n \in J \\ n &= 3 \times n_2 + n_1, 0 \leq n_2 \leq 2, 0 \leq n_1 \leq 2 \\ EXP: U &= Z_2 \times Z_1 \rightarrow J = \{0, 1, 2, 3, 4, 5, 6, 7, 8\} \\ EXP(n_2, n_1) &= 3 \times n_2 + n_1 \text{ for each } (n_2, n_1) \in U. \end{aligned}$$

We can easily show the following:

Proposition 76.

$$DEC \circ EXP = 1 \text{ and } EXP \circ DEC = 1.$$

We put $J_1 = J_2 = J$, $Z_{(1,1)} = Z_{(1,2)} = Z_{(2,1)} = Z_{(2,2)} = Z$. We define maps as follows:

$$\begin{aligned} DEC_1: J_1 &\rightarrow U_1 = Z_{(1,2)} \times Z_{(1,1)}, \\ DEC_2: J_2 &\rightarrow U_2 = Z_{(2,2)} \times Z_{(2,1)}, \\ DEC_1(i) &= (i_2, i_1), i = 3 \times i_2 + i_1, 0 \leq i_2 \leq 2, 0 \leq i_1 \leq 2 \\ DEC_2(j) &= (j_2, j_1), j = 3 \times j_2 + j_1, 0 \leq j_2 \leq 2, 0 \leq j_1 \leq 2 \\ EXP_1: U_1 &= Z_{(1,2)} \times Z_{(1,1)} \rightarrow J_1, \\ EXP_2: U_2 &= Z_{(2,2)} \times Z_{(2,1)} \rightarrow J_2 \\ EXP_1(i_2, i_1) &= 3 \times i_2 + i_1, \\ EXP_2(j_2, j_1) &= 3 \times j_2 + j_1. \end{aligned}$$

Next we define the map:

$$\begin{aligned} 1 \times EXC_{(2,3)} \times 1: Z \times Z \times Z \times Z &\rightarrow Z \times Z \times Z \times Z \\ 1 \times EXC_{(2,3)} \times 1(t_1, t_2, t_3, t_4) &= (t_1, t_3, t_2, t_4) \\ \text{for each } (t_1, t_2, t_3, t_4) &\in Z \times Z \times Z \times Z \end{aligned}$$

Now we define $B_2 = Z_{(1,2)} \times Z_{(2,2)}$, $B_1 = Z_{(1,1)} \times Z_{(2,1)}$ and maps

$$\begin{aligned} BDEC: J_1 \times J_2 &\rightarrow B_2 \times B_1, \\ BEXP: B_2 \times B_1 &\rightarrow J_1 \times J_2 \text{ by} \\ BDEC &= (1 \times EXC_{(2,3)} \times 1) \circ (DEC_1 \times DEC_2), \\ BEXP &= (EXP_1 \times EXP_2) \circ (1 \times EXC_{(2,3)} \times 1). \end{aligned}$$

More clearly we can show that

$$\begin{aligned} BDEC(i, j) &= ((i_2, j_2), (i_1, j_1)), \\ i &= 3 \times i_2 + i_1, 0 \leq i_2 \leq 2, 0 \leq i_1 \leq 2 \\ j &= 3 \times j_2 + j_1, 0 \leq j_2 \leq 2, 0 \leq j_1 \leq 2 \\ BEXP((i_2, j_2), (i_1, j_1)) &= (3 \times i_2 + i_1, 3 \times j_2 + j_1). \end{aligned}$$

By Proposition 76 we can easily show that

Proposition 77.

$$BEXP \circ BDEC = 1 \text{ and } BDEC \circ BEXP = 1.$$

Let $J_4 = J_5 = J$, we define maps,

$$\begin{aligned} DEC_4: J_4 &\rightarrow B_2 = Z_{(1,2)} \times Z_{(2,2)}, \\ DEC_5: J_5 &\rightarrow B_1 = Z_{(1,1)} \times Z_{(2,1)} \text{ by} \\ DEC_4(u) &= (u_2, u_1), \\ u &= 3 \times u_2 + u_1, 0 \leq u_2 \leq 2, 0 \leq u_1 \leq 2, \\ DEC_5(v) &= (v_2, v_1), \\ v &= 3 \times v_2 + v_1, 0 \leq v_2 \leq 2, 0 \leq v_1 \leq 2, \\ EXP_4: B_2 &= Z_{(1,2)} \times Z_{(2,2)} \rightarrow J_4, \\ EXP_5: B_1 &= Z_{(1,1)} \times Z_{(2,1)} \rightarrow J_5, \\ EXP_4(i_2, j_2) &= 3 \times i_2 + j_2, \\ EXP_5(i_1, j_1) &= 3 \times i_1 + j_1. \end{aligned}$$

We define maps:

$$\begin{aligned} DEC^* &= DEC_4 \times DEC_5: J_4 \times J_5 \rightarrow B_2 \times B_1 \\ EXP^* &= EXP_4 \times EXP_5: B_2 \times B_1 \rightarrow J_4 \times J_5. \end{aligned}$$

And also we define maps:

$$\begin{aligned} CDEX &= EXP^* \circ BDEC: J_1 \times J_2 \rightarrow J_4 \times J_5 \\ CEXP &= BEXP \circ DEC^*: J_4 \times J_5 \rightarrow J_1 \times J_2. \end{aligned}$$

Let $TRS: X \times Y \rightarrow Y \times X$ be a trasposed map.

$$TRS(x, y) = (y, x) \text{ for each } (x, y) \in X \times Y.$$

*Emeritus professor Yamaguchi University, Yamaguchi City, 753, Japan.

We define maps

$$CTDEX = TRS \circ CDEC \circ TRS: J_2 \times J_1 \rightarrow J_5 \times J_4$$

$$CTEXP = TRS \circ CEXP \circ TRS: J_5 \times J_4 \rightarrow J_2 \times J_1.$$

More clearly we can show that

$$CDEC(i, j) = (3 \times i_2 + j_2, 3 \times i_1 + j_1)$$

$$CEXP(u, v) = (3 \times u_2 + v_2, 3 \times u_1 + v_1)$$

$$CTDEC(i, j) = (3 \times j_1 + i_1, 3 \times j_2 + i_2)$$

$$CTEXP(u, v) = (3 \times v_1 + u_1, 3 \times v_2 + u_2)$$

$$\text{for } i, j, u, v \in J, \quad i = 3 \times i_2 + i_1, \quad j = 3 \times j_2 + j_1,$$

$$u = 3 \times u_2 + u_1, \quad v = 3 \times v_2 + v_1.$$

By definitions, we can easily show that

Proposition 78.

- (a) $EXP^* \circ DEC^* = 1$ and $DEC^* \circ EXP^* = 1$.
- (b) $CEXP \circ CDEC = 1$ and $CDEC \circ CEXP = 1$.
- (c) $CEXP = CDEC$.
- (d) $CTEXP \circ CTDEC = 1$, $CTDEC \circ CTEXP = 1$.
- (e) $CTEXP = CTDEC$.

We say that $J_4 \times J_5$ is a block coordinate system, and $B_2 \times B_1$ is a block structural system. $B_2 \times B_1$ has more rich mathematical structures than $J_4 \times J_5$.

Usually, we use the following notations:

$$\text{Let } I(0) = \{0, 1, 2\}, I(1) = \{3, 4, 5\}, I(2) = \{6, 7, 8\},$$

$$B(0) = I(0) \times I(0), B(1) = I(0) \times I(1), B(2) = I(0) \times I(2),$$

$$B(3) = I(1) \times I(0), B(4) = I(1) \times I(1), B(5) = I(1) \times I(2),$$

$$B(6) = I(2) \times I(0), B(7) = I(2) \times I(1), B(8) = I(2) \times I(2).$$

These notations are formulated as following simple rules:

$$I(i_2) = \{3 \times i_2 + 0, 3 \times i_2 + 1, 3 \times i_2 + 2\}, \text{ for each } 0 \leq i_2 \leq 2,$$

$$I(i_2) \times I(j_2) = B(3 \times i_2 + j_2) \text{ for each } b_2 = (i_2, j_2) \in B_2.$$

Therefore, we use the notation:

$$I(b_2) = I(i_2) \times I(j_2) = B(EXP(b_2))$$

$$\text{for each } b_2 = (i_2, j_2) \in B_2.$$

Proposition 79.

$$\text{Let } i, j, s \in J, \quad DEC(i) = (i_2, i_1), DEC(j) = (j_2, j_1), DEC(s)$$

$$= (s_2, s_1), b_2 = (i_2, j_2) \in B_2. \text{ We have that}$$

$$(a) \quad DEC(I(i_2)) = \{i_2\} \times Z_1, \quad DEC(J) = Z_2 \times Z_1,$$

$$EXP(\{i_2\} \times Z_1) = I(i_2),$$

$$EXP(Z_2 \times \{i_1\}) = \{i_1, i_1 + 3, i_1 + 6\},$$

$$EXP(Z_2 \times Z_1) = J.$$

$$(b) \quad BDEC(I(b_2)) = \{b_2\} \times B_1,$$

$$BDEC(\{i\} \times J_2) = \{i_2\} \times Z_{(2,2)} \times \{i_1\} \times Z_{(1,1)},$$

$$BDEC(J_1 \times \{j\}) = Z_{(1,2)} \times \{j_2\} \times Z_{(1,1)} \times \{j_1\},$$

$$(c) \quad CDEC(B(s)) = \{s\} \times J_5$$

$$CDEC(\{i\} \times J_2) = B(i).$$

$$CDEC(J_1 \times \{j\}) = \{j_2, j_2 + 3, j_2 + 6\} \times \{j_1, j_1 + 3, j_1 + 6\}.$$

$$(d) \quad CTDEC(B(s)) = J_4 \times \{3 \times s_1 + s_2\},$$

$$CTDEC(\{i\} \times J_2) = \{i_1, i_1 + 3, i_1 + 6\} \times \{i_2, i_2 + 3, i_2 + 6\}$$

$$CTDEC(J_1 \times \{j\}) = B(3 \times j_1 + j_2).$$

Proof.

We show (a).

$$DEC(I(i_2))$$

$$= DEC(\{3 \times i_2 + 0, 3 \times i_2 + 1, 3 \times i_2 + 2\})$$

$$= \{DEC(3 \times i_2), DEC(3 \times i_2 + 1), DEC(3 \times i_2 + 2)\}$$

$$= \{(i_2, 0), (i_2, 1), (i_2, 2)\} = \{i_2\} \times Z_1,$$

$$DEC(J) = DEC(I(0) \cup I(1) \cup I(2))$$

$$= DEC(I(0)) \cup DEC(I(1)) \cup DEC(I(2))$$

$$= \{0\} \times Z_1 \cup \{1\} \times Z_1 \cup \{2\} \times Z_1 = Z_2 \times Z_1,$$

$$EXP(\{i_2\} \times Z_1) = EXP(\{(i_2, 0), (i_2, 1), (i_2, 2)\})$$

$$= \{3 \times i_2 + 0, 3 \times i_2 + 1, 3 \times i_2 + 2\} = I(i_2),$$

$$EXP(Z_2 \times \{i_1\}) = EXP(\{(0, i_1), (1, i_1), (2, i_1)\})$$

$$= \{3 \times 0 + i_1, 3 \times 1 + i_1, 3 \times 2 + i_1\} = \{i_1, i_1 + 3, i_1 + 6\},$$

$$EXP(Z_2 \times Z_1) = EXP(\{0\} \times Z_1 \cup \{1\} \times Z_1 \cup \{2\} \times Z_1)$$

$$= I(0) \cup I(1) \cup I(2) = J.$$

We show (b).

By (a) we have that

$$BDEC(I(b_2)) = BDEC(I(i_2) \times I(j_2))$$

$$= (1 \times EXC_{(2,3)} \times 1)(DEC(I(i_2)) \times DEC(I(j_2)))$$

$$= (1 \times EXC_{(2,3)} \times 1)(\{i_2\} \times Z_{(1,1)} \times \{j_2\} \times Z_{(2,1)})$$

$$= \{i_2\} \times \{j_2\} \times Z_{(1,1)} \times Z_{(2,1)}$$

$$= \{(i_2, j_2)\} \times Z_{(1,1)} \times Z_{(2,1)} = \{b_2\} \times B_1,$$

$$BDEC(\{i\} \times I(j_2))$$

$$= (1 \times EXC_{(2,3)} \times 1)(DEC(i), DEC(I(j_2)))$$

$$= (1 \times EXC_{(2,3)} \times 1)(\{(i_2, i_1)\} \times \{j_2\} \times Z_{(2,1)})$$

$$= \{(i_2, j_2)\} \times \{i_1\} \times Z_{(2,1)} = \{i_2\} \times \{j_2\} \times \{i_1\} \times Z_{(2,1)}.$$

Since $J_2 = I(0) \cup I(1) \cup I(2)$, by (2) we have that

$$BDEC(\{i\} \times J_2)$$

$$= BDEC(\{i\} \times I(0) \cup \{i\} \times I(1) \cup \{i\} \times I(2))$$

$$= BDEC(\{i\} \times I(0)) \cup BDEC(\{i\} \times I(1))$$

$$\cup BDEC(\{i\} \times I(2))$$

$$= \{(i_2, 0)\} \times \{i_1\} \times Z_{(2,1)} \cup \{(i_2, 1)\} \times \{i_1\} \times Z_{(2,1)}$$

$$\cup \{(i_2, 2)\} \times \{i_1\} \times Z_{(2,1)}$$

$$= \{(i_2, 0), (i_2, 1), (i_2, 2)\} \times \{i_1\} \times Z_{(2,1)}$$

$$= \{i_2\} \times Z_{(2,2)} \times \{i_1\} \times Z_{(2,1)}.$$

$$BDEC(I(i_2) \times \{j\})$$

$$= (1 \times EXC_{(2,3)} \times 1)(DEC(I(i_2)), DEC(\{j\}))$$

$$\begin{aligned}
 &= (1 \times \text{EXC}_{(2,3)} \times 1)(\{i_2\} \times Z_{(1,1)} \times \{(j_2, j_1)\}) \\
 &= \{i_2\} \times \{j_2\} \times Z_{(1,1)} \times \{j_1\} = \{(i_2, j_2)\} \times Z_{(1,1)} \times \{j_1\}.
 \end{aligned}$$

Since $J_1 = I(0) \cup I(1) \cup I(2)$, we have

$$\begin{aligned}
 &BDEC(J_1 \times \{j\}) \\
 &= BDEC(I(0) \times \{j\}) \cup BDEC(I(1) \times \{j\}) \\
 &\quad \cup BDEC(I(2) \times \{j\}) \\
 &= \{(0, j_2)\} \times Z_{(1,1)} \times \{j_1\} \cup \{(1, j_2)\} \times Z_{(1,1)} \times \{j_1\} \\
 &\quad \cup \{(2, j_2)\} \times Z_{(1,1)} \times \{j_1\} \\
 &= \{(0, j_2), (1, j_2), (2, j_2)\} \times Z_{(1,1)} \times \{j_1\} \\
 &= Z_{(1,2)} \times \{j_2\} \times Z_{(1,1)} \times \{j_1\}.
 \end{aligned}$$

We show (c).

We can take $b_{20} = (s_2, s_1) \in B_2$ and $\text{EXP}(b_{20}) = s$. By (b) we have that

$$BDEC(I(b_{20})) = \{b_{20}\} \times B_1.$$

By Proposition 79, we have that

$$\begin{aligned}
 CDEC(B(s)) &= CDEC(I(b_{20})) \\
 &= (\text{EXP} \times \text{EXP}) \circ BDEC(I(b_{20})) \\
 &= (\text{EXP} \times \text{EXP})(\{b_{20}\} \times B_1) \\
 &= \{\text{EXP}(b_{20})\} \times \text{EXP}(B_1) = \{s\} \times J_5.
 \end{aligned}$$

$$\begin{aligned}
 CDEC(\{i\} \times J_2) &= (\text{EXP} \times \text{EXP}) \circ BDEC(\{i\} \times J_2) \\
 &= (\text{EXP} \times \text{EXP})(\{i_2\} \times Z_{(2,2)} \times \{i_1\} \times Z_{(1,1)}) \\
 &= \text{EXP}(\{i_2\} \times Z_{(2,2)}) \times \text{EXP}(\{i_1\} \times Z_{(1,1)}) \\
 &= I(i_2) \times I(i_1) = B(3 \times i_2 + i_1) = B(i).
 \end{aligned}$$

$$\begin{aligned}
 CDEC(J_1 \times \{j\}) &= (\text{EXP} \times \text{EXP}) \circ BDEC(J_1 \times \{j\}) \\
 &= (\text{EXP} \times \text{EXP})(Z_{(1,2)} \times \{j_2\} \times Z_{(1,1)} \times \{j_1\}) \\
 &= \text{EXP}(Z_{(1,2)} \times \{j_2\}) \times \text{EXP}(Z_{(1,1)} \times \{j_1\}) \\
 &= \{j_2, j_2 + 3, j_2 + 6\} \times \{j_1, j_1 + 3, j_1 + 6\}.
 \end{aligned}$$

We show (d).

$$\begin{aligned}
 CTDEX(B(s)) &= \text{TRS} \circ CDEC \circ \text{TRS}(I(s_2) \times I(s_1)) \\
 &= \text{TRS} \circ CDEC(I(s_1) \times I(s_2)) \\
 &= \text{TRS} \circ CDEC(B(3 \times s_1 + s_2)) \\
 &= \text{TRS}(\{3 \times s_1 + s_2\} \times J_5) = J_4 \times \{3 \times s_1 + s_2\}.
 \end{aligned}$$

$$\begin{aligned}
 CTDEC(\{i\} \times J_2) &= \text{TRS} \circ CDEC \circ \text{TRS}(\{i\} \times J_2) \\
 &= \text{TRS} \circ CDED(J_1 \times \{i\}) \\
 &= \text{TRS}(\{i_2, i_2 + 3, i_2 + 6\} \times \{i_1, i_1 + 3, i_1 + 6\}) \\
 &= \{i_1, i_1 + 3, i_1 + 6\} \times \{i_2, i_2 + 3, i_2 + 6\}.
 \end{aligned}$$

$$\begin{aligned}
 CTDEX(J_1 \times \{j\}) &= \text{TRS} \circ CDEC \circ \text{TRS}(J_1 \times \{j\}) \\
 &= \text{TRS} \circ CDEC(\{j\} \times J_2) = \text{TRS}(B(j)) \\
 &= \text{TRS}(I(j_2) \times I(j_1)) = I(j_1) \times I(j_2) = B(3 \times j_1 + j_2).
 \end{aligned}$$

By Proposition 79 we have the following:

Proposition 80.

(a) $CDEC: J_1 \times J_2 \rightarrow J_4 \times J_5$ has the properties:

- (i) All rows are transformed into blocks.
- (ii) All blocks are transformed into rows.

(b) $CTDEC: J_1 \times J_2 \rightarrow J_4 \times J_5$ has the properties:

- (i) All columns are transformed into blocks.
- (ii) All blocks are transformed into columns.

14. Basic sets.

Let $T \subset J_1 \times J_2$. A set T is a basic set provided that it satisfies the following condition:

(BA) $|T \cap S| = 1$ for each $S \in BLK$.

Then, we put that $T \cap S = \{t(S)\}$ for each $S \in BLK$.

Here, $BLK = rOW \cup cOL \cup bLK$, which is the collection of all rows, all columns and all 3×3 blocks in $J_1 \times J_2$.

Now, for each $b_2 = (i_2, j_2) \in B_2$ we put

$I(b_2) = I(i_2) \times I(j_2) \subset J_1 \times J_2$. Then $bLK = \{I(b_2) : b_2 \in B_2\}$.

Proposition 81.

If T is a basic set, then we have that

(i) $T = \{t(S) : S \in bLK\}$, where $T \cap S = \{t(S)\}$ for each $S \in bLK$.

(ii) $|T| = 9$.

Proof. Since we have 9 blocks, by (BA) for each block S , $|S \cap T| = 1$. Thus

(1) $S \cap T = \{t(S)\}$ for each block $S \in bLK$.

Claim 1.

$T = \{t(S) : S \in bLK\}$

For each block S , by (1) we have that

(2) $\{t(S) : S \in bLK\} \subset T$.

We show that

(3) $T \subset \{t(S) : S \in bLK\}$.

We assume that (3) does not hold. Then we have a $t_0 \in T$ such that

(4) $t_0 \in T - \{t(S) : S \in bLK\}$.

Since $t_0 \in T \subset J_1 \times J_2$, there is a block $S_0 \in bLK$ such that

(5) $t_0 \in S_0$.

By (1), we have that

(6) $t(S_0) \in S_0 \cap T$.

By (5), (6) we have that

(7) $t_0, t(S_0) \in S_0 \cap T$.

By (BA) and (7) we have that

(8) $t_0 = t(S_0)$.

By (4) we have that $t_0 \notin \{t(S) : S \in bLK\}$, thus

(9) $t_0 \neq t(S_0)$.

(9) contradicts to (8). Hence we have (3).

By (2) and (3) we have Claim 1.

We show that

$$(10) \ t(S_1) \not\equiv t(S_2) \text{ for blocks } S_1, S_2 \in bLK \text{ with } S_1 \equiv S_2.$$

We assume that (10) does not hold.

Then we have blocks $S_1, S_2 \in bLK$ such that

$$(11) \ S_1 \equiv S_2 \text{ and}$$

$$(12) \ t(S_1) = t(S_2).$$

By (1), (12) we have that $S_1 \ni t(S_1) = t(S_1) \in S_2$. Thus

$$(13) \ S_1 \cap S_2 \equiv \phi.$$

By (11) we have that

$$(14) \ S_1 \cap S_2 = \phi.$$

(14) contradicts to (13). Hence, (10) does hold.

By (10) we have that

$$(15) \ |\{t(S): S \in bLK\}| = |bLK| = 9.$$

By (15) and Claim 1 we have that

$$(16) \ |T| = 9.$$

Hence, we have Proposition 81.

Let $B = B_2 \times B_1$ be the block structural system.

Here $B_1 = Z_{(1,1)} \times Z_{(2,1)}$ and $BDEC: J_1 \times J_2 \rightarrow B$, $BEXP: B \rightarrow J_1 \times J_2$ are the block decomposition map, the block expansion map, respectively.

Proposition 82.

Let T be a basic set. There is the unique map $\psi: B_2 \rightarrow B_1$ such that

$$T \cap (I(i_2) \times I(j_2)) = \{BEXP(b_2, \psi(b_2))\}$$

for each $b_2 = (i_2, j_2) \in B_2$.

Proof. Take any $b_2 = (i_2, j_2) \in B_2 = Z_{(1,2)} \times Z_{(2,2)}$. Let $I(b_2) = I(i_2) \times I(j_2)$ be the block in $J_1 \times J_2$. We put $T \cap T(b_2) = \{t(b_2)\}$. Thus

(1) $T \cap I(b_2) = T \cap (I(i_2) \times I(j_2)) = \{t(i_2, j_2)\} = \{t(b_2)\}$ for each $b_2 = (i_2, j_2) \in bLK$.

Since $t(b_2) \in J_1 \times J_2$, we put

$$(2) \ t(b_2) = (i(b_2), j(b_2)), \ 0 \leq i(b_2) \leq 8, \ 0 \leq j(b_2) \leq 8.$$

Since $t(b_2) = (i(b_2), j(b_2)) \in I(i_2) \times I(j_2)$, we have

$$(3) \ i(b_2) \in I(i_2), \ j(b_2) \in I(j_2).$$

Since $I(i_2) = \{3 \times i_2, 3 \times i_2 + 1, 3 \times i_2 + 2\}$ and $I(j_2) = \{3 \times j_2, 3 \times j_2 + 1, 3 \times j_2 + 2\}$, by (3) we have the following decompositions:

$$(4) \ i(b_2) = 3 \times i_2 + i_1(b_2), \ 0 \leq i_1(b_2) \leq 2$$

$$(5) \ j(b_2) = 3 \times j_2 + j_1(b_2), \ 0 \leq j_1(b_2) \leq 2.$$

By (4) and (5) we have that

$$(6) \ DEC(i(b_2)) = (i_2, i_1(b_2)), \text{ and}$$

$$(7) \ DEC(j(b_2)) = (j_2, j_1(b_2)).$$

By (6) and (7) we have that

$$(8) \ BDEC(t(b_2)) = ((i_2, j_2), (i_1(b_2), j_1(b_2))) \in B_2 \times B_1.$$

We put $t_1(b_2) = (i_1(b_2), j_1(b_2)) \in B_1$. By (8) we have

$$(9) \ BDEC(t(b_2)) = (b_2, t_1(b_2)).$$

By (9) and Proposition 76 we have that

$$(10) \ BEXP(b_2, t_1(b_2)) = t(b_2).$$

Therefore, we can define a map $\psi: B_2 \rightarrow B_1$ by

$$(11) \ \psi(b_2) = t_1(b_2) \text{ for each } b_2 \in B_2.$$

By (1), (10), (11) we have that

$$(12) \ T \cap (I(i_2) \times I(j_2)) = \{BEXP(b_2, \psi(b_2))\} \text{ for each } b_2 = (i_2, j_2) \in B_2.$$

Thus, by (12), our map ψ has the required property.

Next, we show the uniqueness of ψ .

Let $\Omega: B_2 \rightarrow B_1$ be a map with

$$(13) \ T \cap (I(i_2) \times I(j_2)) = \{BEXP(b_2, \Omega(b_2))\} \text{ for each } b_2 = (i_2, j_2) \in B_2.$$

Claim 1.

$$\psi = \Omega.$$

Take any $b_2 = (i_2, j_2) \in B_2$. By (12) and (13) we have that

$$(14) \ BEXP(b_2, \psi(b_2)) = BEXP(b_2, \Omega(b_2)).$$

By (14) we have that

$$(15) \ BDEC(BEXP(b_2, \psi(b_2))) = BDEC(BEXP(b_2, \Omega(b_2))).$$

By Proposition 8–3 we have that

$$(16) \ BDEC(BEXP(b_2, \psi(b_2))), = BDEC \circ BEXP(b_2, \psi(b_2)) = (b_2, \psi(b_2)),$$

$$(17) \ BDEC(BEXP(b_2, \Omega(b_2))), = BDEC \circ BEXP(b_2, \Omega(b_2)) = (b_2, \Omega(b_2)).$$

By (15), (16), (17) we have that

$$(18) \ (b_2, \psi(b_2)) = (b_2, \Omega(b_2)).$$

By (18) we have that

$$(19) \ \psi(b_2) = \Omega(b_2).$$

Then, by (19)

$$(20) \ \psi = \Omega.$$

Hence, we have Claim 1.

Therefore, we show Proposition 82.

Let $proj_X: X \times Y \rightarrow X$ and $proj_Y: X \times Y \rightarrow Y$ be projections. That is,

$$proj_X(x, y) = x \text{ and } proj_Y(x, y) = y \text{ for each } (x, y) \in X \times Y.$$

Proposition 83.

Let T be a basic set. There are unique bijective maps

$$\pi_{i_2}: Z_{(2,2)} \rightarrow Z_{(1,1)} \text{ for each } i_2 \in Z_{(1,2)}$$

such that

$$proj_{J_1} \circ BEXP(b_2, \psi(b_2)) = EXP(i_2, \pi_{i_2}(j_2))$$

for each $b_2 = (i_2, j_2) \in B_2$.

Proof. We use the same notations and statements

(1)–(20) in the proof of Proposition 82.

For each $i_2 \in Z_{(1,2)}$ we define a map

$$\pi_{i_2}: Z_{(2,2)} \rightarrow Z_{(1,1)}$$

as follows:

$$(21) \quad \pi_{i_2}(j_2) = \text{pro } j_{Z_{(1,1)}}(\psi(i_2, j_2)) \text{ for each } j_2 \in Z_{(2,2)}.$$

Since $\psi(b_2) = t_1(b_2) = (i_1(b_2), j_1(b_2))$ by (11), we have that

$$(22) \quad \text{pro } j_{Z_{(1,1)}}(\psi(b_2)) = i_1(b_2) \text{ for each } b_2 = (i_2, j_2) \in B_2.$$

By (21), (22) we have that

$$(23) \quad \pi_{i_2}(j_2) = i_1(i_2, j_2)$$

By (4), (23) we have that

$$(24) \quad \text{EXP}(i_2, \pi_{i_2}(j_2)) = 3 \times i_2 + \pi_{i_2}(j_2) \\ = 3 \times i_2 + i_1(i_2, j_2) = i(i_2, j_2).$$

By (2), (10), (11) we have that

$$(25) \quad \text{pro } j_{J_1}(\text{BEXP}(b_2, \psi(b_2))) \\ = \text{pro } j_{J_1}(\text{BEXP}(b_2, t_1(b_2))) = \text{pro } j_{J_1} t(b_2) \\ = \text{pro } j_{J_1}(i(b_2), j(b_2)) = i(b_2) = i(i_2, j_2).$$

By (24), (25) we have that

$$(26) \quad \text{pro } j_{J_1}(\text{BEXP}(b_2, \psi(b_2))) = \text{EXP}(i_2, \pi_{i_2}(j_2)).$$

Thus, π_{i_2} has the required property.

Claim 2.

$\pi_{i_2}: Z_{(2,2)} \rightarrow Z_{(1,1)}$ is injective for each $i_2 \in Z_{(1,2)}$.

We assume that Claim 2 does not hold. Then we have an $i_2 \in Z_{(1,2)}$ such that

$$(27) \quad \pi_{i_2}: Z_{(2,2)} \rightarrow Z_{(1,1)} \text{ is not injective.}$$

By (27) we have $j_2, j_2^* \in Z_{(2,2)}$ such that

$$(28) \quad j_2 \neq j_2^* \text{ and}$$

$$(29) \quad \pi_{i_2}(j_2) = \pi_{i_2}(j_2^*).$$

By (28) we have that

$$(30) \quad I(j_2) \cap I(j_2^*) = \phi.$$

By (30) we have that

$$(31) \quad I(i_2) \times I(j_2) \cap I(i_2) \times I(j_2^*) = \phi.$$

By (31) we have that

$$(32) \quad (T \cap (I(i_2) \times I(j_2))) \cap (T \cap (I(i_2) \times I(j_2^*))) = \phi.$$

We put $b_2 = (i_2, j_2)$, $b_2^* = (i_2, j_2^*) \in B_2$. By (1), (32) we have that

$$(33) \quad \{t(b_2)\} \cap \{t(b_2^*)\} = \phi.$$

By (33) we have that

$$(34) \quad t(b_2) \neq t(b_2^*).$$

By (29) we have that

$$(35) \quad \text{EXP}(i_2, \pi_{i_2}(j_2)) = \text{EXP}(i_2, \pi_{i_2}(j_2^*)).$$

By (26), (35) we have that

$$(36) \quad \text{pro } j_{J_1}(\text{BEXP}(b_2, \psi(b_2)))$$

$$= \text{pro } j_{J_1}(\text{BEXP}(b_2^*, \psi(b_2^*))).$$

By (10), (36) we have that

$$(37) \quad \text{pro } j_{J_1}(t(b_2)) = \text{pro } j_{J_1}(t(b_2^*)).$$

By (2) we have that

$$(38) \quad t(b_2) = (i(b_2), j(b_2)), \quad t(b_2^*) = (i(b_2^*), j(b_2^*)).$$

By (37), (38) we have that

$$(39) \quad i(b_2) = i(b_2^*) = u.$$

Let $r(u) = \{(u, j): j \in J_2\}$ be the u -th row in $J_1 \times J_2$. By (38), (39) we have that

$$(40) \quad t(b_2), t(b_2^*) \in r(u).$$

By (1) we have that

$$(41) \quad t(b_2), t(b_2^*) \in T.$$

By (40), (41) we have that

$$(42) \quad t(b_2), t(b_2^*) \in r(u) \cap T.$$

By (BA) we have that

$$(43) \quad |r(u) \cap T| = 1.$$

By (42), (43) we have that

$$(44) \quad t(b_2) = t(b_2^*).$$

(44) contradicts to (34). Hence, we have Claim 2

Claim 3.

$\pi_{i_2}: Z_{(2,2)} \rightarrow Z_{(1,1)}$ is surjective for each $i_2 \in Z_{(1,2)}$.

Take any $i_2 \in Z_{(1,2)}$ and any $v_1 \in Z_{(1,1)}$. We put

$$(45) \quad v = 3 \times i_2 + v_1 \in J_1.$$

Let $r(v) = \{(v, j): j \in J_2\}$ be the v -th row of $J_1 \times J_2$. By

(BA) we have a $t_v \in J_1 \times J_2$ such that

$$(46) \quad r(v) \cap T = \{t_v\}.$$

Since $t_v \in r(v)$ by (46), we can put

$$(47) \quad t_v = (v, w).$$

Since $w \in J_2$, we have the decomposition

$$(48) \quad w = 3 \times w_2 + w_1, \quad 0 \leq w_2 \leq 2, \quad 0 \leq w_1 \leq 2.$$

By (45), (47), (48) we have that

$$(49) \quad t_v = (v, w) \in I(i_2) \times I(w_2).$$

Since $t_v \in T$ by (46), by (49) we have that

$$(50) \quad t_v \in T \cap (I(i_2) \times I(w_2)).$$

Let $b_2^{**} = (i_2, w_2) \in B_2$. By (1) we have that

$$(51) \quad T \cap (I(i_2) \times I(w_2)) = \{t(b_2^{**})\}.$$

By (50), (51) we have that

$$(52) \quad t_v = t(b_2^{**}).$$

By (52) we have that

$$(53) \quad v = i(b_2^{**}),$$

$$(54) \quad w = j(b_2^{**}).$$

By (45), (53) we have that

$$(55) \quad v_1 = i_1(b_2^{**}).$$

By (48), (54) we have that

$$(56) \quad w_1 = j_1(b_2^{**}).$$

By (11), (55), (56) we have that

$$(57) \quad \psi(b_2^{**}) = (v_1, w_1).$$

Thus, by (21), (57) we have that

$$(58) \quad \begin{aligned} \pi_{i_2}(w_2) &= \mathit{pro} j_{Z(1,1)} \psi(i_2, w_2) = \mathit{pro} j_{Z(1,1)} \psi(b_2^{**}) \\ &= \mathit{pro} j_{Z(1,1)}(v_1, w_1) = v_1. \end{aligned}$$

Then, by (58), we have that π_{i_2} is surjective.

Hence, we have Claim 3.

Claim 4.

The uniqueness of π_{i_2} for each $i_2 \in Z_{(1,2)}$.

Take any $i_2 \in Z_{(1,2)}$. We take any map $\pi_{i_2}^*: Z_{(2,2)} \rightarrow Z_{(1,1)}$ such that

$$(59) \quad \mathit{pro} j_{J_1} \circ BEXP((b_2), \psi(b_2)) = EXP(i_2, \pi_{i_2}^*(j_2)) \text{ for}$$

each $b_2 = (i_2, j_2) \in B_2$.

Take any $j_2 \in Z_{(2,2)}$ and put $b_2 = (i_2, j_2) \in B_2$. We put

$$(60) \quad n = \mathit{pro} j_{J_1} \circ BEXP((b_2), \psi(b_2)).$$

By (26), (59), (60) we have that

$$(61) \quad n = EXP(i_2, \pi_{i_2}(j_2)) = 3 \times i_2 + \pi_{i_2}(j_2),$$

$$(62) \quad n = EXP(i_2, \pi_{i_2}^*(j_2)) = 3 \times i_2 + \pi_{i_2}^*(j_2).$$

Since $\pi_{i_2}(j_2), \pi_{i_2}^*(j_2) \in Z_{(1,1)}$, we have that

$$(63) \quad 0 \leq \pi_{i_2}(j_2) \leq 2, \quad 0 \leq \pi_{i_2}^*(j_2) \leq 2.$$

By (61), (62), (63) we have that

$$(64) \quad \pi_{i_2}(j_2) = \pi_{i_2}^*(j_2).$$

By (64) we have that

$$(65) \quad \pi_{i_2} = \pi_{i_2}^*.$$

Hence, by (65) we have Claim 4.

Therefore, by Claim 1, Claim 2, Claim 3, Claim 4, we have Proposition 83.

For each $j_2 \in Z_{(2,2)}$, we can define a map

$\rho_{j_2}: Z_{(1,2)} \rightarrow Z_{(2,1)}$ by

$$\rho_{j_2}(i_2) = \mathit{pro} j_{Z(2,1)}(\psi(i_2, j_2)) \text{ for each } i_2 \in Z_{(1,2)}.$$

By the similar way as the proof of Proposition 83, we can show Proposition 84.

Proposition 84.

Let T be a basic set. There are unique bijective maps $\rho_{j_2}: Z_{(1,2)} \rightarrow Z_{(2,1)}$ for each $j_2 \in Z_{(2,2)}$ such that

$$\mathit{pro} j_{J_2} \circ BEXP(b_2, \psi(b_2)) = EXP(j_2, \rho_{j_2}(i_2)) \text{ for each } b_2 = (i_2, j_2) \in B_2.$$

We define maps $\xi_1: B_2 \rightarrow J_1$ and $\xi_2: B_2 \rightarrow J_2$ by

$$\xi_1(b_2) = EXP(i_2, \pi_{i_2}(j_2)) \text{ for } b_2 = (i_2, j_2) \in B_2,$$

$$\xi_2(b_2) = EXP(j_2, \rho_{j_2}(i_2)) \text{ for } b_2 = (i_2, j_2) \in B_2.$$

And we define block-graphs of ξ_1 and ξ_2 by

$$BGRP(\xi_1) = \cup \{ \{ \xi_1(i_2, j_2) \} \times I(j_2) : b_2 = (i_2, j_2) \in B_2 \} \subset J_1 \times J_2,$$

$$BGRP(\xi_2) = \cup \{ I(i_2) \times \{ \xi_2(i_2, j_2) \} : b_2 = (i_2, j_2) \in B_2 \} \subset J_1 \times J_2.$$

Proposition 85.

$I(b_2) \cap BGRP(\xi_1) = \{ \xi_1(b_2) \} \times I(j_2)$ for each $b_2 = (i_2, j_2) \in B_2$

Proof. Take any $b_{20} = (i_{20}, j_{20}) \in B_2$. First, we show Claim 1:

Claim 1.

$$I(b_{20}) \cap BGRP(\xi_1) \subset \{ \xi_1(b_{20}) \} \times I(j_{20}).$$

Take any point p such that

$$(1) \quad p = (p_1, p_2) \in I(b_{20}) \cap BGRP(\xi_1).$$

Thus, we have

$$(2) \quad p = (p_1, p_2) \in I(b_{20}) = I(i_{20}) \times I(j_{20}), \text{ and}$$

$$(3) \quad p = (p_1, p_2) \in BGRP(\xi_1) \\ = \cup \{ \{ \xi_1(i_2, j_2) \} \times I(j_2) : b_2 = (i_2, j_2) \in B_2 \}.$$

By (2) we have that

$$(4) \quad p_1 \in I(i_{20})$$

$$(5) \quad p_2 \in I(j_{20}).$$

By (3) we have a $b_{21} = (i_{21}, j_{21}) \in B_2$ such that

$$(6) \quad p = (p_1, p_2) \in \{ \xi_1(i_{21}, j_{21}) \} \times I(j_{21}).$$

By (6) we have that

$$(7) \quad p_1 = \xi_1(i_{21}, j_{21}),$$

$$(8) \quad p_2 \in I(j_{21}).$$

By (5), (8) we have that

$$(9) \quad I(j_{20}) \cap I(j_{21}) \neq \emptyset.$$

By (9) we have that

$$(10) \quad j_{20} = j_{21}.$$

By (7) we have that

$$(11) \quad p_1 = \xi_1(i_{21}, j_{21}) = EXP(i_{21}, \pi_{i_{21}}(j_{21})) \\ = 3 \times i_{21} + \pi_{i_{21}}(j_{21}) \in I(i_{21}).$$

By (4), (11) we have that

$$(12) \quad I(i_{20}) \cap I(i_{21}) \neq \emptyset.$$

By (12) we have that

$$(13) \quad i_{20} = i_{21}.$$

By (10), (13) we have that

$$(14) \quad b_{21} = (i_{21}, j_{21}) = (i_{20}, j_{20}) = b_{20}.$$

By (11), (14) we have that

$$(15) \quad p_1 = \xi_1(b_{21}) = \xi_1(b_{20}).$$

By (5), (15) we have that

$$(16) \quad p = (p_1, p_2) \in \{ \xi_1(b_{20}) \} \times I(j_{20}).$$

By (16) we have Claim 1.

Claim 2.

$$I(b_{20}) \cap BGRP(\xi_1) \supset \{ \xi_1(b_{20}) \} \times I(j_{20})$$

By the definitions we have that

$$(17) \quad \xi_1(b_{20}) = EXP(i_{20}, \pi_{i_{20}}(j_{20}))$$

$$= 3 \times i_{20} + \pi_{i_{20}}(j_{20}) \in I(i_{20}).$$

By (17) we have that

$$(18) \{\xi_1(b_{20})\} \times I(j_{20}) \subset I(i_{20}) \times I(j_{20}) = I(b_{20}).$$

By the definitions, we have

$$(19) \{\xi_1(b_{20})\} \times I(j_{20}) \subset BGRP(\xi_1).$$

By (18), (19) we have Claim 2.

Hence by Claim 1 and Claim 2 we have Proposition 85.

By the similar way as Proposition 85, we can show the following Proposition 86.

Proposition 86.

$I(b_2) \cap BGRP(\xi_2) = I(i_2) \times \{\xi_2(b_2)\}$ for each $b_2 = (i_2, j_2) \in B_2$.

Proposition 87.

(a) $I(b_2) \cap BGRP(\xi_1) \cap BGRP(\xi_2) = \{(\xi_1(b_2), \xi_2(b_2))\}$ for each $b_2 = (i_2, j_2) \in B_2$.

(b) $BGRP(\xi_1) \cap BGRP(\xi_2) = \{(\xi_1(b_2), \xi_2(b_2)) : b_2 \in B_2\}$

Proof. First, we show (a).

Take any $b_2 = (i_2, j_2) \in B_2$. By Proposition 85,

Proposition 86 we have that

$$\begin{aligned} & I(b_2) \cap BGRP(\xi_1) \cap BGRP(\xi_2) \\ &= (I(b_2) \cap BGRP(\xi_1)) \cap (I(b_2) \cap BGRP(\xi_2)) \\ &= (\{\xi_1(b_2)\} \times I(j_2)) \cap (I(i_2) \times \{\xi_2(b_2)\}) \\ &= \{(\xi_1(b_2), \xi_2(b_2))\}. \end{aligned}$$

Thus, we have (a).

Next, we show (b).

Since $J_1 \times J_2 = \cup \{I(b_2) : b_2 \in B_2\}$, we have that

$$\begin{aligned} & BGRP(\xi_1) \cap BGRP(\xi_2) \\ &= (J_1 \times J_2) \cap BGRP(\xi_1) \cap BGRP(\xi_2) \\ &= (\cup \{I(b_2) : b_2 \in B_2\}) \cap BGRP(\xi_1) \cap BGRP(\xi_2) \\ &= \cup \{I(b_2) \cap BGRP(\xi_1) \cap BGRP(\xi_2) : b_2 \in B_2\} \\ &= \{(\xi_1(b_2), \xi_2(b_2)) : b_2 \in B_2\}. \end{aligned}$$

Thus, we have (b).

Hence, we have Proposition 87.

Proposition 88.

$$BGRP(\xi_1) \cap BGRP(\xi_2) = T.$$

Proof. Take any $b_2 = (i_2, j_2) \in B_2$

By Propositions 83 and 84, we have

$$\begin{aligned} (1) \text{ } \text{pro } j_{j_1}(BEXP(b_2, \psi(b_2))) \\ &= EXP(i_2, \pi_{i_2}(j_2)) = \xi_1(b_2) \\ (2) \text{ } \text{pro } j_{j_2}(BEXP(b_2, \psi(b_2))) \\ &= EXP(j_2, \rho_{j_2}(i_2)) = \xi_2(b_2). \end{aligned}$$

By (1), (2) we have that

$$(3) BEXP(b_2, \psi(b_2)) = (\xi_1(b_2), \xi_2(b_2)).$$

By Proposition 82

$$(4) T \cap I(b_2) = \{BEXP(b_2, \psi(b_2))\}.$$

By Proposition 81 we have that

$$(5) T \cap I(b_2) = \{t(I(b_2))\}.$$

By (4) and (5) we have that

$$(6) t(I(b_2)) = BEXP(b_2, \psi(b_2)).$$

Thus by (3), (6), Propositions 81 and 87, we have that

$$\begin{aligned} (7) T &= \{t(I(b_2)) : I(b_2) \in bLK\} \\ &= \{BEXP(b_2, \psi(b_2)) : b_2 \in B_2\} \\ &= \{(\xi_1(b_2), \xi_2(b_2)) : b_2 \in B_2\} \\ &= BGRP(\xi_1) \cap BGRP(\xi_2). \end{aligned}$$

Thus, by (7) we have Proposition 88.

Proposition 89.

$\psi(b_2) = (\pi_{i_2}(j_2), \rho_{j_2}(i_2))$ for each $b_2 = (i_2, j_2) \in B_2$.

Proof.

We can define a map $\psi^* : B_2 \rightarrow B_1$ by

(1) $\psi^*(b_2) = (\pi_{i_2}(j_2), \rho_{j_2}(i_2))$ for each $b_2 = (i_2, j_2) \in B_2$.

Take any $b_2 = (i_2, j_2) \in B_2$, we have that

$$\begin{aligned} (2) BEXP(b_2, \psi^*(b_2)) \\ &= BEXP((i_2, j_2), (\pi_{i_2}(j_2), \rho_{j_2}(i_2))) \\ &= (EXP(i_2, \pi_{i_2}(j_2)), EXP(j_2, \rho_{j_2}(i_2))) \\ &= (\xi_1(b_2), \xi_2(b_2)). \end{aligned}$$

By (3) in the proof of Proposition 88, we have

$$(3) BEXP(b_2, \psi(b_2)) = (\xi_1(b_2), \xi_2(b_2)).$$

By (2), (3) we have that

$$(4) BEXP(b_2, \psi^*(b_2)) = BEXP(b_2, \psi(b_2)).$$

By the uniqueness of Proposition 82 we have that

$$(4) \psi = \psi^*.$$

Hence, we have Proposition 89.

By our observations in this section we have the following:

Proposition 90.

Let $T \subset J_1 \times J_2$ be a basic set. Then we have the followings:

(i) unique bijection maps $\pi_{i_2} : Z_{(2,2)} \rightarrow Z_{(1,1)}$ for $i_2 \in Z_{(1,2)}$

(ii) unique bijective maps $\rho_{j_2} : Z_{(1,2)} \rightarrow Z_{(2,1)}$ for $j_2 \in Z_{(2,2)}$

(iii) the unique map $\psi : B_2 \rightarrow B_1$, defined by

$$\psi(b_2) = (\pi_{i_2}(j_2), \rho_{j_2}(i_2)) \text{ for each } b_2 = (i_2, j_2) \in B_2,$$

such that

(iv) $T \cap I(b_2) = \{BEXP(b_2, \psi(b_2))\}$ for each $b_2 \in B_2$.

Moreover, we have

(v) maps $\xi_1 : B_2 \rightarrow J_1$ and $\xi_2 : B_2 \rightarrow J_2$ defined by

$$\xi_1(b_2) = EXP(i_2, \pi_{i_2}(j_2)) \text{ for } b_2 = (i_2, j_2) \in B_2,$$

$$\xi_2(b_2) = EXP(j_2, \rho_{j_2}(i_2)) \text{ for } b_2 = (i_2, j_2) \in B_2,$$

such that

(vi) $T = BGRP(\xi_1) \cap BGRP(\xi_2)$.

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