Multisoliton solutions of the two-component Camassa-Holm system and their reductions

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Abstract

We develop a systematic procedure for constructing soliton solutions of an integrable two-component Camassa-Holm (CH2) system. The parametric representation of the multisoliton solutions is obtained by using a direct method combined with a reciprocal transformation. The properties of the solutions are then investigated in detail focusing mainly on the smooth one- and two-soliton solutions. The general N-soliton case is described shortly. Subsequently, we show that the CH2 system reduces to the CH equation and the two-component Hunter-Saxton (HS2) system by means of appropriate limiting procedures. The corresponding expressions of the multisoliton solutions are presented in parametric forms, reproducing the existing results for the reduced equations. Last, we discuss the reduction from the HS2 system to the HS equation.

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1. Introduction

In this paper, we consider the following two-component generalization of the Camassa-Holm (CH) equation

$$m_t + um_x + 2mu_x + \rho\rho_x = 0, \tag{1.1a}$$

$$\rho_t + (\rho u)_x = 0, \tag{1.1b}$$

which is abbreviated as the CH2 system. Here, $u = u(x,t), \rho = \rho(x,t)$ and $m = m(x,t) \equiv u - u_{xx} + \kappa^2$ are real-valued functions of time t and a spatial variable x, and the subscripts x and t appended to u and ρ denote partial differentiation. The parameter κ in the expression of m is assumed to be a non-negative real number.

The CH2 system (1.1) has been derived for the first time in [1] in search of the bi-Hamiltonian formulation of integrable nonlinear evolution equations. Actually, the system can be represented as the dual bi-Hamiltonian system for a coupled Korteweg-de Vries equation introduced independently by Zakharov [2] and Ito [3]. Later, a similar system with the coefficient of $\rho\rho_x$ in (1.1a) being minus was studied [4-6]. In particular, a reciprocal transformation between the system and the first negative flow of the AKNS hierarchy was established in [6]. In the physical context, on the other hand, the CH2 system with $\kappa = 0$ was derived by applying an asymptotic analysis to the fully nonlinear Green-Naghdi equations for shallow water waves, where u represents the horizontal velocity and ρ is related to the depth of the fluid in the first approximation [7]. The same system with $\kappa \neq 0$ was also obtained from the basic Euler system for an incompressible fluid with a constant vorticity [8]. One can also consult Ref. [9] as for a brief history of the CH2 system.

One remarkable feature of the CH2 system is that it is a completely integrable system. Indeed, it has a Lax representation given by

$$\Psi_{xx} = \left(-\lambda^2\rho^2 + \lambda m + \frac{1}{4}\right)\Psi, \tag{1.2a}$$

$$\Psi_t = \left(\frac{1}{2\lambda} - u\right)\Psi_x + \frac{u_x}{2}\Psi, \tag{1.2b}$$

where λ is the spectral parameter [7, 8]. It turns out that the compatibility condition of the linear system (1.2) yields (1.1), thus enabling us to apply the inverse scattering transform method (IST) [10, 11]. A number of works have been devoted to the study of the mathematical properties of (1.1). For example, some conditions were provided for the wave breaking and the existence of the traveling waves [7, 12, 13]. The explicit solitary wave solutions were obtained by using the method of dynamical systems [14, 15], and the

general multisoliton solutions were constructed by means of the IST [16]. More precisely, the IST is reformulated as a Riemann-Hilbert problem [11], and the N-soliton solution is given by a parametric form. However, the analysis of multisoliton solutions has not been done as yet.

Various reductions are possible for the CH2 system while preserving its integrability. Specificallly, the reduction to the CH equation is of great importance. This can be accomplished simply by putting $\rho = 0$ in (1.1), giving [17]

$$u_t + 2\kappa^2 u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}. \tag{1.3}$$

The CH equation describes the unidirectional propagation of shallow water waves over a flat bottom. Its structure has been studied extensively from both theoretical and numerical points of view [18, 19]. The Lax representation associated with the CH equation can be obtained simply by putting $\rho = 0$ in (1.2). This enables us to apply the IST which has been successfully used for various integrable soliton equations such as the Korteweg-de Vries (KdV) and nonlinear Schrödinger equations. Unlike the KdV equation which is a typical model of shallow water waves, the CH equation could explain the wave breaking as well as the existence of peaked waves (or peakons) which are inherent in the basic Euler system.

Another reduction is the two-component Hunter-Saxton (HS2) system which can be derived by means of the short-wave limit of the CH2 system. It has the same form as the system (1.1) with the variable m replaced by $-u_{xx} + \kappa^2$. Explicitly, it can be written in the form

$$u_{xxt} - 2\kappa^2 u_x + uu_{xxx} + 2u_x u_{xx} - \rho\rho_x = 0, \quad \rho_t + (\rho u)_x = 0. \tag{1.4}$$

Furthermore, on taking $\rho = 0$, the HS2 system (1.4) reduces to

$$u_{xxt} - 2\kappa^2 u_x + uu_{xxx} + 2u_x u_{xx} = 0. \tag{1.5}$$

In the case of $\kappa = 0$, equation (1.5) becomes the classical Hunter-Saxton (HS) equation which is a model for describing the propagation of weakly nonlinear orientation waves in a massive nematic liquid crystal director field [20]. We refer to (1.5) as the HS equation hereafter.

The purpose of the present paper is to develop a systematic method for obtaining the multisoliton solutions of the CH2 system and investigate their properties. Subsequently, a reduction procedure is performed to obtain the multisoliton solutions of the CH equation and the HS2 system from those of the CH2 system. We impose the boundary conditions

$u(x, t) \to 0$ and $\rho(x, t) \to \rho_0$ as $|x| \to \infty$, where ρ_0 is a positive constant. These boundary conditions are consistent with the hydrodynamic derivation of the system [7, 8]. A direct method is employed to obtain solutions which worked effectively for the construction of the soliton solutions of the CH equation [21] and the modified CH equations [22, 23].

This paper is organized as follows. In section 2, we transform the CH2 system to a system of partial differential equations (PDEs) by means of a reciprocal transformation similar to that employed for the CH and modified CH equations [21-23]. We then perform the bilinearization of the latter system through appropriate dependent variable transformations. Following the standard procedure of the bilinear transformation method [24, 25], we construct the N-soliton solution of the bilinear equations in terms of the tau-functions, where N is an arbitrary positive integer, thus obtaining the parametric representation for the N-soliton solution of the system (1.1). The dispersion relation of the soliton is explored in detail to feature its propagation characteristics. In section 3, we investigate the properties of the soliton solutions. First, we address the one-soliton solutions, showing that the profile of ρ always takes the form of bright soliton whereas that of u takes both bright and dark solitons depending on the dispersion relation of the soliton. Subsequently, the asymptotic analysis of the N-soliton solution is performed to derive the formula for the phase shift. Last, the interaction process of two solitons is exemplified for both overtaking and head-on collisions. In section 4, we carry out various reductions of the CH2 system. Specifically, by introducing appropriate scaling variables, we demonstrate that the CH2 system reduces to the CH equation in the limit $\rho_0 \to 0$, and recover the N-soliton solution of the CH equation as well as the formula for the phase shift. We also show that the short-wave limit of the CH2 system leads to the HS2 system, and the N-soliton solution of the latter system is recovered from that of the former system. Then, we give a brief summary about the reduction to the HS equation. Section 5 is devoted to some concluding remarks. In appendix A, we detail the bilinearization of the CH2 system. In appendix B, we provide a proof of the bilinear identities for the tau-functions associated with the N-soliton solution of the CH2 system.

2. Exact method of solution

In this section, we develop a systematic method for constructing the multisoliton solutions of the CH2 system. To this end, we employ an exact method of solution which is referred to as the direct method [24] or the bilinear transformation method [25]. When compared with the IST, this method is an especially powerful technique for obtaining particular solutions like soliton and periodic wave solutions. After transforming the system (1.1)

to an equivalent system of PDEs by a reciprocal transformation, we bilinearize the latter system and then solve it in terms of the tau-functions, thus giving rise to the parametric representation of the N-soliton solution.

2.1. Reciprocal transformation

First of all, we introduce the reciprocal transformation $(x, t) \rightarrow (y, \tau)$ according to

$$dy = \rho \, dx - \rho u \, dt, \qquad d\tau = dt. \tag{2.1a}$$

Then, the x and t derivatives transform as

$$\frac{\partial}{\partial x} = \rho \frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} - \rho u \frac{\partial}{\partial y}. \tag{2.1b}$$

Applying the transformation (2.1) to the system (1.1), we obtain the system of PDEs

$$\left(\frac{m}{\rho^2} \right)_\tau + \rho_y = 0, \tag{2.2a}$$

$$\rho_\tau + \rho^2 u_y = 0. \tag{2.2b}$$

It then follows from (2.1b) that the variable $x = x(y, \tau)$ obeys a system of linear PDEs

$$x_y = \frac{1}{\rho}, \tag{2.3a}$$

$$x_\tau = u. \tag{2.3b}$$

The system of equations (2.3) is integrable since its compatibility condition $x_{\tau y} = x_{y\tau}$ is assured by virtue of (2.2b).

Now, the quantity $m = u - u_{xx} + \kappa^2$ in (1.1) can be rewritten in terms of the new coordinate system as

$$m = u + \rho(\ln \rho)_{\tau y} + \kappa^2, \tag{2.4}$$

where we have used (2.2b) to replace u_y by $-\rho_\tau / \rho^2$. Let us introduce the new dependent variable $Y = Y(y, \tau)$ by the relation

$$\frac{m}{\rho^2} - \frac{\kappa^2}{\rho_0^2} = Y_y. \tag{2.5}$$

Subsituting (2.5) into (2.2a) and then integrating the resultant expression by y under the boundary conditions $Y_\tau \rightarrow 0$ and $\rho \rightarrow \rho_0$ as $|y| \rightarrow \infty$, we obtain

$$\rho = \rho_0 - Y_\tau. \tag{2.6}$$

The following proposition is the starting point in the present analysis.

Proposition 2.1. *The variables x and Y satisfy the system of PDEs*

$$x_y(\rho_0 - Y_\tau) = 1, \tag{2.7}$$

$$(\rho_0 - Y_\tau)\left(\frac{\kappa^2}{\rho_0^2} + Y_y\right) = x_\tau x_y - [(\rho_0 - Y_\tau)x_{\tau y}]_y + \kappa^2 x_y. \tag{2.8}$$

Proof. Equation (2.7) follows immediately from (2.3a) and (2.6). If we substitute m from (2.5) into (2.4) and use (2.3a) and (2.3b) to express ρ and u in terms of x_y and x_τ, respectively, (2.4) becomes

$$\frac{\kappa^2}{\rho_0^2} + Y_y = x_\tau x_y^2 - x_{\tau yy} + \frac{x_{\tau y} x_{yy}}{x_y} + \kappa^2 x_y^2.$$

Dividing this expression by x_y and using (2.7), we arrive at (2.8). □

2.2. Bilinearization

In applying the bilinear transformation method to the given nonlinear equations, the first step is to transform the equations into the bilinear equations, which we shall now demonstrate. To this end, we introduce the dependent variable transformations

$$x = \frac{y}{\rho_0} + \ln\frac{\tilde{f}}{f} + d, \tag{2.9}$$

$$Y = i\ln\frac{\tilde{g}}{g}, \tag{2.10}$$

where $f, \tilde{f}, g$ and $\tilde{g}$ are tau-functions and d is an arbitrary constant. One advantage of the form (2.10) is that the structure of the system of bilinear equations becomes transparent when compapred with the introduction of another form like $Y = 2\tan^{-1}(\text{Im}\,g/\text{Re}\,g)$. This facilitates the analysis, in particular the construction of solutions. Obviously, the definition of Y from (2.5) implies that it can be taken as a real quantity which is achieved simply if one chooses the tau-function $\tilde{g}$ as a complex conjugate of g. This recipe can be used successfully in constructing real soliton solutions, as will be manifested in theorem 2.2.

Now, we establish the following proposition.

Proposition 2.2. *Consider the following system of bilinear equations for $f, \tilde{f}, g$ and $\tilde{g}$:*

$$D_y \tilde{f} \cdot f + \frac{1}{\rho_0}(\tilde{f}f - \tilde{g}g) = 0, \tag{2.11}$$

$$\mathrm{i}D_\tau \tilde{g} \cdot g + \rho_0(\tilde{f}f - \tilde{g}g) = 0, \tag{2.12}$$

$$D_\tau D_y \tilde{f} \cdot f + \frac{1}{\rho_0}D_\tau \tilde{f} \cdot f + \kappa^2 D_y \tilde{f} \cdot f = 0, \tag{2.13}$$

$$D_\tau D_y \tilde{g} \cdot g - \mathrm{i}\frac{\kappa^2}{\rho_0^2}D_\tau \tilde{g} \cdot g + \mathrm{i}\rho_0 D_y \tilde{g} \cdot g = 0, \tag{2.14}$$

where the bilinear operators are defined by

$$D_y^m D_\tau^n f \cdot g = (\partial_y - \partial_{y'})^m (\partial_\tau - \partial_{\tau'})^n f(y,\tau)g(y',\tau')|_{y'=y,\,\tau'=\tau}, \qquad (m, n = 0, 1, 2, ...). \tag{2.15}$$

Then, the solutions of this system of equations solve the equations (2.7) and (2.8).

The proof of proposition 2.2 will be detailed in appendix A.

2.3. Parametric representations of the solutions

Theorem 2.1. *The two-component CH system (1.1) admits the parametric representations of the solutions*

$$u(y,\tau) = \left(\ln \frac{\tilde{f}}{f} \right)_\tau, \tag{2.16}$$

$$\rho(y,\tau) = \rho_0 - \mathrm{i} \left(\ln \frac{\tilde{g}}{g} \right)_\tau, \tag{2.17}$$

$$x(y,\tau) = \frac{y}{\rho_0} + \ln \frac{\tilde{f}}{f} + d. \tag{2.18}$$

Proof. The expression (2.16) follows by introducing (2.9) into (2.3*b*) whereas the expression (2.17) comes from (2.6) and (2.10). The expression (2.18) is just (2.9). □

Remark 2.1. The parametric representations of $1/\rho$ and m/ρ^2 in terms of the tau-functions are also available from (2.3*a*), (2.5), (2.9) and (2.10). Explicitly, they read

$$\frac{1}{\rho} = \frac{1}{\rho_0} + \left(\ln \frac{\tilde{f}}{f} \right)_y, \tag{2.19}$$

$$\frac{m}{\rho^2} = \frac{\kappa^2}{\rho_0^2} + \mathrm{i} \left(\ln \frac{\tilde{g}}{g} \right)_y. \tag{2.20}$$

2.4. N-soliton solution

Theorem 2.2. *The tau-functions $f, \tilde{f}, g$ and $\tilde{g}$ constituting the N-soliton solution of the system of bilinear equations (2.11)-(2.14) are given by the expressions*

$$f = \sum_{\mu=0,1} \exp\left[\sum_{j=1}^{N} \mu_j \left(\xi_j + \phi_j\right) + \sum_{1 \leq j < l \leq N} \mu_j \mu_l \gamma_{jl}\right], \tag{2.21a}$$

$$\tilde{f} = \sum_{\mu=0,1} \exp\left[\sum_{j=1}^{N} \mu_j \left(\xi_j - \phi_j\right) + \sum_{1 \leq j < l \leq N} \mu_j \mu_l \gamma_{jl}\right], \tag{2.21b}$$

$$g = \sum_{\mu=0,1} \exp\left[\sum_{j=1}^{N} \mu_j \left(\xi_j + \mathrm{i}\psi_j\right) + \sum_{1 \leq j < l \leq N} \mu_j \mu_l \gamma_{jl}\right], \tag{2.22a}$$

$$\tilde{g} = \sum_{\mu=0,1} \exp\left[\sum_{j=1}^{N} \mu_j \left(\xi_j - \mathrm{i}\psi_j\right) + \sum_{1 \leq j < l \leq N} \mu_j \mu_l \gamma_{jl}\right], \tag{2.22b}$$

where

$$\xi_j = k_j \left(y - c_j \tau - y_{j0}\right), \qquad (j = 1, 2, ..., N), \tag{2.23a}$$

$$\mathrm{e}^{\gamma_{jl}} = \frac{\kappa^2(c_j - c_l)^2 - \rho_0(k_j - k_l)c_j c_l(c_j k_j - c_l k_l)}{\kappa^2(c_j - c_l)^2 - \rho_0(k_j + k_l)c_j c_l(c_j k_j + c_l k_l)}, \qquad (j, l = 1, 2, ..., N; j \neq l), \tag{2.23b}$$

$$\mathrm{e}^{-\phi_j} = \sqrt{\frac{(1 - \rho_0 k_j)c_j - \rho_0 \kappa^2}{(1 + \rho_0 k_j)c_j - \rho_0 \kappa^2}}, \qquad (j = 1, 2, ..., N), \tag{2.23c}$$

$$\mathrm{e}^{-\mathrm{i}\psi_j} = \sqrt{\frac{\left(\frac{\kappa^2}{\rho_0} - \mathrm{i}\rho_0 k_j\right)c_j + \rho_0^2}{\left(\frac{\kappa^2}{\rho_0} + \mathrm{i}\rho_0 k_j\right)c_j + \rho_0^2}}, \qquad (j = 1, 2, ..., N), \tag{2.23d}$$

and c_j is the velocity of jth soliton in the (y, τ) coordinate system which is given by the solution of the quadratic equation

$$(1 - \rho_0^2 k_j^2)c_j^2 - 2\rho_0 \kappa^2 c_j - \rho_0^4 = 0, \qquad (j = 1, 2, ..., N). \tag{2.23e}$$

Here, k_j and y_{j0} are arbitrary complex parameters satisfying the conditions $k_j \neq k_l$ for $j \neq l$. The notation $\sum_{\mu=0,1}$ implies the summation over all possible combinations of $\mu_1 = 0, 1, \mu_2 = 0, 1, ..., \mu_N = 0, 1$.

A proof of theorem 2.2 will be given in appendix B in which the tau-functions (2.21) and (2.22) are shown to satisfy the system of bilinear equations (2.11)-(2.14) by means of mathematical induction.

Remark 2.2. The bilinear equations (2.13) and (2.14) arise from the reduction of the BKP family of integrable soliton equations [26, 27]. The tau-functions associated with the N-soliton solutions of these equations have the same forms as those given by (2.21) and (2.22). Within this framework, however, the parameters c_j and k_j in (2.23b) can be taken independently. On the other hand, for the present N-soliton solutions, both parameters are related to each other by the quadratic equation (2.23e). This follows from the requirement that the tau-functions solve the bilinear equations (2.11) and (2.12) simultaneously.

The parametric representation of the N-soliton solution given by (2.16)-(2.18) with the tau-functions (2.21) and (2.22) is characterized by the $2N$ complex parameters k_j and y_{j0} $(j = 1, 2...., N)$. The parameters k_j determine the amplitude and the velocity of the solitons, whereas the parameters y_{j0} determine the position (or phase) of the solitons. If we impose the conditions $\tilde{f} = f^*$ and $\tilde{g} = g^*$ where the asterisk denotes complex conjugate, then the solutions become real functions of x and t. Note, however that they would yield multi-valued functions unless certain conditions are imposed on the parameters $k_j (j = 1, 2, .., N)$. The similar situation has already been encountered in investigating the structure of the soliton solutions of the CH and modified CH equations [21-23]. We will address this issue in the next section where the detailed analysis of the soliton solutions will be performed.

Before proceeding, we investigate the characteristics of the velocity of the soliton in the (y, τ) coordinate system. As will be discussed in section 3.1, the corresponding velocity in (x, t) coordinate system is given simply by c_j/ρ_0. The quadratic equation (2.23e) has two roots

$$c_j = \frac{\rho_0}{1 - (\rho_0 k_j)^2} (\kappa^2 + d_j) = \frac{\rho_0^3}{d_j - \kappa^2}, \qquad (j = 1, 2, ..., N), \qquad (2.24a)$$

where

$$d_j = \epsilon_j \sqrt{\kappa^4 + \rho_0^2 - \rho_0^4 k_j^2}, \qquad (\epsilon_j = \pm 1, \quad j = 1, 2, ..., N). \qquad (2.24b)$$

To assure the reality of c_j, one must impose the condition for the parameter $\rho_0 k_j$. Actually, it must lie in the interval

$$0 < \rho_0 k_j < \sqrt{\kappa^4 + \rho_0^2}/\rho_0, \qquad (j = 1, 2, ..., N), \qquad (2.25)$$

where we have assumed $k_j > 0$ $(j = 1, 2, ..., N)$. Figure 1 plots the velocities $c_+ \equiv c_j(\epsilon_j = +1)$ and $c_- \equiv c_j(\epsilon_j = -1)$ as a function of $\rho_0 k \equiv \rho_0 k_j$. The velocity c_+ is positive for
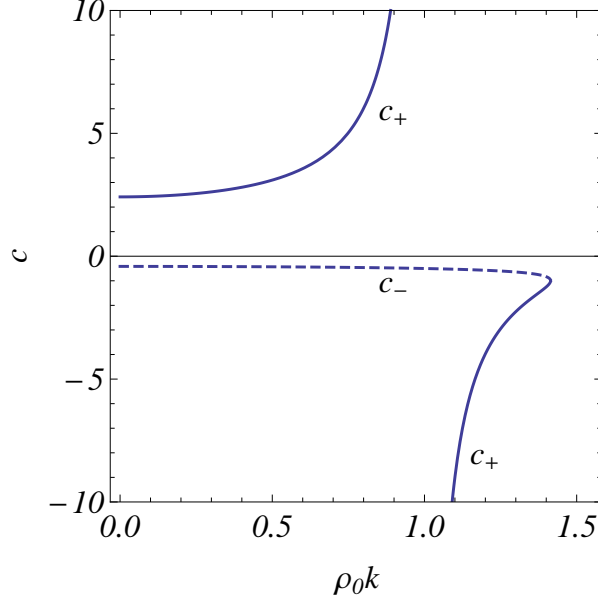
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Figure 1. The velocity $c = c_\pm$ of the soliton as a function of $\rho_0 k$ for $\rho_0 = 1$ and $\kappa = 1$: c_+(solid curve), c_-(dashed curve).

$0 < \rho_0 k < 1$ and negative for $1 < \rho_0 k < \sqrt{\kappa^4 + \rho_0^2}/\rho_0$. It exhibits the singularity at $\rho_0 k = 1$. Specifically,

$$\rho_0 \left(\kappa^2 + \sqrt{\kappa^4 + \rho_0^2} \right) < c_+ < \infty, \ (0 < \rho_0 k < 1), \tag{2.26a}$$

$$-\infty < c_+ < -\rho_0^3/\kappa^2, \ \left(1 < \rho_0 k < \sqrt{\kappa^4 + \rho_0^2}/\rho_0 \right). \tag{2.26b}$$

On the other hand, the velocity c_- is a continuous function of $\rho_0 k$ and takes negative values in the interval (2.25), as indicated by the inequality

$$-\rho_0^3/\kappa^2 < c_- < -\rho_0 \left(\sqrt{\kappa^4 + \rho_0^2} - \kappa^2 \right), \ \left(0 < \rho_0 k < \sqrt{\kappa^4 + \rho_0^2}/\rho_0 \right). \tag{2.27}$$

In particular, $c_- = -\rho_0^3/(2\kappa^2)$ at $\rho_0 k = 1$. It turns out that the soliton with the velocity c_- always propagates to the left whereas the soliton with the velocity c_+ propagates to the right and left depending on the value of $\rho_0 k$. Thus, the two-soliton solution exhibits both the overtaking and head-on collisions.

Using (2.24), the expressions (2.23c) and (2.23d) become

$$e^{-\phi_j} = \frac{|(1 - \rho_0 k_j)c_j - \rho_0 \kappa^2|}{\rho_0 \sqrt{\kappa^4 + \rho_0^2}} = \frac{\{(1 - \rho_0 k_j)c_j - \rho_0 \kappa^2\} \text{sgn } c_j}{\rho_0 \sqrt{\kappa^4 + \rho_0^2}}, \tag{2.28}$$

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$$\mathrm{e}^{-\mathrm{i}\psi_j} = \frac{\kappa^2 c_j + \rho_0^3 - \mathrm{i}\rho_0^2 k_j c_j}{\sqrt{\kappa^4 + \rho_0^2 \, |c_j|}}. \tag{2.29}$$

where the last expression in (2.28) is obtained by employing (2.24) again with sgn being the sign function. Substituting c_j from (2.24) into (2.28), one can show that $\mathrm{e}^{-\phi_j} < 1$ and hence $\phi_j > 0$. In view of the relation $d_j^2 - d_l^2 = \rho_0^4(-k_j^2 + k_l^2)$ which comes from (2.24b), one can derive the formula

$$\kappa^2(d_j - d_l)^2 + \rho_0^4(k_j \pm k_l)(k_j d_l \pm k_l d_j) + \kappa^2 \rho_0^4(k_j \pm k_l)^2$$

$$= \frac{1}{2}(d_j + d_l + 2\kappa^2) \left\{ (d_j - d_l)^2 + \rho_0^4(k_j \pm k_l)^2 \right\}.$$

Inserting this into (2.23b), we obtain a simplified expression for it

$$\mathrm{e}^{\gamma_{jl}} = \frac{(d_j - d_l)^2 + \rho_0^4(k_j - k_l)^2}{(d_j - d_l)^2 + \rho_0^4(k_j + k_l)^2}. \tag{2.30}$$

It will be used in proving the N-soliton solution. See appendix B.

Remark 2.3. Equation (1.1a) with a term $-\rho\rho_x$ instead of $+\rho\rho_x$ coupled with equation (1.1b), i.e.

$$m_t + um_x + 2mu_x - \rho\rho_x = 0, \quad \rho_t + (\rho u)_x = 0, \tag{2.31}$$

has been introduced in purely mathematical contexts [4-6]. It exhibits peculiar features when compared with features of the system (1.1). In particular, it admits peakons and kinks as well as smooth solitons [6]. The smooth N-soliton solutions with $N \le 4$ have been obtained by using the Darboux transformation [28]. The exact method of solution developed here enables us to construct the general N-soliton solution in a simple manner, which we shall summarize shortly. The expressions corresponding to (2.1)-(2.6) follow by the replacement of the variables in accordance with the rule $\rho \to \mathrm{i}\rho\,(\rho_0 \to \mathrm{i}\rho_0), y \to \mathrm{i}y, Y \to \mathrm{i}Y$ while other variables remain unchanged. The parametric representation of the solutions then takes the form

$$u(y, \tau) = \left(\ln \frac{\tilde{f}}{f} \right)_\tau, \quad \rho(y, \tau) = \rho_0 - \left(\ln \frac{\tilde{g}}{g} \right)_\tau, \quad x(y, \tau) = \frac{y}{\rho_0} + \ln \frac{\tilde{f}}{f} + d. \tag{2.32}$$

The tau-functions associated with the N-soliton solution can be obtained from (2.21)-(2.23) if one replaces the parameters as $k_j \to -\mathrm{i}k_j, c_j \to \mathrm{i}c_j, y_{j0} \to \mathrm{i}y_{j0}\ (j = 1, 2, ..., N)$, in addition to the replacements of the variables prescribed above. The soliton solutions have a rich mathematical structure and their properties deserve further study. The results of the detailed analysis will be reported elsewhere.

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3. Properties of soliton solutions

In this section, we first explore the properties of the one-soliton solution in detail and then perform an asymptotic analysis of the general N-soliton solution. Consequently, the formula for the phase shift of each soliton will be derived. The two-soliton case is discussed in some detail.

3.1. One-soliton solution

The tau-functions corresponding to the one-soliton solution are given by (2.21) and (2.22) with $N = 1$

$$f = 1 + e^{\xi+\phi}, \qquad \tilde{f} = 1 + e^{\xi-\phi}, \tag{3.1}$$

$$g = 1 + e^{\xi+i\psi}, \qquad \tilde{g} = 1 + e^{\xi-i\psi}, \tag{3.2}$$

with

$$\xi = k\left(y - c\tau - y_0\right), \tag{3.3a}$$

$$c = c_\pm = \frac{\rho_0^3}{\pm\sqrt{\kappa^4 + \rho_0^2 - \rho_0^4 k^2} - \kappa^2}, \tag{3.3b}$$

$$e^{-\phi} = \frac{\left|(1 - \rho_0 k)c - \rho_0\kappa^2\right|}{\rho_0\sqrt{\kappa^4 + \rho_0^2}}, \tag{3.3c}$$

$$e^{-i\psi} = \frac{\kappa^2 c + \rho_0^3 - i\rho_0^2 kc}{\sqrt{\kappa^4 + \rho_0^2}\,|c|}, \tag{3.3d}$$

where we have put $\xi = \xi_1, k = k_1, c = c_1, \phi = \phi_1, \psi = \psi_1$ and $y_0 = y_{10}$ for simplicity.

The parametric representation of the one-soliton solution is obtained by introducing (3.1) and (3.2) with (3.3) into (2.16)-(2.18). It can be written in the form

$$u = \frac{kc\,\sinh\phi}{\cosh\xi + \cosh\phi}, \tag{3.4a}$$

$$\rho = \rho_0 + \frac{kc\,\sin\psi}{\cosh\xi + \cos\psi}, \tag{3.4b}$$

$$X \equiv x - \tilde{c}t - x_0 = \frac{\xi}{\rho_0 k} + \ln\frac{1 - \tanh\frac{\phi}{2}\tanh\frac{\xi}{2}}{1 + \tanh\frac{\phi}{2}\tanh\frac{\xi}{2}}, \tag{3.4c}$$

with

$$\sinh\phi = \frac{k|c|}{\sqrt{\kappa^4 + \rho_0^2}}, \qquad \cosh\phi = \sqrt{1 + \frac{k^2 c^2}{\kappa^4 + \rho_0^2}}, \tag{3.4d}$$

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$$\sin \psi = \frac{\rho_0^2 kc}{\sqrt{\kappa^4 + \rho_0^2 \, |c|}}, \qquad \cos \psi = \frac{\kappa^2 c + \rho_0^3}{\sqrt{\kappa^4 + \rho_0^2 \, |c|}}, \qquad (3.4e)$$

where $\tilde{c} = c/\rho_0$ is the velocity of the soliton in the (x, t) coordinate system, $x_0 = y_0/\kappa$ and the constant d in (2.18) has been chosen such that $\xi = 0$ corresponds to $X = 0$. The traveling wave coordinate X defined by (3.4c) is particularly useful for the description of the one-soliton solution since it becomes stationary in this coordinate system. One can use the formula $\tanh(\phi/2) = \sinh \phi/(\cosh \phi + 1)$ to rewrite (3.4c) in terms of $\sinh \phi$ and $\cosh \phi$.

It now follows from (3.4d) and (3.4e) that $c \sin \psi = \rho_0^2 \sinh \phi$. Since $\phi > 0$, the sign of c must coincide with that of $\sin \psi$. This condition coupled with (3.4e) is used to determine the permissible value of ψ. Explicitly,

$$c_+ \, (0 < \rho_0 k < 1) : 0 < \psi < \pi/2, \quad c_+ \, (1 < \rho_0 k < \sqrt{\kappa^4 + \rho_0^2}/\rho_0) : \pi < \psi < 3\pi/2,$$

$$c_- \, (0 < \rho_0 k < \sqrt{\kappa^4 + \rho_0^2}/\rho_0) : 3\pi/2 < \psi < 2\pi. \qquad (3.4f)$$

Let us now describe some important properties of the solution.

(a) Smoothness of the solution

We compute the y derivative of x from (3.4c) to obtain

$$x_y = \frac{1}{\rho_0} - \frac{k \sinh \phi}{\cosh \xi + \cosh \phi}. \qquad (3.5)$$

Since $k > 0$ and $\phi > 0$, one has the inequality $x_y \geq x_y|_{\xi=0}$. Substituting (3.4d) for $\sinh \phi$ and $\cosh \phi$ and using (2.23e), we obtain

$$x_y|_{\xi=0} = \frac{1}{\rho_0} - \frac{k \sinh \phi}{1 + \cosh \phi}$$

$$= \frac{1}{\rho_0} \left[1 - \frac{1}{|c|} \left(|c - \rho_0 \kappa^2| - \rho_0 \sqrt{\kappa^4 + \rho_0^2} \right) \right] = \frac{1}{|c|} \left(\sqrt{\rho_0^2 + \kappa^4} + \kappa^2 \mathrm{sgn} \, c \right). \qquad (3.6)$$

The last expression follows from the previous one by considering the cases $c > 0$ and $c < 0$ separately with the help of the inequalities (2.26) and (2.27) for $c_\pm$. Note, in particular that $c_+ > \rho_0 \kappa^2$ for $0 < \rho_0 k < 1$ which is a unique positive branch of the dispersion curve, as is evident from Figure 1. Thus, if c is finite, then $x_y > 0$, and the map (2.1) becomes one-to-one, assuring that the solution is smooth and nonsingular. Actually, one can show that the derivatives du/dX and $d\rho/dX$ are finite for arbitrary $X \in \mathbb{R}$. Furthermore, it turns out from (3.3b) and (3.6) that the smoothness of the solution prevails in the zero

dispersion limit $\kappa \to 0$. However, the limit operation $\rho_0 \to 0$ with κ being fixed at a constant value requires a delicate analysis. See section 4.1.

(b) Amplitude-velocity relation

The amplitude-velocity relation of the soliton is an important characteristic of the wave. It can be derived simply from the explicit form (3.4) of the solution. To this end, let A_ρ be the amplitude of the wave measured from the constant level $\rho = \rho_0$ and A_u be that of the fluid velocity, i.e., $A_\rho = \rho(X = 0) - \rho_0$, and $A_u = |u(X = 0)|$. We find that

$$A_\rho = \left(\sqrt{\kappa^4 + \rho_0^2} |\tilde{c}| - \kappa^2 \tilde{c} - \rho_0^2 \right) / \rho_0, \tag{3.7a}$$

$$A_u = \left| |\tilde{c} - \kappa^2| - \sqrt{\kappa^4 + \rho_0^2} \right|, \tag{3.7b}$$

where $\tilde{c} = c/\rho_0$. Note that

$$u(X = 0) = kc \tanh \frac{\phi}{2} = \left(|\tilde{c} - \kappa^2| - \sqrt{\kappa^4 + \rho_0^2} \right) \operatorname{sgn} \tilde{c}.$$

Invoking the expression of the velocity c from (3.3b), we can see that $A_\rho > 0$ for arbitrary $c = c_\pm$ whereas $u(X = 0) > 0$ for $c > 0$ and $u(X = 0) < 0$ for $c < 0$. These results show that the profile of ρ is always of bright type, but that of u depends on the propagation direction of the soliton. Actually, if c is positive (negative), then u is curved upward (downward).

Figure 2 depicts the typical profile of u and ρ for the right-going soliton (a), and the left-going soliton (b) and (c), respectively

3.2. N-soliton solution

Here, we investigate the asymptotic behavior of the N-soliton solution for large time. Let $\tilde{c}_n (= c_n/\rho_0)$, $(n = 1, 2, ..., N)$ be the velocity of the nth soliton in the (x, t) coordinate system, and order them in accordance with the relation $\tilde{c}_N < \tilde{c}_{N-1} < ... < \tilde{c}_1$. We take the limit $t \to -\infty$ with the phase variable ξ_n of the nth soliton being fixed. Then, the other phase variables behave like $\xi_1, \xi_2, ..., \xi_{n-1} \to +\infty$, and $\xi_{n+1}, \xi_{n+2}, ..., \xi_N \to -\infty$. Performing an asymptotic analysis for the tau-functions (2.21) and (2.22), the leading-order approximations for them are found to be

$$f \sim \left(\prod_{1 \le j < l \le n-1} e^{\gamma_{jl}} \right) \exp \left[\sum_{j=1}^{n-1} (\xi_j + \phi_j) \right] \left(1 + e^{\xi_n + \phi_n + \delta_n^{(-)}} \right), \tag{3.8a}$$
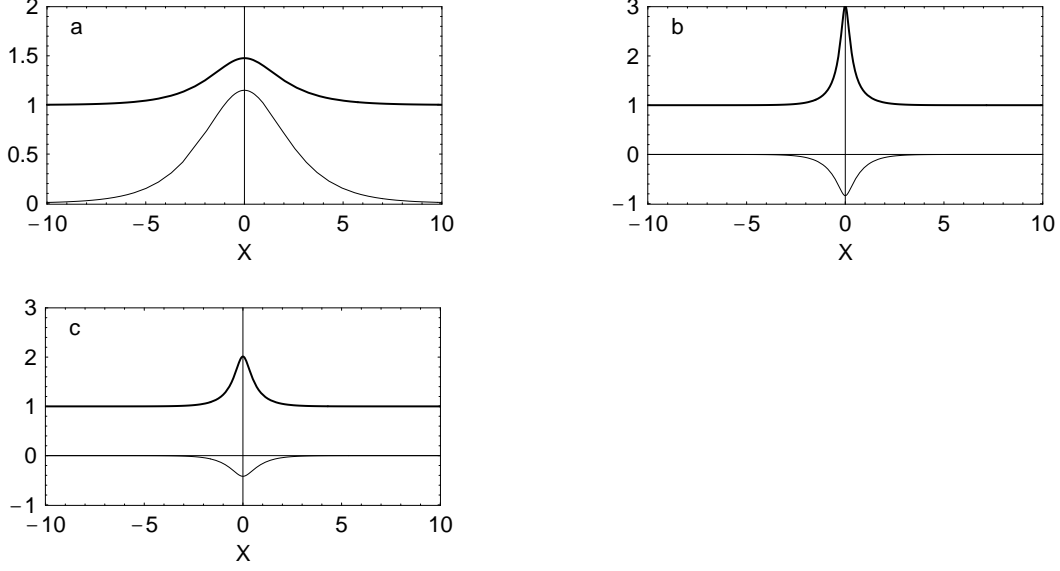
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Figure 2. One-soliton solution. u: thin solid curve, ρ: bold solid curve. a: $\kappa = 1, \rho_0 = 1, k = 0.4, \tilde{c} = \tilde{c}_+ = 2.81$, b: $\kappa = 1, \rho_0 = 1, k = 1.4, \tilde{c} = \tilde{c}_+ = -1.25$, c: $\kappa = 1, \rho_0 = 1, k = 1.4, \tilde{c} = \tilde{c}_- = -0.83$.

$$\tilde{f} \sim \left(\prod_{1 \leq j < l \leq n-1} e^{\gamma_{jl}} \right) \exp \left[\sum_{j=1}^{n-1} (\xi_j - \phi_j) \right] \left(1 + e^{\xi_n - \phi_n + \delta_n^{(-)}} \right), \qquad (3.8b)$$

$$g \sim \left(\prod_{1 \leq j < l \leq n-1} e^{\gamma_{jl}} \right) \exp \left[\sum_{j=1}^{n-1} (\xi_j + \mathrm{i}\psi_j) \right] \left(1 + e^{\xi_n + \mathrm{i}\psi_n + \delta_n^{(-)}} \right), \qquad (3.9a)$$

$$\tilde{g} \sim \left(\prod_{1 \leq j < l \leq n-1} e^{\gamma_{jl}} \right) \exp \left[\sum_{j=1}^{n-1} (\xi_j - \mathrm{i}\psi_j) \right] \left(1 + e^{\xi_n - \mathrm{i}\psi_n + \delta_n^{(-)}} \right), \qquad (3.9b)$$

where

$$\delta_n^{(-)} = \sum_{j=1}^{n-1} \gamma_{nj} = \sum_{j=1}^{n-1} \ln \left[\frac{(d_n - d_j)^2 + \rho_0^4 (k_n - k_j)^2}{(d_n - d_j)^2 + \rho_0^4 (k_n + k_j)^2} \right]. \qquad (3.10)$$

Substituting (3.8) and (3.9) into (2.16)-(2.18), we obtain the asymptotic form of u, ρ and x

$$u \sim \frac{k_n c_n \sinh \phi_n}{\cosh \left(\xi_n + \delta_n^{(-)} \right) + \cosh \phi_n}, \qquad (3.11)$$

$$\rho \sim \rho_0 + \frac{k_n c_n \sin \psi_n}{\cosh \left(\xi_n + \delta_n^{(-)} \right) + \cos \psi_n}, \qquad (3.12)$$

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$$x - \tilde{c}_n t - x_{n0} \sim \frac{\xi_n}{\rho_0 k_n} + \ln \frac{1 - \tanh \frac{\phi_n}{2} \tanh \frac{\left(\xi_n + \delta_n^{(-)}\right)}{2}}{1 + \tanh \frac{\phi_n}{2} \tanh \frac{\left(\xi_n + \delta_n^{(-)}\right)}{2}} - 2 \sum_{j=1}^{n-1} \phi_j. \tag{3.13}$$

In the limit $t \to +\infty$, on the other hand, we see that $\xi_1, \xi_2, ..., \xi_{n-1} \to -\infty$, and $\xi_{n+1}, \xi_{n+2}, ..., \xi_N \to +\infty$. Applying the similar analysis yields the asymptotic forms corresponding to (3.8)-(3.13)

$$f \sim \left(\prod_{n+1 \le j < l \le N} e^{\gamma_{jl}} \right) \exp \left[\sum_{j=n+1}^{N} (\xi_j + \phi_j) \right] \left(1 + e^{\xi_n + \phi_n + \delta_n^{(+)}} \right), \tag{3.14a}$$

$$\tilde{f} \sim \left(\prod_{n+1 \le j < l \le N} e^{\gamma_{jl}} \right) \exp \left[\sum_{j=n+1}^{N} (\xi_j - \phi_j) \right] \left(1 + e^{\xi_n - \phi_n + \delta_n^{(+)}} \right), \tag{3.14b}$$

$$g \sim \left(\prod_{n+1 \le j < l \le N} e^{\gamma_{jl}} \right) \exp \left[\sum_{j=n+1}^{N} (\xi_j + \mathrm{i}\psi_j) \right] \left(1 + e^{\xi_n + \mathrm{i}\psi_n + \delta_n^{(+)}} \right), \tag{3.15a}$$

$$\tilde{g} \sim \left(\prod_{n+1 \le j < l \le N} e^{\gamma_{jl}} \right) \exp \left[\sum_{j=n+1}^{N} (\xi_j - \mathrm{i}\psi_j) \right] \left(1 + e^{\xi_n - \mathrm{i}\psi_n + \delta_n^{(+)}} \right), \tag{3.15b}$$

where

$$\delta_n^{(+)} = \sum_{j=n+1}^{N} \gamma_{nj} = \sum_{j=n+1}^{N} \ln \left[\frac{(d_n - d_j)^2 + \rho_0^4 (k_n - k_j)^2}{(d_n - d_j)^2 + \rho_0^4 (k_n + k_j)^2} \right], \tag{3.16}$$

and

$$u \sim \frac{k_n c_n \sinh \phi_n}{\cosh \left(\xi_n + \delta_n^{(+)} \right) + \cosh \phi_n}, \tag{3.17}$$

$$\rho \sim \rho_0 + \frac{k_n c_n \sin \psi_n}{\cosh \left(\xi_n + \delta_n^{(+)} \right) + \cos \psi_n}, \tag{3.18}$$

$$x - \tilde{c}_n t - x_{n0} \sim \frac{\xi_n}{\rho_0 k_n} + \ln \frac{1 - \tanh \frac{\phi_n}{2} \tanh \frac{\left(\xi_n + \delta_n^{(+)}\right)}{2}}{1 + \tanh \frac{\phi_n}{2} \tanh \frac{\left(\xi_n + \delta_n^{(+)}\right)}{2}} - 2 \sum_{j=n+1}^{N} \phi_j. \tag{3.19}$$

These results show that as $t \to \pm\infty$, the N-soliton solution is represented by a superposition of N independent solitons each of which has the form of the one-soliton solution given by (3.4). The net effect of the collision of solitons appears as a phase shift. To

see this, let x_{nc} be the center position of the nth soliton. It then follows from (3.13) and (3.19) that the trajectory of x_{nc} is given by

$$x_{nc} \sim \tilde{c}_n t - \frac{\delta_n^{(-)}}{\rho_0 k_n} - 2 \sum_{j=1}^{n-1} \phi_j, \quad (t \to -\infty), \tag{3.20a}$$

$$x_{nc} \sim \tilde{c}_n t - \frac{\delta_n^{(+)}}{\rho_0 k_n} - 2 \sum_{j=n+1}^{N} \phi_j, \quad (t \to +\infty). \tag{3.20b}$$

We define the phase shift of the nth soliton which propagates to the right by $\Delta_n^R = x_{nc}(t \to +\infty) - x_{nc}(t \to -\infty)$, and that propagates to the left by $\Delta_n^L = x_{nc}(t \to -\infty) - x_{nc}(t \to +\infty)$. Using (2.23c), (3.10), (3.16) and (3.20), we find that

$$\Delta_n^R = \frac{1}{\rho_0 k_n} \left[\sum_{j=1}^{n-1} \ln \left[\frac{(d_n - d_j)^2 + \rho_0^4 (k_n - k_j)^2}{(d_n - d_j)^2 + \rho_0^4 (k_n + k_j)^2} \right] - \sum_{j=n+1}^{N} \ln \left[\frac{(d_n - d_j)^2 + \rho_0^4 (k_n - k_j)^2}{(d_n - d_j)^2 + \rho_0^4 (k_n + k_j)^2} \right] \right]$$

$$+ \sum_{j=n+1}^{N} \ln \left[\frac{(1 - \rho_0 k_j)\tilde{c}_j - \kappa^2}{(1 + \rho_0 k_j)\tilde{c}_j - \kappa^2} \right] - \sum_{j=1}^{n-1} \ln \left[\frac{(1 - \rho_0 k_j)\tilde{c}_j - \kappa^2}{(1 + \rho_0 k_j)\tilde{c}_j - \kappa^2} \right]. \tag{3.21}$$

The expression of Δ_n^L is equal to $-\Delta_n^R$.

3.3. Two-soliton solution

The two-soliton solution is the most fundamental element in understanding the dynamics of solitons since each soliton exhibits pair wise interactions with every other soliton, as indicated by the formulas of the phase shift. There exist two types of interactions for the CH2 system, i.e., the overtaking and head-on collisions. We describe them separately.

The tau-functions for the two-soliton solution are given by (2.21)-(2.23) and (2.30) with $N = 2$. They read

$$f = 1 + e^{\xi_1 + \phi_1} + e^{\xi_2 + \phi_2} + \delta \, e^{\xi_1 + \xi_2 + \phi_1 + \phi_2}, \tag{3.22a}$$

$$\tilde{f} = 1 + e^{\xi_1 - \phi_1} + e^{\xi_2 - \phi_2} + \delta \, e^{\xi_1 + \xi_2 - \phi_1 - \phi_2}, \tag{3.22b}$$

$$g = 1 + e^{\xi_1 + i\psi_1} + e^{\xi_2 + i\psi_2} + \delta \, e^{\xi_1 + \xi_2 + i\psi_1 + i\psi_2}, \tag{3.23a}$$

$$\tilde{g} = 1 + e^{\xi_1 - i\psi_1} + e^{\xi_2 - i\psi_2} + \delta \, e^{\xi_1 + \xi_2 - i\psi_1 - i\psi_2}, \tag{3.23b}$$

where

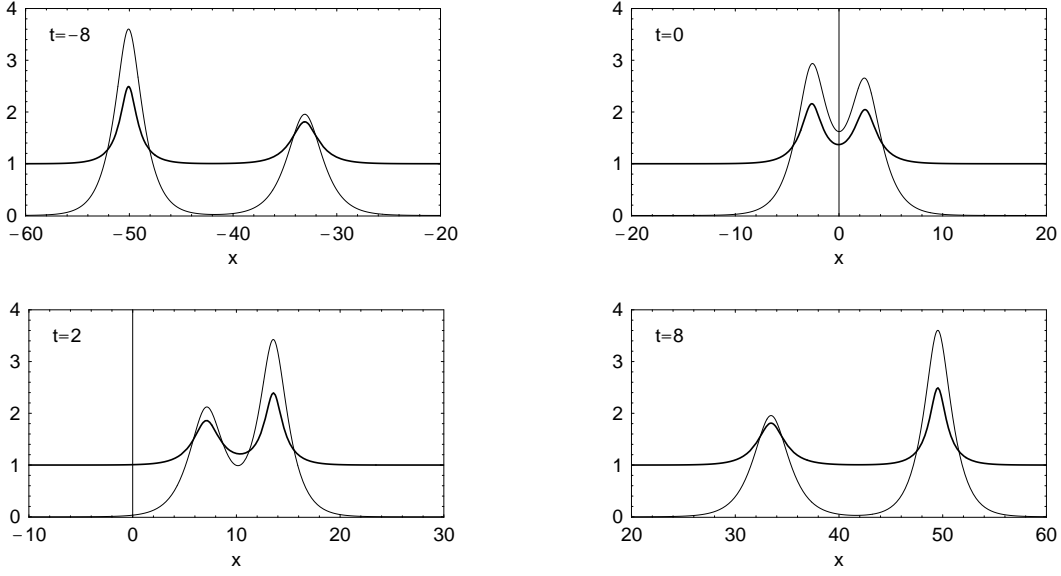$$\xi_j = k_j \left(y - c_j \tau - y_{j0} \right), \qquad (j = 1, 2), \tag{3.24a}$$

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Figure 3. The overtaking collision of two solitons. u: thin solid curve, ρ: bold solid curve. $\kappa = 1, \rho_0 = 1, k_1 = 0.8, k_2 = 0.7, \tilde{c}_{1+} = 6.02, \tilde{c}_{2+} = 4.37$.

$$\delta = e^{\gamma_{12}} = \frac{(d_1 - d_2)^2 + \rho_0^4(k_1 - k_2)^2}{(d_1 - d_2)^2 + \rho_0^4(k_1 + k_2)^2}, \tag{3.24b}$$

$$e^{-\phi_j} = \sqrt{\frac{(1 - \rho_0 k_j)c_j - \rho_0 \kappa^2}{(1 + \rho_0 k_j)c_j - \rho_0 \kappa^2}}, \qquad (j = 1, 2), \tag{3.24c}$$

$$e^{-i\psi_j} = \sqrt{\frac{\left(\frac{\kappa^2}{\rho_0} - i\rho_0 k_j\right)c_j + \rho_0^2}{\left(\frac{\kappa^2}{\rho_0} + i\rho_0 k_j\right)c_j + \rho_0^2}}, \qquad (j = 1, 2). \tag{3.24d}$$

Recall from (2.24) that the velocity of jth soliton in (x, t) coordinate system is given by

$$\tilde{c}_j = c_j/\rho_0 = \frac{\rho_0^2}{d_j - \kappa^2}, \quad d_j = \epsilon_j \sqrt{\kappa^4 + \rho_0^2 - \rho_0^4 k_j^2}, \quad (j = 1, 2). \tag{3.25}$$

Substituting (3.22)-(3.25) into (2.16)-(2.18), we obtain the parametric representation of the two-soliton solution. Since the velocity $\tilde{c}_j$ takes either the positive or negative values, this solution enables us to describe both the overtaking and head-on collisions between two solitons.
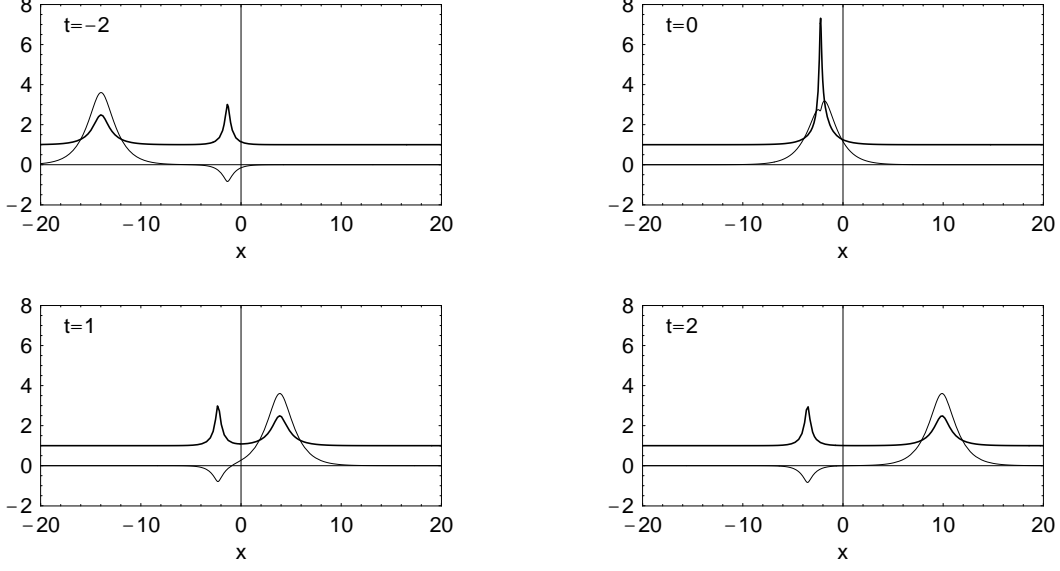
(a) Overtaking collision

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Figure 4. The head-on collision of two solitons. u: thin solid curve, ρ: bold solid curve. $\kappa = 1, \rho_0 = 1, k_1 = 0.8, k_2 = 1.4, \tilde{c}_{1+} = 6.02, \tilde{c}_{2+} = -1.25$

We consider the case $c_j = c_{j+}, 0 < \rho_0 k_j < 1$ so that $0 < \tilde{c}_{2+} < \tilde{c}_{1+}$. Figure 3 illustrates the overtaking collision of two solitons for four distinct values of t. The solitonic feature of the solution is obvious from the figure which confirms an asymptotic analysis presented in §3.1. The phase shift of each soliton is given by (3.21). Explicitly,

$$\Delta_1^R = -\frac{1}{\rho_0 k_1} \ln \left[\frac{(d_1 - d_2)^2 + \rho_0^4(k_1 - k_2)^2}{(d_1 - d_2)^2 + \rho_0^4(k_1 + k_2)^2} \right] + \ln \left[\frac{(1 - \rho_0 k_2)\tilde{c}_2 - \kappa^2}{(1 + \rho_0 k_2)\tilde{c}_2 - \kappa^2} \right], \qquad (3.26a)$$

$$\Delta_2^R = \frac{1}{\rho_0 k_2} \ln \left[\frac{(d_1 - d_2)^2 + \rho_0^4(k_1 - k_2)^2}{(d_1 - d_2)^2 + \rho_0^4(k_1 + k_2)^2} \right] - \ln \left[\frac{(1 - \rho_0 k_1)\tilde{c}_1 - \kappa^2}{(1 + \rho_0 k_1)\tilde{c}_1 - \kappa^2} \right], \qquad (3.26b)$$

with

$$d_1 = \sqrt{\kappa^4 + \rho_0^2 - \rho_0^4 k_1^2}, \quad d_2 = \sqrt{\kappa^4 + \rho_0^2 - \rho_0^4 k_2^2}. \qquad (3.26c)$$

(b) Head-on collision

An example of the head-on collision is shown in Figure 4, where the velocity of each soliton is chosen as $c_{2+} < 0 < c_{1+}$. The formula of the phase shift for the right-running soliton is the same as (3.26a) whereas that of the left-running soliton is given by $\Delta_2^L = -\Delta_2^R$.

Remark 3.1.

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As noticed in [7], the CH2 system (1.1) with $\kappa = 0$ does not admit peakons. The same will be true in the case of $\kappa \neq 0$. Recall, however that another integrable CH2 system (2.31) exhibits peakons when the parameter κ is related to the boundary value ρ_0 of ρ as $\rho_0 = \kappa^2$. See, for example [28].

4. Reductions to the CH equation, the HS2 system and the HS equation

$$
\begin{array}{ccc}
\boxed{\text{CH2}} & \xrightarrow{\text{SL}} & \boxed{\text{CH}} \\
\downarrow{\scriptstyle\text{SWL}} & & \downarrow{\scriptstyle\text{SWL}} \\
\boxed{\text{HS2}} & \xrightarrow{\text{SL}} & \boxed{\text{HS}}
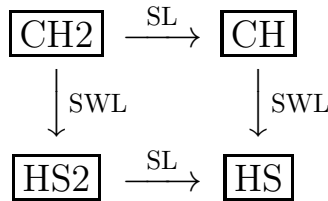\end{array}
$$

Figure 5. The reduction process for the CH2 system in which SL and SWL abbraviate the scaling and short-wave limits, respectively.

In this section, we first show that the CH2 system and its N-soliton solution reduce to the CH equation and the corresponding N-soliton solution under an appropriate limiting procedure, or more precisely, the scaling limit. Then, we demonstrate that the short-wave limit of the CH2 system yields the HS2 system. The reduction to the HS equation is outlined shortly.

The primary difference between the scaling limit and short-wave limit is that in the former limit, no scalings are prescribed for the space and time variables whereas in the latter limit, the rapidly-varying space variable $\hat{x}$ and slowly-varying time variable $\hat{t}$ are introduced via the relations $\hat{x} = x/\epsilon$ and $\hat{t} = \epsilon t$, where ϵ is a scaling parameter. The reduction process developed here is displayed in Figure 5 in which the two different avenues leading to the HS equation are indicated.

4.1. Reduction to the CH equation

The CH equation (1.2) is derived formally from the CH2 system by putting $\rho = 0$. In this setting, one must impose the boundary condition $\rho_0 = 0$. The N-soliton solution of the CH equation is reduced from that of the CH2 system by taking the limit $\rho_0 \to 0$. This limiting procedure is, however highly non-trivial, as will be shown below.

First, we introduce the following scaling variables with an overbar

$$
u = \bar{u}, \ \rho = \rho_0 \bar{\rho}, \ m = \bar{m}, \ x = \bar{x}, \ y = \frac{\rho_0}{\kappa} \bar{y}, \ t = \bar{t}, \ \tau = \bar{\tau}, \ d = \bar{d},
$$

$$k_j = \frac{\kappa}{\rho_0}\,\bar{k}_j, \ c_j = \frac{\rho_0}{\kappa}\,\bar{c}_j, \ y_{j0} = \frac{\rho_0}{\kappa}\,\bar{y}_{j0}, \ (j = 1, 2, ..., N). \tag{4.1}$$

Then, the leading-order asymptotics of c_j from (2.24), and γ_{jl}, ϕ_j, and ψ_j from (2.23) are found to be

$$c_j \sim \frac{2\rho_0\kappa^2}{1 - (\kappa\bar{k}_j)^2}, \ (j = 1, 2, ..., N), \tag{4.2a}$$

$$e^{\gamma_{jl}} = \left(\frac{\bar{k}_j - \bar{k}_l}{\bar{k}_j + \bar{k}_l}\right)^2 \equiv e^{\bar{\gamma}_{jl}}, \ (j, l = 1, 2, ..., N; j \neq l), \tag{4.2b}$$

$$e^{-\phi_j} \sim \frac{1 - \kappa\bar{k}_j}{1 + \kappa\bar{k}_j} \equiv e^{-\bar{\phi}_j}, \ (j = 1, 2, ..., N), \tag{4.2c}$$

$$e^{-i\psi_j} \sim 1 - i\frac{\rho_0}{\kappa}\,\bar{k}_j, \ (j = 1, 2, ..., N). \tag{4.2d}$$

We note that a limiting form $\bar{c}_j \sim -\rho_0^2/(2\kappa)$ of the velocity which arises from (2.24) with $\epsilon_j = -1 \ (j = 1, 2)$ is not relevant since in accodance with the scaling (4.1), this expression leads to $\bar{c}_j/\rho_0 \sim -\rho_0/(2\kappa) \to 0 \ (\rho_0 \to 0)$, showing that the velocity in the $(\bar{x}, \bar{t})$ coordinate system degenerates to zero.

The asymptotics of the tau-functions f and $\tilde{f}$ from (2.21) and g and $\tilde{g}$ from (2.22) become

$$f \sim \sum_{\mu=0,1} \exp\left[\sum_{j=1}^{N} \mu_j\left(\bar{\xi}_j + \bar{\phi}_j\right) + \sum_{1 \le j < l \le N} \mu_j\mu_l\bar{\gamma}_{jl}\right] \equiv \bar{f}, \tag{4.3a}$$

$$\tilde{f} \sim \sum_{\mu=0,1} \exp\left[\sum_{j=1}^{N} \mu_j\left(\bar{\xi}_j - \bar{\phi}_j\right) + \sum_{1 \le j < l \le N} \mu_j\mu_l\bar{\gamma}_{jl}\right] \equiv \bar{\tilde{f}}, \tag{4.3b}$$

$$g = \bar{f}_0 + i\frac{\rho_0}{\kappa}\,\bar{f}_{0,\bar{y}} + O(\rho_0^2), \tag{4.4a}$$

$$\tilde{g} = \bar{f}_0 - i\frac{\rho_0}{\kappa}\,\bar{f}_{0,\bar{y}} + O(\rho_0^2), \tag{4.4b}$$

where

$$\bar{f}_0 = \sum_{\mu=0,1} \exp\left[\sum_{j=1}^{N} \mu_j\bar{\xi}_j + \sum_{1 \le j < l \le N} \mu_j\mu_l\bar{\gamma}_{jl}\right], \tag{4.5a}$$

$$\bar{\xi}_j = \bar{k}_j\left(\bar{y} - \bar{c}_j\bar{\tau} - \bar{y}_{j0}\right), \quad \bar{c}_j = \frac{2\kappa^3}{1 - (\kappa\bar{k}_j)^2}, \qquad (j = 1, 2, ..., N). \tag{4.5b}$$

Introducing (4.1), (4.3) and (4.4) into (2.16)-(2.18) and taking the limit $\rho_0 \to 0$, we obtain the limiting forms of u, ρ and x

$$\bar{u} = \left(\ln\frac{\bar{\tilde{f}}}{\bar{f}}\right)_{\bar{\tau}}, \tag{4.6}$$

$$\rho \sim \rho_0 \left(1 - \frac{2}{\kappa}(\ln \bar{f}_0)_{\bar{y}\tau}\right) \equiv \rho_0 \bar{\rho}, \tag{4.7}$$

$$\bar{x} = \frac{\bar{y}}{\kappa} + \ln \frac{\bar{\bar{f}}}{\bar{f}} + \bar{d}. \tag{4.8}$$

The parametric representation of the N-soliton solution given by (4.6) and (4.8) with the tau-functions (4.3) coincides perfectly with that of the CH equation presented in [21]. In particular, the one-soliton solution (3.4) reduces to

$$\bar{u} = \frac{2\kappa \bar{c}\bar{k}^2}{1 + \kappa^2 \bar{k}^2 + (1 - \kappa^2 \bar{k}^2)\cosh \bar{\xi}}, \tag{4.9a}$$

$$\bar{X} = \bar{x} - \bar{\bar{c}} - \bar{x}_0 = \frac{\bar{\xi}}{\kappa \bar{k}} + \ln \frac{(1 - \kappa \bar{k})\,e^{\bar{\xi}} + 1 + \kappa \bar{k}}{(1 + \kappa \bar{k})\,e^{\bar{\xi}} + 1 - \kappa \bar{k}}, \tag{4.9b}$$

with

$$\bar{\xi} = \bar{k}(\bar{y} - \bar{c}\bar{\tau} - \bar{y}_0) \quad \bar{c} = \frac{2\kappa^3}{1 - (\kappa \bar{k})^2}, \quad \bar{\bar{c}} = \bar{c}/\kappa, \tag{4.9c}$$

reproducing the one-soliton solution of the CH equation.

The limiting form of the phase shift which is denoted by $\bar{\Delta}_n^R$ can be derived from (3.21) by using (4.2a). It reads

$$\bar{\Delta}_n^R = \frac{1}{\kappa \bar{k}_n} \left[\sum_{j=1}^{n-1} \ln \left(\frac{\bar{k}_n - \bar{k}_j}{\bar{k}_n + \bar{k}_j}\right)^2 - \sum_{j=n+1}^{N} \ln \left(\frac{\bar{k}_n - \bar{k}_j}{\bar{k}_n + \bar{k}_j}\right)^2\right]$$

$$+ \sum_{j=n+1}^{N} \ln \left(\frac{1 - \kappa \bar{k}_j}{1 + \kappa \bar{k}_j}\right)^2 - \sum_{j=1}^{n-1} \ln \left(\frac{1 - \kappa \bar{k}_j}{1 + \kappa \bar{k}_j}\right)^2. \tag{4.10}$$

This is just the formula for the phase shift of the N-soliton solution of the CH equation presented in [21].

Remark 4.1.
If we put $\bar{r} = \kappa - 2(\ln \bar{f}_0)_{\bar{y}\tau}$, then

$$\bar{\rho} = \frac{\bar{r}}{\kappa}, \quad \bar{m} = \bar{r}^2. \tag{4.11}$$

The first equation in (4.11) follows immediately from (4.7), and the second equation can be derived by taking the scaling limit of (2.20). The reciprocal transformation (2.1a) reproduces the corresponding one for the CH equation [21]

$$d\bar{y} = \bar{r}\, d\bar{x} - \bar{r}\bar{u}\, d\bar{t}, \quad d\bar{\tau} = d\bar{t}. \tag{4.12}$$

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In terms of the scaling variables (4.1), the bilinear equations (2.11)-(2.13) reduce respectively to

$$\kappa D_{\bar{y}}\bar{\tilde{f}} \cdot \bar{f} + \bar{\tilde{f}}\bar{f} - \bar{f}_0^2 = 0, \tag{4.13}$$

$$D_{\bar{\tau}}D_{\bar{y}}\bar{f}_0 \cdot \bar{f}_0 + \kappa(\bar{\tilde{f}}\bar{f} - \bar{f}_0^2) = 0, \tag{4.14}$$

$$\kappa D_{\bar{\tau}}D_{\bar{y}}\bar{\tilde{f}} \cdot \bar{f} + D_{\bar{\tau}}\bar{\tilde{f}} \cdot \bar{f} + \kappa^3 D_{\bar{y}}\bar{\tilde{f}} \cdot \bar{f} = 0. \tag{4.15}$$

The scaling limit of (2.14) is performed after eliminating the derivative $D_\tau\tilde{g} \cdot g$ in (2.14) by means of (2.12). We then find that the limiting form of (2.14) coincides with (4.14). One can show that the tau-functions $\bar{f}$ and $\bar{\tilde{f}}$ from (4.3) and $\bar{f}_0$ from (4.5) solve the above bilinear equations.

4.2. Reduction to the HS2 system

The HS2 system arises from the short-wave limit of the CH2 system. In this case, we introduce the scaling variables with a hat

$$u = \epsilon^2 \hat{u}, \ \rho = \epsilon\hat{\rho}, \ m = \hat{m}, \ x = \epsilon\hat{x}, \ y = \epsilon^2\hat{y}, \ t = \frac{\hat{t}}{\epsilon}, \ \tau = \frac{\hat{\tau}}{\epsilon}. \tag{4.16}$$

Rescaling the CH2 system (1.1) by (4.16) and taking the limit $\epsilon \to 0$, we obtain the HS2 system

$$\hat{m}_{\hat{t}} + \hat{u}\hat{m}_{\hat{x}} + 2\hat{m}\hat{u}_{\hat{x}} + \hat{\rho}\hat{\rho}_{\hat{x}} = 0, \tag{4.17a}$$

$$\hat{\rho}_{\hat{t}} + (\hat{\rho}\hat{u})_{\hat{x}} = 0, \tag{4.17b}$$

where $\hat{m} = -\hat{u}_{\hat{x}\hat{x}} + \kappa^2$, which coincides with (1.4) upon removing the hat attached to the variables.

The N-soliton solution of the HS2 system can be recovered from that of the CH2 system by means of a scaling limit. The appropriate scaling variables are found to be

$$k_j = \frac{\hat{k}_j}{\epsilon^2}, \ c_j = \epsilon^3\hat{c}_j, \ y_{j0} = \epsilon^2\hat{y}_{j0}, \ (j = 1, 2, ..., N), \ \rho_0 = \epsilon\hat{\rho}_0, \ d = \epsilon\hat{d}. \tag{4.18}$$

In the limit $\epsilon \to 0$, the soliton parameters corresponding to those given by (4.2) have the leading-order asymptotics

$$c_j \sim -\frac{\epsilon^3}{\hat{\rho}_0\hat{k}_j^2}(\kappa^2 + \hat{d}_j), \quad \hat{d}_j = \epsilon_j\sqrt{\kappa^4 - \hat{\rho}_0^4\hat{k}_j^2}, \quad (j = 1, 2, ..., N), \tag{4.19a}$$

$$e^{\gamma_{jl}} \sim \frac{(\hat{d}_j - \hat{d}_l)^2 + \hat{\rho}_0^4(\hat{k}_j - \hat{k}_l)^2}{(\hat{d}_j - \hat{d}_l)^2 + \hat{\rho}_0^4(\hat{k}_j + \hat{k}_l)^2} \equiv e^{\hat{\gamma}_{jl}}, \quad (j, l = 1, 2, ..., N; j \neq l), \tag{4.19b}$$

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$$\mathrm{e}^{-\phi_j} \sim 1 + \epsilon \, \frac{\hat{k}_j \hat{c}_j}{\kappa^2}, \;\; (j = 1, 2, ..., N), \tag{4.19c}$$

$$\mathrm{e}^{-\mathrm{i}\psi_j} \sim \sqrt{\frac{\left(\frac{\kappa^2}{\hat{\rho}_0} - \mathrm{i}\hat{\rho}_0 \hat{k}_j\right) \hat{c}_j + \hat{\rho}_0^2}{\left(\frac{\kappa^2}{\hat{\rho}_0} + \mathrm{i}\hat{\rho}_0 \hat{k}_j\right) \hat{c}_j + \hat{\rho}_0^2}} \equiv \mathrm{e}^{-\mathrm{i}\hat{\psi}_j}, \;\; (j = 1, 2, ..., N). \tag{4.19d}$$

The tau-functions (2.21) and (2.22) have the leading-order asymptotics

$$f \sim \hat{f} + \frac{\epsilon}{\kappa^2} \, \hat{f}_{\hat{\tau}}, \quad \tilde{f} \sim \hat{f} - \frac{\epsilon}{\kappa^2} \, \hat{f}_{\hat{\tau}}, \tag{4.20}$$

$$g \sim \sum_{\mu=0,1} \exp\left[\sum_{j=1}^{N} \mu_j \left(\hat{\xi}_j + \mathrm{i}\hat{\psi}_j\right) + \sum_{1 \le j < l \le N} \mu_j \mu_l \hat{\gamma}_{jl}\right] \equiv \hat{g}, \tag{4.21a}$$

$$\tilde{g} \sim \sum_{\mu=0,1} \exp\left[\sum_{j=1}^{N} \mu_j \left(\hat{\xi}_j - \mathrm{i}\hat{\psi}_j\right) + \sum_{1 \le j < l \le N} \mu_j \mu_l \hat{\gamma}_{jl}\right] \equiv \hat{\tilde{g}}, \tag{4.21b}$$

where

$$\hat{f} = \sum_{\mu=0,1} \exp\left[\sum_{j=1}^{N} \mu_j \hat{\xi}_j + \sum_{1 \le j < l \le N} \mu_j \mu_l \hat{\gamma}_{jl}\right], \tag{4.22a}$$

$$\hat{\xi}_j = \hat{k}_j(\hat{y} - \hat{c}_j \hat{\tau} - \hat{y}_{j0}), \quad \hat{c}_j = -\frac{1}{\hat{\rho}_0 \hat{k}_j^2} \left(\kappa^2 + \epsilon_j \sqrt{\kappa^4 - \hat{\rho}_0^4 \hat{k}_j^2}\right), \quad (j = 1, 2, ..., N). \tag{4.22b}$$

The parametric representation for the N-soliton solution of the HS2 system follows by introducing (4.20) and (4.21) into (2.16)-(2.18) and taking the limit $\epsilon \to 0$. Explicitly,

$$\hat{u} = -\frac{2}{\kappa^2} (\ln \hat{f})_{\hat{\tau}\hat{\tau}}, \tag{4.23}$$

$$\hat{\rho} = \hat{\rho}_0 - \frac{2}{\kappa^2} \mathrm{i} \left(\ln \frac{\hat{\tilde{g}}}{\hat{g}}\right)_{\hat{\tau}}, \tag{4.24}$$

$$\hat{x} = \frac{\hat{y}}{\hat{\rho}_0} - \frac{2}{\kappa^2} (\ln \hat{f})_{\hat{\tau}} + \hat{d}. \tag{4.25}$$

The limiting forms of (2.19) and (2.20) turn out to be

$$\frac{1}{\hat{\rho}} = \frac{1}{\hat{\rho}_0} - \frac{2}{\kappa^2} (\ln \hat{f})_{\hat{\tau}\hat{y}}, \tag{4.26}$$

$$\frac{\hat{m}}{\hat{\rho}^2} = \frac{\kappa^2}{\hat{\rho}_0^2} + \mathrm{i} \left(\ln \frac{\hat{\tilde{g}}}{\hat{g}}\right)_{\hat{y}}. \tag{4.27}$$

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We write the one-soliton solution for reference.

$$\hat{u} = -\frac{1}{2\kappa^2}\frac{(\hat{k}\hat{c})^2}{\cosh^2\frac{\hat{\xi}}{2}}, \quad \hat{\rho} = \frac{1}{\frac{1}{\hat{\rho}_0} + \frac{\hat{k}^2\hat{c}}{2\kappa^2}\frac{1}{\cosh^2\frac{\hat{\xi}}{2}}}, \tag{4.28a}$$

$$\hat{X} = \hat{x} - \hat{\tilde{c}}\hat{t} - \hat{x}_0 = \frac{\hat{\xi}}{\hat{\rho}_0\hat{k}} + \frac{\hat{k}\hat{c}}{\kappa^2}\tanh\frac{\hat{\xi}}{2}, \tag{4.28b}$$

with

$$\hat{\xi} = \hat{k}(\hat{y} - \hat{c}\hat{\tau} - \hat{y}_0), \quad \hat{c} = -\frac{1}{\hat{\rho}_0\hat{k}^2}\left(\kappa^2 \pm \sqrt{\kappa^4 - \hat{\rho}_0^4\hat{k}^2}\right), \quad \hat{\tilde{c}} = \hat{c}/\hat{\rho}_0. \tag{4.28c}$$

Notice that the velocities $\hat{\tilde{c}}$ from (4.28c) are negative for both plus and minus signs so that the soliton propagates to the left as opposed to the soliton solution of the CH2 system for which the bi-directional propagation is possible. Furthermore, in contrast to the CH2 case, the profile of $\hat{\rho}$ takes the form of a dark soliton. We also remark that all the results reduced from the CH2 system reproduce the corresponding ones obtained recently by an analysis of the HS2 system [29].

Remark 4.2.
Under the scaling (4.16), the reciprocal transformation (2.1) and equations (2.2)-(2.5) remain the same form. The bilinear equations (2.11), (2.12) and (2.14) reduce respectively to

$$D_{\hat{\tau}}D_{\hat{y}}\hat{f}\cdot\hat{f} - \frac{\kappa^2}{\hat{\rho}_0^2}(\hat{f}^2 - \hat{\tilde{g}}\hat{g}) = 0, \tag{4.29}$$

$$iD_{\hat{\tau}}\hat{\tilde{g}}\cdot\hat{g} + \hat{\rho}_0(\hat{f}^2 - \hat{\tilde{g}}\hat{g}) = 0, \tag{4.30}$$

$$D_{\hat{\tau}}D_{\hat{y}}\hat{\tilde{g}}\cdot\hat{g} - i\frac{\kappa^2}{\hat{\rho}_0^2}D_{\hat{\tau}}\hat{\tilde{g}}\cdot\hat{g} + i\hat{\rho}_0 D_{\hat{y}}\hat{\tilde{g}}\cdot\hat{g} = 0, \tag{4.31}$$

whereas the bilinear equation (2.13) reduces to (4.29) when coupled with (2.11).

4.3. Reduction to the HS equation

The HS equation (1.5) can be reduced from either the short-wave limit of the CH equation or the scaling limit of the HS2 system, as shown in Figure 5. The former reduction has been performed in [30]. To attain the latter reduction, we employ the same scaling variables as those given by (4.1) and find that the resulting expressions reproduce those obtained in [30]. The reduction process can be established in parallel with that for the CH2 system, and hence the detail of the computation is omitted here.

The parametric representation of the N-soliton solution can be obtained by taking the scaling limit of (4.22), (4.23) and (4.25). It leads, after removing the hat appended to the variables for simplicity, to

$$u = -\frac{2}{\kappa^2} \left(\ln f \right)_{\tau\tau}, \tag{4.32a}$$

$$x = \frac{y}{\kappa} - \frac{2}{\kappa^2} \left(\ln f \right)_\tau + d, \tag{4.32b}$$

with

$$f = \sum_{\mu=0,1} \exp\left[\sum_{j=1}^{N} \mu_j \xi_j + \sum_{1 \leq j < l \leq N} \mu_j \mu_l \gamma_{jl} \right], \tag{4.33a}$$

$$\xi_j = k_j (y - c_j \tau - y_{j0}), \quad c_j = -\frac{2\kappa}{k_j^2}, \ (j = 1, 2, ..., N), \tag{4.33b}$$

$$e^{\gamma_{jl}} = \left(\frac{k_j - k_l}{k_j + k_l} \right)^2, \ (j, l = 1, 2, ..., N; j \neq l). \tag{4.33c}$$

The one-soliton solution is given by

$$u = -\frac{2}{k^2} \frac{1}{\cosh^2 \frac{\xi}{2}}, \quad X = x - \tilde{c}t - x_0 = \frac{\xi}{\kappa k} - \frac{2}{\kappa k} \tanh \frac{\xi}{2}, \tag{4.34a}$$

with

$$\xi = k(y - c\tau - y_0), \quad c = -\frac{2\kappa}{k^2}, \quad \tilde{c} = \frac{c}{\kappa}. \tag{4.34b}$$

The above parametric solution takes the form of a cusp soliton. This can be confirmed simply by computing the derivative $u_X (= u_\xi / X_\xi)$ from (4.34), giving $u_X = 4\kappa/(k \sinh \xi)$. Thus, $\lim_{X \to \pm 0} u_X = \pm\infty$, showing that the slope of the soliton becomes infinite at the crest.

5. Concluding remarks

An intriguing feature of the CH equation is the existence of peakons which mimic Stokes' limiting solitary waves in the classical shallow water wave theory [31]. The peakons can be reduced from the smooth solitons by taking the zero dispersion limit $\kappa \to 0$. See, for example [32, 33]. Since the CH2 system under consideration is an integrable generalization of the CH equation, one can expect that it exhibits peakons as well. The detailed analysis of the one-soliton solution (3.4) reveals that the peakon can not be produced from the smooth soliton in any limiting procedure. On the other hand, another integrable CH2 system (2.31) admits peakons [6]. However, the general N-peakon solution is still unavailable for this system. In addition, whether peakons can be reduced from smooth solitons

or not has not been resolved. The complete classification of traveling wave solutions of the CH2 system has not been performed yet for both periodic and nonperiodic boundary conditions. Specifically, as for the existence of multi-valued solutions, no decisive answer exists even today. These interesting problems will be considered in a future work.

Acknowledgements

Appendix A. Proof of Proposition 2.2.

First, we show that the solutions of the bilinear equations (2.11) and (2.12) solve (2.7). Upon substituting (2.9) and (2.10) into (2.7), the equation to be proved becomes $P = 0$, where

$$P \equiv \left\{ \frac{1}{\rho_0} + \left(\ln \frac{\tilde{f}}{f} \right)_y \right\} \left\{ \rho_0 - \mathrm{i} \left(\ln \frac{\tilde{g}}{g} \right)_\tau \right\} - 1.$$

Invoking the definition of the bilinear operator (2.15), P is rewritten in the form

$$P = \left\{ \left(\frac{1}{\rho_0} \tilde{f} f + D_y \tilde{f} \cdot f \right) (\rho_0 \tilde{g} g - \mathrm{i} D_\tau \tilde{g} \cdot g) - \tilde{f} \tilde{g} f g \right\} / (\tilde{f} \tilde{g} f g).$$

This expression becomes zero by virtue of (2.11) and (2.12).

To preceed, we introduce (2.9) and (2.10) into (2.8), and obtain

$$\left\{ \rho_0 - \mathrm{i} \left(\ln \frac{\tilde{g}}{g} \right)_\tau \right\} \left\{ \frac{\kappa^2}{\rho_0^2} + \mathrm{i} \left(\ln \frac{\tilde{g}}{g} \right)_y \right\} = \left(\ln \frac{\tilde{f}}{f} \right)_\tau \left\{ \frac{1}{\rho_0} + \left(\ln \frac{\tilde{f}}{f} \right)_y \right\}$$

$$- \left[\left\{ \rho_0 - \mathrm{i} \left(\ln \frac{\tilde{g}}{g} \right)_\tau \right\} \left(\ln \frac{\tilde{f}}{f} \right)_{\tau y} \right]_y + \kappa^2 \left\{ \frac{1}{\rho_0} + \left(\ln \frac{\tilde{f}}{f} \right)_y \right\}.$$

In view of (2.11) and (2.12), the second term on the right-hand side of the above equation is modified as

$$\left[\left\{ \rho_0 - \mathrm{i} \left(\ln \frac{\tilde{g}}{g} \right)_\tau \right\} \left(\ln \frac{\tilde{f}}{f} \right)_{\tau y} \right]_y = \left(\ln \frac{\tilde{g} g}{\tilde{f} f} \right)_{\tau y}.$$

Inserting this relation and using (2.11) and (2.12), the equation to be proved reduces to $Q = 0$, where

$$Q \equiv \left(\ln \frac{\tilde{g}g}{\tilde{f}f} \right)_{\tau y} + \mathrm{i}\rho_0 \frac{\tilde{f}f}{\tilde{g}g} \left(\ln \frac{\tilde{g}}{g} \right)_y - \frac{1}{\rho_0} \frac{\tilde{g}g}{\tilde{f}f} \left(\ln \frac{\tilde{f}}{f} \right)_\tau + \frac{\kappa^2}{\rho_0} \left(\frac{\tilde{f}f}{\tilde{g}g} - \frac{\tilde{g}g}{\tilde{f}f} \right).$$

It now follows from the the definition of the bilinear operators that

$$(\ln \tilde{f}f)_{\tau y} = \frac{D_\tau D_y \tilde{f} \cdot f}{\tilde{f}f} - \frac{1}{(\tilde{f}f)^2}(D_\tau \tilde{f} \cdot f)(D_y \tilde{f} \cdot f),$$

$$(\ln \tilde{g}g)_{\tau y} = \frac{D_\tau D_y \tilde{g} \cdot g}{\tilde{g}g} - \frac{1}{(\tilde{g}g)^2}(D_\tau \tilde{g} \cdot g)(D_y \tilde{g} \cdot g).$$

Substituting these identities into the first term of Q and rewriting the second and third terms by means of the bilinear operators, Q recasts to

$$Q = \frac{D_\tau D_y \tilde{g} \cdot g}{\tilde{g}g} + \mathrm{i} \frac{D_y \tilde{g} \cdot g}{(\tilde{g}g)^2} (\mathrm{i}D_\tau \tilde{g} \cdot g + \rho_0 \tilde{f}f) - \frac{D_\tau D_y \tilde{f} \cdot f}{\tilde{f}f}$$

$$+ \frac{D_\tau \tilde{f} \cdot f}{(\tilde{f}f)^2} \left(D_y \tilde{f} \cdot f - \frac{1}{\rho_0} \tilde{g}g \right) + \frac{\kappa^2}{\rho_0} \left(\frac{\tilde{f}f}{\tilde{g}g} - \frac{\tilde{g}g}{\tilde{f}f} \right).$$

This expression turns out to be zero by virtue of (2.11)-(2.14).

Appendix B. Proof of Theorem 2.2

In this appendix, we show that the tau-functions (2.21) and (2.22) solve the system of bilinear equations (2.11)-(2.14). We use a mathematical induction similar to that has been employed for the proof of the N-soliton solution of the nonlinear network equations [34]. Since the proof can be performed in a similar manner for all equations, we describe the proof of (2.11) in some detail, and outline the proof for other three equations.

First, we substitute the tau-functions f and $\tilde{f}$ from (2.21) into the bilinear equation (2.11) and use the formula

$$D_\tau^m D_y^n \exp\left[\sum_{i=1}^N \mu_i \xi_i \right] \cdot \exp\left[\sum_{i=1}^N \nu_i \xi_i \right]$$

$$= \left\{ -\sum_{i=1}^N (\mu_i - \nu_i)k_i c_i \right\}^m \left\{ \sum_{i=1}^N (\mu_i - \nu_i)k_i \right\}^n \exp\left[\sum_{i=1}^N (\mu_i + \nu_i)\xi_i \right], \quad (m, n = 0, 1, 2, ...),$$

$$(B.1)$$

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to show that the equation to be proved becomes

$$
\sum_{\mu,\nu=0,1}\left[\left\{\sum_{i=1}^{N}(\mu_i-\nu_i)k_i+\frac{1}{\rho_0}\right\}\exp\left[-\sum_{i=1}^{N}(\mu_i-\nu_i)\phi_i\right]-\frac{1}{\rho_0}\exp\left[-\mathrm{i}\sum_{i=1}^{N}(\mu_i-\nu_i)\psi_i\right]\right]
$$

$$
\times\exp\left[\sum_{i=1}^{N}(\mu_i+\nu_i)\xi_i+\sum_{1\le i<j\le N}(\mu_i\mu_j+\nu_i\nu_j)\gamma_{ij}\right]=0. \tag{B.2}
$$

Let $P_{m,n}$ be the coefficient of the factor $\exp\left[\sum_{i=1}^{n}\xi_i+\sum_{i=n+1}^{m}2\xi_i\right]$ $(1\le n<m\le N)$ on the left-hand side of (B.2). This coefficient is obtained if one performs the summation with respect to μ_i and ν_i under the conditions $\mu_i+\nu_i=1$ $(i=1,2,...,n)$, $\mu_i=\nu_i=1$ $(i=n+1,n+2,...,m)$, $\mu_i=\nu_i=0$ $(i=m+1,m+2,...,N)$. We then introduce the new summation indices σ_i by the relations $\mu_i=(1+\sigma_i)/2$, $\nu_i=(1-\sigma_i)/2$ for $i=1,2,...,n$, where σ_i takes either the value $+1$ or -1, so that $\mu_i\mu_j+\nu_i\nu_j=(1+\sigma_i\sigma_j)/2$.

Consequently, $P_{m,n}$ can be rewritten in the form

$$
P_{m,n}=\sum_{\sigma=\pm1}\left[\left\{\sum_{i=1}^{n}\sigma_ik_i+\frac{1}{\rho_0}\right\}\exp\left[-\sum_{i=1}^{n}\sigma_i\phi_i\right]-\frac{1}{\rho_0}\exp\left[-\mathrm{i}\sum_{i=1}^{n}\sigma_i\psi_i\right]\right]
$$

$$
\times\exp\left[\frac{1}{2}\sum_{1\le i<j\le n}(1+\sigma_i\sigma_j)\gamma_{ij}+\sum_{i=1}^{m}\sum_{\substack{j=n+1\\(j\ne i)}}^{m}\gamma_{ij}\right]. \tag{B.3}
$$

If we invoke (2.24) and (2.28)-(2.30) as well as the definition of σ_i, we deduce

$$
\exp\left[-\sum_{i=1}^{n}\sigma_i\phi_i\right]=\prod_{i=1}^{n}\left[\frac{\operatorname{sgn}c_i}{\sqrt{\rho_0^2+\kappa^4}}\frac{d_i-\kappa^2\rho_0\sigma_ik_i}{1+\rho_0\sigma_ik_i}\right], \tag{B.4}
$$

$$
\exp\left[-\mathrm{i}\sum_{i=1}^{n}\sigma_i\psi_i\right]=\prod_{i=1}^{n}\left[\frac{\operatorname{sgn}c_i(d_i-\mathrm{i}\rho_0^2\sigma_ik_i)}{\sqrt{\rho_0^2+\kappa^4}}\right], \tag{B.5}
$$

$$
\exp\left[\frac{1}{2}\sum_{1\le i<j\le n}(1+\sigma_i\sigma_j)\gamma_{ij}\right]=\prod_{1\le i<j\le n}\left[\frac{(d_i-d_j)^2+\rho_0^4(\sigma_ik_i-\sigma_jk_j)^2}{(d_i-d_j)^2+\rho_0^4(\sigma_ik_i+\sigma_jk_j)^2}\right]. \tag{B.6}
$$

Substituting (B. 4)-(B. 6) into (B. 3), $P_{m,n}$ becomes

$$
P_{m,n}=c_{m,n}\sum_{\sigma=\pm1}\left[\left(\sum_{i=1}^{n}\rho_0\sigma_ik_i+1\right)\prod_{i=1}^{n}\frac{d_i-\kappa^2\rho_0\sigma_ik_i}{1+\rho_0\sigma_ik_i}-\prod_{i=1}^{n}(d_i-\mathrm{i}\rho_0^2\sigma_ik_i)\right]
$$

$$\times \prod_{1 \le i < j \le n} \left[(d_i - d_j)^2 + \rho_0^4 (\sigma_i k_i - \sigma_j k_j)^2 \right], \tag{B.7}$$

where $c_{m,n}$ is a multiplicative factor independent of the summation indices σ_i ($i = 1, 2, ..., n$). To put (B. 7) into a more tractable form, we introduce the new variables r and θ_i by $d_i + \mathrm{i}\rho_0^2 k_i = r\mathrm{e}^{\mathrm{i}\theta_i} = r z_i$, where $z_i = \mathrm{e}^{\mathrm{i}\theta_i}$, $r = \sqrt{d_i^2 + \rho_0^4 k_i^2} = \sqrt{\kappa^4 + \rho_0^2}$. Note that r is a constant independent of k_i. To proceed, we substitute the relation

$$\frac{d_i - \kappa^2 \rho_0 \sigma_i k_i}{1 + \rho_0 \sigma_i k_i} = \frac{-\kappa^2 d_i + \kappa^4 + \rho_0^2 - \rho_0^3 \sigma_i k_i}{d_i - \kappa^2}, \tag{B.8}$$

which follows from (2.24) into the first term on the right-hand side of (B. 7) and then rewrite $P_{m,n}$ in terms of the new variables z_i. Dropping a factor independent of the summation indices σ_i, the equation to be proved reduces to the following algebraic identity in $z_1, z_2, ..., z_n$:

$$P_n(z_1, z_2, ..., z_n)$$

$$\equiv \sum_{\sigma = \pm 1} \left[\left\{ \frac{r}{2\mathrm{i}\rho_0} \sum_{i=1}^{n} \left(z_i^{\sigma_i} - z_i^{-\sigma_i} \right) + 1 \right\} \prod_{j=1}^{n} \left\{ -\frac{\kappa^2}{2r} \left(z_j + z_j^{-1} \right) + 1 - \frac{\rho_0}{2\mathrm{i}r} \left(z_j^{\sigma_j} - z_j^{-\sigma_j} \right) \right\} \right.$$

$$\left. - \prod_{j=1}^{n} \left\{ \frac{1}{2} \left(z_j + z_j^{-1} \right) - \frac{\kappa^2}{r} \right\} z_j^{-\sigma_j} \right] \prod_{1 \le i < j \le n} \left(z_i^{\sigma_i} - z_j^{\sigma_j} \right) \left(z_i^{-\sigma_i} - z_j^{-\sigma_j} \right) = 0, \ (n = 1, 2, ..., N). \tag{B.9}$$

The proof proceeds by mathematical induction. The identity (B. 9) can be confirmed for $n = 1, 2$ by a direct computation. Assume that $P_{n-2} = P_{n-1} = 0$. Then,

$$P_n|_{z_1=1} = 2 \left(1 - \frac{\kappa^2}{r} \right) \prod_{i=2}^{n} (1 - z_i)(1 - z_i^{-1}) P_{n-1}(z_2, z_3, ..., z_n) = 0. \tag{B.10}$$

$$P_n|_{z_1=z_2} = -2(z_1 - z_1^{-1})^2 \left\{ \frac{1}{2}(z_1 + z_1^{-1}) - \frac{\kappa^2}{r} \right\}^2$$

$$\times \prod_{j=3}^{n} (z_1 - z_j)(z_1^{-1} - z_j^{-1})(z_1^{-1} - z_j)(z_1 - z_j^{-1}) P_{n-2}(z_3, z_4, ..., z_n) = 0. \tag{B.11}$$

The function P_n is symmetric with respect to $z_1, z_2, ..., z_n$ and invariant under the transformation $z_i \to z_i^{-1}$ for arbitrary i. When coupled with the above two properties (B. 10) and (B. 11), one can see that P_n is factored by a function

$$\prod_{i=1}^{n} (z_i - 1)(z_i^{-1} - 1) \prod_{1 \le i < j \le n} (z_i - z_j)(z_i - z_j^{-1})(z_i^{-1} - z_j)(z_i^{-1} - z_j^{-1}). \tag{B.12}$$

It turns out from this expression that

$$\prod_{i=1}^{n} z_i^2 \prod_{1 \le i < j \le n} (z_i z_j)^2 P_n = A_n \prod_{i=1}^{n} (z_i - 1)(1 - z_i) \prod_{1 \le i < j \le n} (z_i - z_j)^2 (z_i z_j - 1)^2, \qquad (B.13)$$

where A_n is a polynomial of $z_1, z_2, ..., z_n$. The left-hand side of (B. 13) is a polynomial whose degree in $z_1, z_2, ..., z_n$ is at most $2n^2 + 2n$ whereas that of the right-hand side is $3n^2 - n$ at least. This is impossible for $n \ge 4$ except $P_n \equiv 0$. The identity $P_3 = 0$ can be checked by a direct computation, implying that the identity (B. 9) holds for all n.

The bilinear equations (2.12), (2.13) and (2.14) reduce, after substituting the tau-functions (2.21) and (2.22), to the algebraic identities $Q_n = 0, R_n = 0$ and $S_n = 0$, respectively, where

$$Q_n(z_1, z_2, ..., z_n)$$

$$\equiv \sum_{\sigma = \pm 1} \left[\left[\prod_{i=1}^{n} \left\{ \frac{1}{2}(z_i + z_i^{-1}) - \frac{\kappa^2}{r} \right\} + \frac{1}{2} \sum_{i=1}^{n} \left(z_i^{\sigma_i} - z_i^{-\sigma_i} \right) \prod_{\substack{j=1 \\ (j \ne i)}}^{n} \left\{ \frac{1}{2}(z_j + z_j^{-1}) - \frac{\kappa^2}{r} \right\} \right] \prod_{j=1}^{n} z_j^{-\sigma_j} \right.$$

$$\left. - \prod_{i=1}^{n} \left\{ -\frac{\kappa^2}{2r}(z_i + z_i^{-1}) + 1 - \frac{\rho_0}{2ir} \left(z_i^{\sigma_i} - z_i^{-\sigma_i} \right) \right\} \right] \prod_{1 \le i < j \le n} \left(z_i^{\sigma_i} - z_j^{\sigma_j} \right) \left(z_i^{-\sigma_i} - z_j^{-\sigma_j} \right),$$

$$(n = 1, 2, ..., N). \qquad (B.14)$$

$$R_n(z_1, z_2, ..., z_n)$$

$$\equiv \sum_{\sigma = \pm 1} \left[\left\{ 1 + \frac{r}{2i\rho_0} \sum_{i=1}^{n} \left(z_i^{\sigma_i} - z_i^{-\sigma_i} \right) \right\} \left[\sum_{i=1}^{n} \left(z_i^{\sigma_i} - z_i^{-\sigma_i} \right) \prod_{\substack{j=1 \\ (j \ne i)}}^{n} \left\{ \frac{1}{2}(z_j + z_j^{-1}) - \frac{\kappa^2}{r} \right\} \right] \right.$$

$$\left. - \frac{\kappa^2 r}{\rho_0^2} \sum_{i=1}^{n} \left(z_i^{\sigma_i} - z_i^{-\sigma_i} \right) \prod_{i=1}^{n} \left\{ \frac{1}{2}(z_i + z_i^{-1}) - \frac{\kappa^2}{r} \right\} \right]$$

$$\times \prod_{i=1}^{n} \left\{ -\frac{\kappa^2}{2r}(z_i + z_i^{-1}) + 1 - \frac{\rho_0}{2ir} \left(z_i^{\sigma_i} - z_i^{-\sigma_i} \right) \right\}$$

$$\times \prod_{1 \le i < j \le n} \left(z_i^{\sigma_i} - z_j^{\sigma_j} \right) \left(z_i^{-\sigma_i} - z_j^{-\sigma_j} \right), \quad (n = 1, 2, ..., N). \qquad (B.15)$$

$$S_n(z_1, z_2, ..., z_n)$$

$$\equiv \sum_{\sigma = \pm 1} \left[\left\{ \frac{\kappa^2}{r} + \frac{1}{2} \sum_{i=1}^{n} \left(z_i^{\sigma_i} - z_i^{-\sigma_i} \right) \right\} \sum_{i=1}^{n} \left(z_i^{\sigma_i} - z_i^{-\sigma_i} \right) \prod_{\substack{j=1 \\ (j \ne i)}}^{n} \left\{ \frac{1}{2}(z_j + z_j^{-1}) - \frac{\kappa^2}{r} \right\} \right]$$

$$+ \sum_{i=1}^{n} \left(z_i^{\sigma_i} - z_i^{-\sigma_i} \right) \prod_{i=1}^{n} \left\{ \frac{1}{2}(z_i + z_i^{-1}) - \frac{\kappa^2}{r} \right\} \Bigg]$$

$$\times \prod_{i=1}^{n} z_i^{-\sigma_i} \prod_{1 \leq i < j \leq n} \left(z_i^{\sigma_i} - z_j^{\sigma_j} \right) \left(z_i^{-\sigma_i} - z_j^{-\sigma_j} \right), \quad (n = 1, 2, ..., N). \qquad (B.16)$$

The polynomials Q_n, R_n and S_n are shown to be factored by a function (B. 12). Applying the similar induction argument to that used in proving (B. 9), one can establish the identities $Q_n = 0, R_n = 0$ and $S_n = 0$. This completes the proof of theorem 2.2.

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