An Algorithm to form a D-stable Coefficient Set and its Application to the Determination of ℓ^2 D-stability Radius

Fumitake Fujii (Department of Mechanical Engineering)
Kenzo Wada (Department of Mechanical Engineering)

Abstract

This paper concerns the zeros of a polynomial inside a given circle in the left half complex plane. Marden provided an algorithm to determine the number of zeros which a given polynomial has inside the unit circle by using only its coefficients. In this short article, it will be shown that the result can be extended to more general case where the center of the circle is on the real axis and its radius is arbitrary. The execution of the proposed algorithm will produce a set of inequalities which serve as an algebraic expression of a D-stable set in the coefficient space. It should be emphasized that those expressions are suitable for controller design consideration in the coefficient space. The determination of ℓ^2 D-stability radius as an application of the result and its related theorem will also be dictated.

Key Words: Marden's algorithm, coefficient space, zeros of a polynomial, Rouchè's theorem, ℓ^2 D-stability

1 Introduction

For linear time-invariant plants in state space, pole assignment is an accepted technique for designing a regulator which ensures ideal transient response[2]. But it is often adequate to place all closed loop poles inside a pre-specified region instead of specifying exact location of each. This kind of problem is referred to as the "D-stabilization" or "D-pole assignment", where D represents a domain of interest like inside a circle[8], inside an ellipse or left of a sector[3]. D-stabilization to a circular region for plants with structured uncertainty has been extensively studied in the state space for the past decade[9, 10, 11, 12]. On the other hand, it is known that an $n-1$-th order controller can assign the corresponding closed loop poles of a control system with n-th order fixed coefficient plant transfer function at arbitrary location if the plant has mutually coprime numerator and denominator polynomials. However, the problem becomes surprisingly difficult when interval perturbations in the plant coefficients should be considered, that is, when the plant needs to be robustly D-stabilized. This is because the relationship between coefficients of a polynomial and the corresponding root location is hard to be established analytically for high order polynomials. Thus finding a set or a boundary surface in the coefficient space which guarantees closed loop pole location inside a desirable domain is crucial for achieving robust D-stabilization. For instance, Keel and Bhattacharyya[7] proposed fixed order pole-assignment problem for a linear uncertain plant in the coefficient space and presented a linear programming solution using "desired interval coefficient set" which is obtained by a repetitive root calculation. Soh *et al.*[6] also gave robust pole assignment solution for an interval plant using the formulation of nonlinear programming. They presented three different approaches to calculate the D-stable coefficient set: a repetitive root calculation using the famous result of Kharitonov, the sensitivity Jacobian and a convex polytope of the polynomial coefficients corresponding to the disjoint real interval roots.

This paper will present an algorithm to obtain a D-stable set in the coefficient space when the domain of interest is the interior of a circle in the left half complex plane. The development will be shown to result in the extension of the the Marden's algorithm[1] which determines the number of zeros of a given polynomial $f(s)$ inside a unit circle in the complex plane. The key of Marden's result lies in a definition of the associated polynomial given in accordance with $f(s)$. It will be shown that our extension is also achieved by the specially constructed associated polynomial using the inversion of zeros of $f(s)$ about the circle in concern. The resulting set consists of n possibly nonlinear simultaneous inequalities, where n is the order of the polynomial. It is worth to note that the obtained algebraic set is quite suitable to be combined with the already existing methods on robust D-stabilizing controller design. Our result will also allow the calculation of ℓ^2 D-stability radius for a given D-stable polynomial. Its formulation and a related theorem will be explained in section 3.

2 Extension of the Marden's algorithm

This section states how the algorithm given by Marden[1] can be extended to a general case where the domain of interest is a circle located on a left half complex plane with arbitrary radius. Let the polynomial in concern be given by

$$\phi_0(s) = \alpha_0 + \alpha_1 s + \cdots + \alpha_n s^n \tag{1}$$

and the target domain D be a circle as shown in Fig. 1, which is defined by

$$D \triangleq \{s \in C \,||s+\alpha| \le r\} \quad (0 < r < |\alpha|). \tag{2}$$

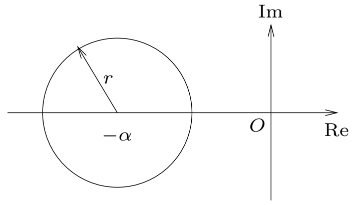

Figure 1: The target domain D

In the following discussion, we will restrict our attention to the case where the coefficients $\alpha_i (i = 0, 1, \cdots, n)$ of (1) are all real. The center $-\alpha$ of the circle (2) should be assumed to be real because symmetricity on the location of zeros of a real polynomial forces to choose a domain symmetric about the real axis. Next, let us recall the definition of an inversion[15].

Definition 1 (Inversion) *A point P' on a half-line $\overline{OP}$ in the complex plane is said to be an inversion of P about the circle Q if $\overline{OP} \cdot \overline{OP'} = r^2$ where O is the center of Q and r is its radius.*

A real polynomial $\phi_0(s)$ can be rewritten in the factored form

$$\begin{aligned} \phi_0(s) &= \alpha_n \prod_{j=1}^{n} (s - s_j) && (3) \\ &= \alpha_n \prod_{j=1}^{n} (s - \overline{s_j}), && (4) \end{aligned}$$

where s_j $(j = 1, 2, \ldots, n)$ denotes the zero of (1). The inversion of s_j about the boundary circle of domain (2) is given by

$$s_j^* = -\alpha + \frac{r^2}{\overline{s_j} + \alpha}, \tag{5}$$

and let us define a polynomial $\phi_0^*(s)$ associated with $\phi_0(s)$ by

$$\begin{aligned} \phi_0^*(s) &\triangleq \frac{(s+\alpha)^n}{r^n} \phi_0 \left(-\alpha + \frac{r^2}{s+\alpha} \right) && (6) \\ &= \frac{(s+\alpha)^n}{r^n} \alpha_n \prod_{j=1}^{n} \left(-\alpha + \frac{r^2}{s+\alpha} - \overline{s_j} \right) \\ &= \frac{\phi_0(-\alpha)}{r^n} \prod_{j=1}^{n} (s - s_j^*) && (7) \\ &= \beta_0 + \beta_1 s + \cdots + \beta_n s^n. \end{aligned}$$

As one can see from the expression (7), the associated polynomial defined by (6) has s_j^* $(j = 1, 2, \ldots, n)$: the inversion of s_j about the boundary circle ∂D, as its zero.

Note: In the Marden's algorithm[1] where ∂D is a unit circle at the origin, the associated polynomial $f^*(s)$ accordingly obtained from $f(s)$ was defined by using z_k^*: the inversion of the zero z_k of $f(s)$ about the unit circle, as shown in equation (42,14). The definition of the associated polynomial $\phi_0^*(s)$ is similarly constructed with the use of the expression (5).

Lemma 1 *$\phi_0(s)$ and $\phi_0^*(s)$ have a same absolute value on the boundary of the circular region D, that is, the following equation holds*

$$|\phi_0^*(-\alpha+re^{j\theta})| = |\phi_0(-\alpha+re^{j\theta})| \quad (\theta \in [0, 2\pi]). \tag{8}$$

Proof) A direct calculation using (3) and (7) shows

$$\phi_0^*(-\alpha + re^{j\theta}) = e^{jn\theta} \cdot \overline{\phi_0(-\alpha + re^{j\theta})}, \tag{9}$$

and the result follows. ■

The dependency of Marden's algorithm on Rouche's theorem together with lemma 1 implies that the body of Marden's procedure can be applied as it is to the determination of the number of zeros which $\phi_0(s)$ has inside the circle D. Thus we have just obtained the following theorem.

Theorem 1 (Extended result of Marden) *For a given real polynomial $\phi_0(s)$, let us define a polynomial sequence*

$$\phi_j(s) = \sum_{k=0}^{n-j} \alpha_k^{(j)} s^k \quad (j = 0, 1, \cdots, n) \tag{10}$$

where

$$\begin{cases} \phi_{j+1}(s) &= \beta_{n-j}^{(j)}\phi_j(s) - \alpha_{n-j}^{(j)}\phi_j^*(s) \\ \alpha_k^{(j+1)} &= \beta_{n-j}^{(j)}\alpha_k^{(j)} - \alpha_{n-j}^{(j)}\beta_k^{(j)}. \end{cases} \tag{11}$$

$\phi_j^*(s)$ in (11) is similarly generated from $\phi_j(s)$ by replacing ϕ_0 and n in (6) with ϕ_j and $n-j$, respectively. Also, let δ_{j+1} be a real number defined by

$$\delta_{j+1} = |\beta_{n-j}^{(j)}|^2 - |\alpha_{n-j}^{(j)}|^2. \tag{12}$$

If

- *$\phi_j(s)$ has p_j zeros inside D and no zeros on the boundary ∂D for each j,*
- *$\delta_{j+1} \neq 0$,*

then $\phi_{j+1}(s)$ has

$$p_{j+1} = \frac{1}{2}\{n - j - [(n-j) - 2p_j]sgn(\delta_{j+1})\} \tag{13}$$

zeros inside D and no zeros on the boundary ∂D where D is a circular region defined by (2). Furthermore, if we define P_k as

$$P_k \stackrel{\triangle}{=} \delta_1\delta_2\cdots\delta_k \quad (k = 1, 2, \cdots, n), \tag{14}$$

the number of zeros of $\phi_0(s)$ inside the circle D can be calculated by

$$p = \frac{1}{2}\left(n - \sum_{k=1}^{n} sgnP_k\right). \tag{15}$$

Proof) It can be proved in the same manner as done in the original proof of Marden's result by noticing that lemma 1 holds. ■

What can be drawn from theorem 1 is that $\phi_0(s)$ has all its n zeros inside D and no zeros on the boundary if and only if

$$sgnP_k = -1 \quad (k = 1, 2, \cdots, n). \tag{16}$$

For it is equivalent to the verification of the following n inequalities

$$\delta_1 < 0 \tag{17}$$
$$\delta_k > 0 \quad (k = 2, 3, \cdots, n), \tag{18}$$

and recalling that each δ_k is a function of the coefficients of the polynomial (1), (17)(18) provides an algebraic expression of a D-stable set in the coefficient space within which the coefficients of (1) are allowed to vary without losing its D-stability. We will refer to this D-stable coefficient set as V in the following discussion.

We state here a numerical example on the execution of the presented algorithm.
Example 1 [Execution of the algorithm]
Consider the problem of determining the number of zeros that the polynomial

$$\phi_0(s) = a_2 s^2 + a_1 s + a_0 \tag{19}$$

has on the circular region D

$$D = \{s \| s+3| < 1\} \tag{20}$$

by using its coefficients a_2, a_1, a_0.

First, generate the polynomial sequence in accordance with (11) by

$$\begin{aligned}
\phi_0^*(s) &= (s+3)^2 \times \left\{ a_2 \left(\frac{-3s-8}{s+3} \right)^2 \right. \\
&\qquad \left. + a_1 \left(\frac{-3s-8}{s+3} \right) + a_0 \right\} \\
&= (9a_2 - 3a_1 + a_0)s^2 \\
&\qquad + (48a_2 - 17a_1 + 6a_0)s \\
&\qquad + (64a_2 - 24a_1 + 9a_0) \\
&= \beta_2 s^2 + \beta_1 s + \beta_0 \qquad (21) \\
\phi_1(s) &= \beta_2 \phi_0(s) - a_2 \phi_0^*(s) \\
&= (\beta_2 a_1 - a_2 \beta_1)s + (\beta_2 a_0 - a_2 \beta_0) \\
&= \alpha_1^{(1)} s + \alpha_0^{(1)}. \qquad (22)
\end{aligned}$$

Similarly, $\phi_1^*(s)$ and $\phi_2(s)$ can be calculated as

$$\begin{aligned}
\phi_1^*(s) &= (-3\alpha_1^{(1)} + \alpha_0^{(1)})s + (-8\alpha_1^{(1)} + 3\alpha_0^{(1)}) \\
&= \beta_1^{(1)} s + \beta_0^{(1)} \qquad (23) \\
\phi_2(s) &= \beta_1^{(1)} \alpha_0^{(1)} - \alpha_1^{(1)} \beta_0^{(1)} \qquad (24)
\end{aligned}$$

and the algorithm is terminated. From theorem 1, one can conclude that the given polynomial (19) has all its zeros inside the circle D if

$$\begin{aligned}
\delta_1 &= |\beta_2|^2 - |a_2|^2 < 0 \qquad (25) \\
\delta_2 &= |\beta_1^{(1)}|^2 - |\alpha_1^{(1)}|^2 > 0 \qquad (26)
\end{aligned}$$

holds. Conditions (25) and (26) can be expressed in terms of the coefficients a_2, a_1 and a_0 like

$$\begin{aligned}
&(8a_2 - 3a_1 + a_0)(10a_2 - 3a_1 + a_0) < 0 \qquad (27) \\
&(a_0^2 + 6a_1^2 + 32a_2^2 - 5a_0 a_1 - 28a_1 a_2 + 12a_2 a_0) \\
&\times (a_0^2 + 12a_1^2 + 128a_2^2 \\
&- 7a_0 a_1 - 80a_1 a_2 + 24a_2 a_0) > 0. \qquad (28)
\end{aligned}$$

■

3 Calculation of ℓ^2 D-stability radius

For a stability analysis of a polynomial in the coefficient space, the question "how large perturbation on the coefficients of a nominally stable polynomial is permitted while maintaining its stability" is usually of great concern. To answer this question quantitatively, the "ℓ^2 stability radius" is often used[16]. The same is true for a robust D-stability of a polynomial, and "ℓ^2 D-stability radius" can be defined similarly. This section dictates several results on the ℓ^2 D-stability radius by using the previously obtained algebraic expression of a D-stable set.

The set of surfaces defined by

$$E = \{(\alpha_n, \cdots, \alpha_0) \mid \delta_k = 0 \quad (\forall k = 1, 2, \cdots, n)\}$$

is a boundary of a D-stable coefficient set V in the coefficient space because zeros of a polynomial is a continuous function of its coefficients. Thus for a given D-stable polynomial

$$f(s) = a_n s^n + \cdots + a_1 s + a_0,$$

finding a minimum of

$$\min_{(\alpha_n, \cdots, \alpha_1, \alpha_0)} \left\{ \sum_{i=0}^{n} (\alpha_i - a_i)^2 \right\}^{\frac{1}{2}}$$
$$\text{s.t. } (\alpha_n, \cdots, \alpha_1, \alpha_0) \in E \tag{29}$$

appears to be an ℓ^2 D-stability radius evaluated in the coefficient space.
Example 2 [Calculation of ℓ^2 D-stability radius]
Let the left hand side of (27) and (28) be $h_1(a_0, a_1, a_2)$ and $h_2(a_0, a_1, a_2)$, respectively. Assume for simplicity that a nominally D-stable real polynomial is given by

$$\phi_0(s) = a_0^n + a_1^n s + a_2^n s^2 = (s+\beta)^2 \quad (-\beta + j0 \in D).$$

Calculation of the ℓ^2 D-stability radius about the circle D defined by (20) can be formulated as a following constrained minimization problem

$$\min_{i=1,2} \left\{ (a_0 - a_0^n)^2 + (a_1 - a_1^n)^2 + (a_2 - a_2^n)^2 \right\}^{\frac{1}{2}}$$
$$\text{s.t. } h_i(a_0, a_1, a_2) = 0. \tag{30}$$

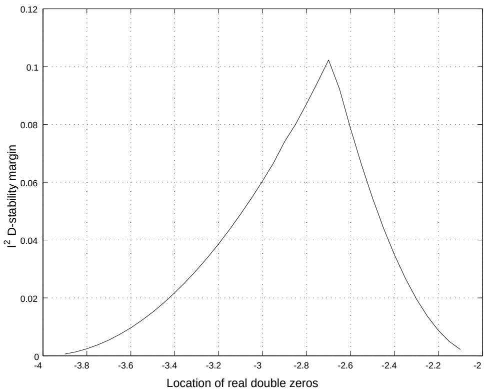

Figure 2: Relationship between the ℓ^2 D-stability radius and the location of the nominal double zeros

The radius obtained by solving (30) is clearly a function of the location of the nominal zeros (in this case $-\beta$). It is then quite natural to ask which β will lead to the supremum of the D-stability radius. Fig. 2 is obtained by iteratively solving the constrained optimization problem (30) by changing the value of $-\beta$ from -3.9 to -2.1 with the increment by 0.05. It is interesting to note that the supremum is taken at $-\beta = -2.7$, not being equal to the center of the circle $-3.0 + j0$ as one might intuitively imagine.

The next theorem claims where the minimum of ℓ^2 D-stability radius for given interval polynomial family is taken.

Theorem 2 [Vertex result on ℓ^2 D-stability radius]

Assume the interval polynomial family $P(s)$

$$\begin{aligned} P(s) &= a_n s^n + \cdots + a_1 s + a_0 \qquad (31) \\ & \qquad a_i \in [a_i^-, a_i^+] \ (\forall i = 0, 1, \cdots, n) \end{aligned}$$

is robustly D-stable against all possible coefficient perturbations. Then the minimum ℓ^2 D-stability radius of the given interval polynomial family will be taken on one of the 2^{n+1} vertices of the hypercube H

$$H = \left\{ (a_n, a_{n-1}, \cdots, a_0) \mid a_i \in [a_i^-, a_i^+] \ (\forall i) \right\}. \qquad (32)$$

Proof) Let the vector of coefficients be denoted like

$$\boldsymbol{a} = [a_n, a_{n-1}, \ldots, a_0]^T,$$

and the vertices of the hypercube H be denoted by $\boldsymbol{v}_k$ ($k = 1, 2, \cdots, 2^{n+1}$). We prove the theorem by contradiction. Now suppose the minimum ℓ^2 D-stability radius is taken on a polynomial $\boldsymbol{c} \in H$ where $\boldsymbol{c} \neq \boldsymbol{v}_k (\forall k)$. Let $\rho(\boldsymbol{x})$ denote the ℓ^2 D-stability radius of the polynomial $\boldsymbol{x}$. Then the assumption leads the following inequality

$$\rho(\boldsymbol{c}) < \min(\rho(\boldsymbol{v}_1), \ldots, \rho(\boldsymbol{v}_{2^{n+1}})). \qquad (33)$$

So there is at least one $\boldsymbol{p} \in \boldsymbol{R}^{n+1}$ with $\|\boldsymbol{p}\| = \rho(\boldsymbol{c})$ such that the polynomial $\boldsymbol{c} + \boldsymbol{p}$ is not D-stable. On the other hand, the inequality $\rho(\boldsymbol{x}) < y$ assures robust D-stability of a polynomial $\boldsymbol{x}$ against any coefficient perturbations $\boldsymbol{q}$ satisfying $\|\boldsymbol{q}\| < y$. Thus we can infer the 2^{n+1} polynomials $\boldsymbol{v}_1 + \boldsymbol{p}, \ldots, \boldsymbol{v}_{2^{n+1}} + \boldsymbol{p}$ are all D-stable from the relation (33). Recall the circular domain D in concern is a Kharitonov region[5], D-stability of 2^{n+1} aforementioned vertex polynomials will lead to the robust D-stability of a hypercube defined by

$$H' = \left\{ (a_n, \cdots, a_0) + \boldsymbol{p} \mid a_i \in [a_i^-, a_i^+] \ (\forall i) \right\}. \qquad (34)$$

It can be seen easily that $\boldsymbol{c} + \boldsymbol{p}$ belongs to H', leading to a contradiction to the statement above. Thus the proof is completed. ■

4 Conclusion

In this paper, a solution was given for the problem of determining the number of zeros of a polynomial $\phi_0(s)$ inside a circle D in the left half complex plane by using its coefficients, based on an algorithm originally proposed by Marden whose equivalent known as Jury's stability test. The enhancement is accomplished by defining an associated polynomial $\phi_0^*(s)$ using the inversion of the zeros of $\phi_0(s)$ about the boundary of D, which is given by (6). As you can see, Marden's algorithm owes its theoretical basis to Rouche's theorem. This fact together with lemma 1 shows that the aforementioned problem can be solved in a similar manner. Our extension also allows to express a D-stable coefficient set algebraically in the coefficient space as a set of inequalities. It serves as a region within which the coefficients of the closed loop characteristic polynomial should be located in spite of the coefficient perturbation and turns out to be useful when one considers a controller design problem for an interval plant in the coefficient space. In addition, the expression also permits the calculation of ℓ^2 D-stability radius. It has also been shown that the vertex result holds on the determination of a minimum ℓ^2 D-stability radius of a robustly D-stable interval polynomial as stated in section 3.

References

[1] M. Marden : "The Geometry of the zeros of a polynomial in a complex variable," American Mathematical Society, 1945

[2] K. Ogata : "Modern Control Engineering – Second Edition," Prentice-Hall International, 1990

[3] K. Joya : "A study on robust D-stability – A Parametric Approach," Ph.D Dissertation, Tokyo Institute of Technology, 1992

[4] E. I. Jury : Theory and application of the z-transform method, John Wiley, New York, 1964

[5] I. R. Petersen : "A Class of Stability Regions for Which a Kharitonov-Like Theorem Holds," IEEE trans. on Automatic Control, Vol.34, No.10, pp.1111-1115, 1989

[6] Y. C. Soh, R. J. Evans, I. R. Petersen and R. E. Betz : Robust Pole Assignment, Automatica, Vol.23, No.5, pp.601-610, 1987

[7] L. H. Keel and S. P. Bhattacharrya : A Linear Programming Approach To Controller Design, Proc. 36th Conf. Decision & Control, pp.2139-2148, 1997

[8] K.Furuta and S.B.Kim : "Pole Assignment in a Specified Disk," IEEE Transactions on Automatic Control, Vol.32, No.5, pp. 423–427, 1987

[9] Y.T.Juang and K.H.Chen : "Robust Pole-Assignment of Linear Dynamic Systems," Control-Theory and Advance Technology (C-TAT), Vol.5, No.1, pp. 67–74, 1989

[10] J.H.Chou : "Pole-assignment robustness in a specified disk," Systems & Control Letters, Vol.16, pp. 41–44, 1991

[11] R.J.Wang and W.J.Wang : "Pole Assignment by State Feedback for Uncertain Linear Systems," C-TAT, Vol.10, No.4, pp. 1797–1807, 1995

[12] S.Phoojaruenchanachai and K.Furuta : "Pole Placement in Specified Disc for a Class of Uncertain Systems," Proc. SICE, pp.915–918, 1989

[13] H.Rotstein, R.S.Pena, J.Bandoni, A.Desages and J.Romagnoli : "Robust Characteristic Polynomial Assignment," Automatica, Vol.27, No.4, 1991

[14] J. Ackermann : "Robust Control - Systems with Uncertain Physical Parameters -," Springer-Verlag, 1997

[15] T. Takagi : Lecture notes of algebra (in Japanese), Kyoritsu Publishing, 1965

[16] Ya.Z.Tsypkin and B.T.Polyak : "Frequency Domain Criteria for l^p-Robust Stability of Continuous Linear Systems," IEEE Transactions on Automatic Control, Vol.36, No.12, pp.1464-1469, 1991

(Received August 29,2003)