Doctor Dissertation

On the energy estimates of the wave equation with time dependent propagation speed

（時間に依存する伝播速度をもつ波動方程式のエネルギー評価について）

September 2016

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I would like to dedicate the dissertation to my beloved family, supervisor and friends

Abstract

In this thesis, we study the following Cauchy problem for n-dimensional linear wave equation with time-dependent propagation speed $a = a(t)$:

$$\begin{cases} (\partial_t^2 - a(t)^2\Delta)\, u = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ (u(0,x), (\partial_t u)(0,x)) = (u_0(x), u_1(x)), \quad x \in \mathbb{R}^n, \end{cases} \tag{1}$$

where $\mathbb{R}_+ = [0,\infty)$, $a \in C^m(\mathbb{R}_+)$ with $m \geq 2$, $0 < a(t) < a_1$ with a positive constant a_1 and Δ denotes the Laplace operator defined by $\Delta = \sum_{j=1}^n \partial_{x_j}^2$. Then the total energy to the solution of (1) is defined by the sum of the elastic energy and the kinetic energy as follows:

$$E(t) = E(t; u_0, u_1) = \frac{1}{2}a(t)^2\|\nabla u(t,\cdot)\|^2 + \frac{1}{2}\|\partial_t u(t,\cdot)\|^2,$$

where $\|\cdot\|$ denotes the usual L^2 norm in $\mathbb{R}^n$, $\nabla_x = (\partial_{x_1}, \ldots, \partial_{x_n})$.

If the propagation speed a is a constant, then the total energy is conserved with respect to t, that is, the following equality holds:

$$E(t) \equiv E(0).$$

Moreover, it is known that the following estimates are established:

$$\lim_{t\to\infty} \frac{1}{2}a(t)^2\|\nabla u(t,\cdot)\|^2 = \lim_{t\to\infty} \frac{1}{2}\|\partial_t u(t,\cdot)\|^2, \tag{2}$$

that is, both the elastic and the kinetic energies converge to the same quantity as $t \to \infty$. However, such properties are not trivial for time dependent propagation speed. If the energy conservation does not hold, we introduce the following energy estimates, which is called *the generalized energy conservation*:

$$C_0 a(t)^2 E(0) \leq E(t) \leq C_1 E(0) \tag{GEC}$$

with positive constants C_0 and C_1. Actually, (GEC) can be proved immediately by the classical energy method even though a is not a constant if $a'/a \in L^1(R_+)$. However, the classical energy method is useless for the proof of (GEC) if $a'/a \notin L^1(R_+)$; this is what we concern in this thesis. Moreover, we also study the following energy estimate:

$$E(t) \leq C \tag{BE}$$

with a positive constant C, which is called *the boundedness of the energy.* Here the constant C may depend not only on $a(t)$ and $E(0)$ but also on the higher order derivatives of the initial data in the sense of Sobolev and the Gevrey norm of the initial data. Let us introduce the following conditions to the variable propagation speed $a(t)$:

$$\int_t^\infty |a(s) - a_\infty(1+s)^{-\delta}|ds \leq C_0(1+t)^\alpha \tag{3}$$

and

$$|a^{(k)}(t)| \leq C_k(1+t)^{-\delta-\beta k} \tag{4}$$

with $k = 1, \ldots, m$, $a_\infty > 0$, $\alpha < 0$, $\beta < 1$ and $\delta > 0$. Here (3) is called the stabilization property, which describes the order of the difference between $a(t)$ and a monotone decreasing function $a_\infty(1+t)^{-\delta}$, and (4) describes the order of the oscillation and the smoothness of $a(t)$. Then one of our main theorem is given as follows:
Theorem 1. *(i) If* (3) *and* (4) *hold for* $\beta \geq \alpha + \delta + (1 - \alpha - \delta)/m$, *then* (GEC) *is established. Moreover,* (3) *and* (4) *does not conclude* (GEC) *in general if* $\beta < \alpha + \delta$. *(ii) If* $u_0(x) \in \gamma_1^\nu$, $u_1(x) \in \gamma_0^\nu$ *with* $\nu > 1$, (3) *and* (4) *hold for* $\beta \geq \alpha\nu/(\nu + 1) + \delta + (1 - \alpha - \delta)/m$, *then* (BE) *is established, where* γ_0^ν *and* γ_1^ν *are the Gevrey classes of order* ν. *Moreover,* (3) *and* (4) *does not conclude* (BE) *in general if* $\beta < \alpha\nu/(\nu - 1) + \delta$.

Here we note that we have examples of $a(t)$, which cannot be applied any theorems in the preceding works, but can be done Theorem 1 to conclude (GEC) and (BE). The second theorem of this thesis derives a particular effect of the variable propagation speed for the behavior of the elastic energy.

Theorem 2. *Suppose that the conditions of Theorem 1 (i) are fulfilled. If* $\delta > 1$ *and* $\delta \geq -2\alpha$, *then the elastic energy to the solution of* (1) *has the following estimate:*

$$\frac{1}{2}a(t)^2\|\nabla u(t,\cdot)\|^2 \leq C_1(1+t)^{2-2\delta}E(0).$$

Thus the elastic energy has a better estimate than the estimate from (GEC).

We remark that the main theorems of this thesis Theorems 2.1, 2.4 and 2.7 consider more general model of $a(t)$.

Our thesis consists of the following four sections:

In Section 1, we introduce some basic knowledges for the energy estimates of the Cauchy problem of wave equations with time dependent propagation speed, the preceding works related to our main theorems and the motivation to our problems. In section 2, we introduce the main results of this thesis Theorems 2.1, 2.4 and 2.7, and corresponding examples Examples 2.3, 2.6 and 2.8. In section 3, we prove Theorems 2.1 and 2.4. The key idea for the proof of Theorem 2.1 is to estimate the Fourier image of the solution by different way in the high frequency part Z_H, and the low frequency part Z_ψ of the phase space, which are called the hyperbolic zone, and pseudo-differential zone respectively. To be more precise, we construct an approximated solution in Z_H by diagonalization technique for 2×2 matrix valued function making use of the C^m-property (4). On the other hand, the stabilization property (3) is applicable for the estimate in Z_ψ. The proof of Theorem 2.4 is concluded by constructing series of functions of variable propagation speed and initial data, which provide the estimates of non-(GEC) and non-(BE). In section 4, we prove Theorem 2.7. The fundamental idea for the proof is to introduce a new zone Z_{st} in the middle frequency part, which is called the stabilized zone. Consequently, we have more precise estimate of the solution in Z_{st} by using a new technique for the representation of the solution, and thus we can conclude the proof of Theorem 2.7.

Symbols

$\mathbb{R}$: real number field

$\mathbb{R}_+ = [0, \infty)$

$\mathbb{R}^n$: n-dimensional Euclidean space

$\mathbb{C}$: complex number field

$\mathbb{C}^n$: n-dimensional complex Euclidean space

$|x| = \sqrt{|x_1|^2 + \cdots + |x_n|^n}$ ($x \in \mathbb{C}^n$ or $x \in \mathbb{R}^n$)

$\partial_t = \dfrac{\partial}{\partial t}$, $\partial_{x_j} = \dfrac{\partial}{\partial x_j}$

$\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and multi-index of non-negative integers $\alpha = (\alpha_1, \ldots, \alpha_n)$

$\nabla = (\partial_{x_1}, \ldots, \partial_{x_n})$

$\Delta = \sum_{j=1}^n \partial_{x_j}^2$ (Laplace operator in $\mathbb{R}^n$)

$C^m(I)$ (with non-negative integer m): the space of all m-times continuously differentiable functions on I

$C^\infty(I) = \bigcap_{m=0}^\infty C^m(I)$

$C^\sigma(I)$ (with real number $\sigma \in (0, 1)$): the space of all σ-Hölder continuous functions on I

$L^p(\Omega)$: the space of all p-th power Lebesgue integrable functions in Ω

H^s: Sobolev space of order s

$\dot{H}^s$: homogeneous Sobolev space of order s

$W^{p,N}$: the space of all functions f satisfying $\sum_{|\alpha| \leq N} \|\partial^\alpha f(\cdot)\|_{L^p(\mathbb{R}^n)} < \infty$

$(\cdot, \cdot)_X$: inner product of the inner product space X

$\| \cdot \|_X$: norm of the norm space X

$\| \cdot \| = \| \cdot \|_{L^2(\mathbb{R}^n)}$

$f \lesssim g$: for positive functions f and g there exists a positive constant C such that $f \leq Cg$

$f \simeq g \Leftrightarrow f \lesssim g$ and $g \lesssim f$

C, C_k ($k = 0, 1, \ldots$): positive constants

Acknowledgement

- Alhamdulillah, all praises to Allah for the strengths and His blessing in completing this study.
- I am deeply grateful to my supervisor, Professor Fumihiko HIROSAWA, for his great inspiration, excellent guidance, deep thoughts and friendship during the course of my graduate studies.
- My deep gratitude to the dissertation committee:
  - 1. Prof. Makoto MASUMOTO
  - 2. Prof. Isao KIUCHI
  - 3. Prof. Noriaki SUETAKE
  - 4. Ass. Prof. Yasushi HATAYA
- My appreciation to The Indonesian Endowment Fund for Education (Lembaga Pengelola Dana Pendidikan), for his scholarship support through the LPDP scholarship cooperated with Sebelas Maret University.
- I would like to thanks to International student affair staffs: Mr. Takumi KADOTA and Mr. Kiyoshi OKAJIMA for enabling me to visit their office to help me in academic affair and daily life in Japan.
- Finally, very special thanks to my beloved parents, my dear husband (Herman Saputro), my Sons (Razan Lauzai Muaddab, Rakan Lauzai Zuhair and Raihan Lauzai Satohiro). Thanks for all the love, patience, prayers, and encouragement during the last three years.

Contents

1 Introduction

In this thesis we study the following Cauchy problem for n-dimensional linear wave equation with time dependent propagation speed $a = a(t)$:

$$\begin{cases} (\partial_t^2 - a(t)^2\Delta)\, u = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ (u(0,x), (\partial_t u)(0,x)) = (u_0(x), u_1(x)), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $a \in C^1(\mathbb{R}_+)$ and $0 < a(t) \le a_1$ with a positive constant a_1.

The following L^p-L^q type estimate is a fundamental problem of (1.1):

$$\|\nabla u(t,\cdot)\|_{L^q(\mathbb{R}^n)} + \|\partial_t u(t,\cdot)\|_{L^q(\mathbb{R}^n)} \le C(1+t)^{s_0 - \frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\|\nabla u_0(\cdot)\|_{W^{p,N_p}} + \|u_1(\cdot)\|_{W^{p,N_p}}\right), \tag{1.2}$$

where $s_0 \ge 0$, $1/p + 1/q = 1, 1 < p \le 2$ and N_p is an integer satisfying $N_p \ge n\,(1/p - 1/q)$. There are many results, which are concerned with (1.2), for instance, [14, 15, 16, 17]. In particular, if $p = q = 2$, then the estimate (1.2) is an estimate of the total energy to the solution of (1.1) defined by the sum of the elastic energy and the kinetic energy:

$$E(t) = E(t; u_0, u_1) := \frac{1}{2}a(t)^2\|\nabla u(t,\cdot)\|^2 + \frac{1}{2}\|\partial_t u(t,\cdot)\|^2.$$

In this thesis, we focus on L^2-L^2 type estimate of (1.2) with $s_0 = 0$. If the propagation speed is a constant, then the total energy is conserved with respect to t, that is, the following equality holds:

$$E(t) \equiv E(0).$$

Moreover, it is known that the following estimates are establised:

$$\lim_{t\to\infty} \frac{1}{2}a(t)^2\|\nabla u(t,\cdot)\|^2 = \lim_{t\to\infty} \frac{1}{2}\|\partial_t u(t,\cdot)\|^2 = \frac{E(0)}{2},$$

that is, both the elastic energy and the kinetic energy converge to the same quantity as $t \to \infty$. However, such properties are not trivial for time dependent propagation speed. If the energy conservation does not hold, we introduce the following energy estimates:

Definition 1.1. The total energy $E(t)$ to the solution of (1.1) satisfies *the generalized energy conservation* if the following estimate holds:

$$E(t) \simeq E(0). \tag{GEC}$$

Definition 1.2. The total energy $E(t)$ to the solution of (1.1) satisfies *the boundedness of the energy* if the following estimate holds:

$$E(t) \le C. \tag{BE}$$

Remark 1.3. If (GEC) holds, then (BE) is valid with $C = C_1E(0)$. However, (GEC) does not follow from (BE) in general. Indeed, it is possible that the energy cannot be estimated by the initial energy $E(0)$, but can be done by a constant C under some additional assumptions to the initial data for some non-linear problems or linear problem (1.1) with a singular propagation speed (see [1, 2, 3, 4, 5, 9, 11, 13, 18]). Here the constant C in (BE) depends on the following properties of the initial data: the smoothness in Sobolev or the Gevrey class, decay order $|x| \to \infty$, and so on.

Let us consider the sufficient conditions on $a(t)$ which ensure (GEC). One of the trivial results from the standard energy method is given as follows:

Proposition 1.4. *If $a'/a \in L^1(\mathbb{R}_+)$, then* (GEC) *is established.*

It is a natural question whether (GEC) holds or not for $a'/a \notin L^1(\mathbb{R}_+)$. If we restrict ourselves to the following estimate:

$$0 < a_0 \le a(t) \le a_1 \tag{1.3}$$

with positive constants a_0 and a_1, then Reissig and Smith gave the following answers to the question in [14];

Theorem 1.5 ([14])**.** *Let $a \in C^2(\mathbb{R}_+)$ satisfy* (1.3)*. If the following estimates hold:*

$$\left|a^{(k)}(t)\right| \le C_k(1+t)^{-k} \tag{1.4}$$

for $k = 1, 2$, then (GEC) *is established. Moreover, for any $\beta < 1$ there exists $a \in C^\infty(\mathbb{R}_+)$ satisfying* (1.3) *and*

$$\left|a^{(k)}(t)\right| \le C_k(1+t)^{-\beta k} \tag{1.5}$$

for any $k \in \mathbb{N}$ such that (GEC) *does not hold.*

Theorem 1.5 concludes that the size of the parameter β in (1.5), which describes an order of the oscillation of the propagation speed, is crucial for (GEC), and the critical number is $\beta = 1$. Moreover, $a'/a \in L^1(\mathbb{R}_+)$ is not a necessary condition for (GEC); indeed $a(t) = 2+\cos(\log(e+t))$ satisfies both (1.4) for $k = 1, 2$ and $a'/a \notin L^1(\mathbb{R}_+)$. Generally, we cannot expect (GEC) if $a(t)$ doest not satisty (1.5) with $\beta < 1$. However, (GEC) can be valid even though (1.5) is not satistied for β satistying $\beta < 1$ if there exists some constants a_∞ and $\alpha \in [0, 1)$ such that the following estimate, which is called *the stabilization condition*, holds:

$$\int_0^t |a(s) - a_\infty|\, ds \le C_0(1+t)^\alpha. \tag{1.6}$$

Indeed, the following theorem is known:

Theorem 1.6 ([10])**.** *Let $m \ge 2$ and $a \in C^m(\mathbb{R}_+)$ satisfy* (1.3)*. If* (1.5) *for $k = 1, \dots, m$ and* (1.6) *hold for*

$$\beta \ge \beta_m := \alpha + \frac{1-\alpha}{m}, \tag{1.7}$$

then (GEC) *is valid. If $\beta < \alpha$, then the assumptions* (1.5) *and* (1.6) *do not conclude* (GEC) *in general.*

Theorem 1.6 tells us that the condition to $a(t)$ for (GEC) is described by an interaction of the parameters α, β and m, which represent the order of the stabilization to a constant a_∞, the oscillating speed and the smoothness of $a(t)$ respectivety. Moreover, we observe the followings:

(i) β_m is monotone decreasing, and monotone increasing with respect to m, and α respectively. That is, β can be chosen smaller as α smaller or m larger for (1.7). It follows that a stronger stabilization or higher reguralrity admit faster oscillation to $a(t)$ for (GEC).

(ii) $\lim_{m\to\infty} \beta_m = \alpha$, it follows that (1.7) is almost optimal as $m \to \infty$.

(iii) $\inf_{m\geq 2, \alpha\in[0,1)}\{\beta_m\} = 0$, it follows that β must be non-negative for (GEC). Therefore, Theorem 1.6 does not give any answer whether (GEC) is valid or not if (1.5) does not hold for $\beta < 0$.

One of the main purposes of this thesis is to extend the result of Theorem 1.6 faster oscillating propagation speed $a(t)$ which can not be apllied Theorem 1.6.

2 Main Theorems

Let ν and s be real numbers satisfying $\nu > 1$ and $s \ge 0$. The Gevrey class γ_s^ν is the set of all functions of $f \in \dot{H}^s$ satisfying

$$\int_{\mathbb{R}^n} \exp\left(2\rho|\xi|^{\frac{1}{\nu}}\right)|\xi|^{2s}|\hat{f}(\xi)|^2 d\xi < \infty$$

with a positive constants ρ, where $\hat{f}$ denotes the Fourier transformation of f with respect to $x \in \mathbb{R}^n$ and $\dot{H}^s$ is the homogeneous Sobolev space of order s.

We suppose that $a \in C^m(\mathbb{R}_+)$ for $m \ge 2$. Let $\lambda = \lambda(t)$ be a positive and monotone decreasing functions satisfying

$$\lambda(t) \in C^1(\mathbb{R}_+),\ \ a(t) - \lambda(t) \in L^1(\mathbb{R}_+) \ \text{ and } \ a(t) \simeq \lambda(t). \tag{2.1}$$

For a non-negative function $\eta(t)$ and a positive monotone decreasing function $\Xi(t)$ satisfying

$$\Xi(t) = o\left(\int_t^\infty \lambda(s)ds\right) \ \ (t \to \infty), \tag{2.2}$$

we introduce the following conditions:

$$\left|a^{(k)}(t)\right| \le C_k \lambda(t)\eta(t)^k \ \ (k = 1, \dots, m) \tag{2.3}$$

and

$$\int_t^\infty |a(s) - \lambda(s)|\, ds \le \Xi(t). \tag{2.4}$$

Then our first theorem is given as follows:

Theorem 2.1 ([6]). *Let $a \in C^m(\mathbb{R}_+)$ with $m \ge 2$. Suppose that* (2.1), (2.2), (2.3) *and* (2.4) *are valid. If there exist positive constants K_0 and K_1 such that*

$$\frac{\eta(t)\Xi(t)}{\lambda(t)} \le K_0 \tag{2.5}$$

and

$$\Xi(t)^{m-1}\int_0^t \lambda(s)\left(\frac{\eta(s)}{\lambda(s)}\right)^m ds \le K_1, \tag{2.6}$$

then the following estimates are established:

$$\lambda(t)^2 E(0) \lesssim E(t) \lesssim C E(0). \tag{GEC$'$}$$

In particular, if $(u_0, u_1) \in \gamma_1^\nu \times \gamma_0^\nu$ and there exist positive constants $\tilde{K}_0$ and $\tilde{K}_1$ such that

$$\frac{\eta(t)\Xi(t)^\kappa}{\lambda(t)} \le \tilde{K}_0 \tag{2.7}$$

and

$$\Xi(t)^{\kappa(m-1)}\int_0^t \lambda(s)\left(\frac{\eta(s)}{\lambda(s)}\right)^m ds \le \tilde{K}_1 \tag{2.8}$$

for a real number κ satisfying

$$\kappa > \frac{\nu}{\nu - 1}, \tag{2.9}$$

then (BE) *is established.*

Remark 2.2. It is common to introduce the Gevrey class on the well-posedness for the Cauchy problem with singular propagation speed, where $a(t)$ is singular if it is non-Lipschitz continuous or having zeros. In [4], which is a pioneer work for this kind of problem, it is proved that if $a(t) > 0$ and $a(t) \in C^\sigma(\mathbb{R}_+)$ with $\sigma \in (0,1)$, then (1.1) is well-posed in the Gevrey class γ_0^ν with $\nu < 1/(1-\sigma)$. After that, the relations between various types of singularities of $a(t)$ and the Gevrey order ν for the well-posedness of (1.1) were studied in many papers, for instance [1, 3, 5, 11, 13]. In particular, a sort of stabilization properties corresponding to (1.6) and (2.4) are introduced in [1, 11, 13].

Example 2.3. We restrict ourselves to considering the following models:

$$\Xi(t) = C_0(1+t)^\alpha, \quad \eta(t) = (1+t)^{-\beta} \quad \text{and} \quad \lambda(t) = a_\infty(1+t)^{-\delta} \tag{2.10}$$

with $\alpha < 0$, $\delta \geq 0$ and a positive constant a_∞. Here (2.2) requires the following inequality:

$$\alpha < -\delta + 1. \tag{2.11}$$

Then (2.5), and (2.6) are valid for

$$\beta \geq \beta_{\infty,\delta} := \alpha + \delta,$$

and

$$\beta \geq \beta_{m,\delta} := \alpha + \delta + \frac{1-\alpha-\delta}{m}$$

respectively. Noting $\beta_{m,\delta} > \beta_{\infty,\delta}$ and $\lim_{m\to\infty} \beta_{m,\delta} = \beta_{\infty,\delta}$, the condition (2.6) approaches (2.5) as $m \to \infty$. Analogously, (2.7), and (2.8) are valid for

$$\beta > \tilde{\beta}_{\infty,\delta} := \frac{\alpha\nu}{\nu-1} + \delta,$$

and

$$\beta > \tilde{\beta}_{m,\delta} := \frac{\alpha\nu}{\nu-1} + \delta + \frac{1-\alpha-\delta}{m}$$

respectively. Moreover, we see that $\tilde{\beta}_{m,\delta} > \tilde{\beta}_{\infty,\delta}$ and $\lim_{m\to\infty} \tilde{\beta}_{m,\delta} = \tilde{\beta}_{\infty,\delta}$.

The following theorem ensures the optimality of the conditions (2.5) and (2.7), or the conditions (2.6) and (2.8) approach the optimal ones as $m \to \infty$.

Theorem 2.4 ([6]). *Let $n = 1$ and $\nu > 1$. For any $\beta \in \mathbb{R}$ satisfying*

$$\beta < \min\{0, \beta_{\infty,\delta}\}, \tag{2.12}$$

there exist $a_j \in C^\infty(\mathbb{R}_+)$ and $(u_{j,0}(x), u_{j,1}(x))$ $(j = 1, 2, \dots)$ satisfying (2.1), (2.3) and (2.4) with $a_j(t) = a(t)$, (2.10), (2.11) and

$$E(0; u_{j,0}, u_{j,1}) \leq 1 \tag{2.13}$$

such that

$$\lim_{j\to\infty} \sup_{t>0} E(t; u_{j,0}, u_{j,1}) = \infty. \tag{2.14}$$

Moreover, if

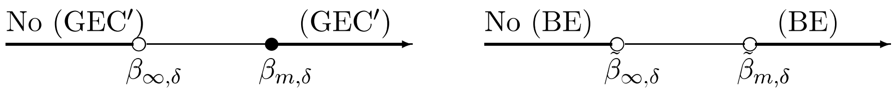
$$\beta < \min\left\{0, \tilde{\beta}_{\infty,\delta}\right\}, \tag{2.15}$$

then (2.14) is established though the following estimate is valid:

$$\sup_j \left\{ \int_{\mathbb{R}} \exp\left(2\rho|\xi|^{\frac{1}{\nu}}\right) \left(a_j(0)^2|\xi|^2 \left|\hat{u}_{j,0}(\xi)\right|^2 + |\hat{u}_{j,1}(\xi)|^2\right) d\xi \right\} \le 1 \tag{2.16}$$

for a positive constant ρ.

Remark 2.5. If $\beta \ge \beta_{m,\delta}$, then (GEC$'$) is valid by Theorem 2.1; thus (2.14) is not realized since (2.13) holds. Analogously, if $\beta > \tilde{\beta}_{m,\delta}$, then (2.14) is not true since (2.16) holds. It is open problems whether (GEC$'$) or (BE) hold or not if $\beta_{\infty,\delta} \le \beta < \beta_{m,\delta}$, and $\tilde{\beta}_{\infty,\delta} \le \beta \le \tilde{\beta}_{m,\delta}$ respectively.



Example 2.6. Let us introduce an example of $a(t)$, which can be applied Theorem 2.1. Let $\chi \in C^\infty(\mathbb{R})$ be a 1-periodic function satisfying

$$0 \le \chi(\tau) \le 1 \text{ and } \chi(\tau) \equiv 0 \text{ near } \tau = 0. \tag{2.17}$$

We suppose that γ, δ, ε and κ are real numbers satisfying

$$\gamma > 1,\ \varepsilon \ge 0,\ \kappa < 0 \text{ and } \max\{0, -1+\kappa+\gamma\} \le \delta \le \gamma.$$

Denoting

$$\rho_j := \frac{[(1+j)^{-\kappa-\varepsilon}]}{[(1+j)^{-\kappa}]},$$

we define $a(t)$ by

$$a(t) := \begin{cases} (1+t)^{-\delta} + (1+j)^{-\gamma}\chi\left([(1+j)^{-\kappa}](t-j)\right) & t \in [j, j+\rho_j), \\ (1+t)^{-\delta} & t \in [j+\rho_j, j+1) \end{cases} \tag{2.18}$$

for $j = 0, 1, \ldots$.

Setting $\lambda(t) = (1+t)^{-\delta}$, for $t \in [j, j+1)$ we have

$$\max_{t\in[j,j+1]} \left\{\left|a^{(k)}(t)\right|\right\} \lesssim (1+t)^{-\delta-k} + (1+j)^{-\gamma-k\kappa} \lesssim (1+t)^{-\delta-k\left(\kappa+\frac{\gamma-\delta}{m}\right)} \quad (k = 1, \cdots, m)$$

and

$$\int_t^\infty |a(s) - \lambda(s)|\, ds \lesssim \sum_{l=j}^{\infty} (1+l)^{-\gamma-\varepsilon} \simeq (1+j)^{-\gamma-\varepsilon+1} \simeq (1+t)^{-\gamma-\varepsilon+1}.$$

Therefore, (2.18) is an example satisfying (2.10) with $\alpha = -\gamma-\varepsilon+1$ and $\beta = \kappa + (\gamma-\delta)/m$.

In Theorem 2.1 and 2.4, we have introduced some results for the estimates of the total energy of the solution to (1.1). In the next theorem we particularly consider the estimate of the elastic energy.

Theorem 2.7 ([7]). *Let $a \in C^m(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$ with $m \geq 2$, and suppose that the same assumption in Theorem 2.1 are fulfilled. If $\lambda(t)$ satisfies*

$$\int_t^\infty a(s)\,ds = O(t\lambda(t)) \quad (t \to \infty) \tag{2.19}$$

and

$$1 \lesssim (1+t)\sqrt{\lambda(t)} \text{ or } (1+t)\sqrt{\lambda(t)} \text{ is monotone decreasing,} \tag{2.20}$$

then there exists a positive constant N such that the following estimate for the elastic energy is valid:

$$\frac{1}{2}a(t)^2\|\nabla u(t,\cdot)\|^2 \lesssim (1+t)^2\lambda(t)^2\left(\|\nabla F(|\nabla|)u_0(\cdot)\|^2 + \|F(|\nabla|)u_1(\cdot)\|^2\right), \tag{2.21}$$

where F is defined by

$$F(r) = \begin{cases} 1 & \text{for } 0 \leq r < 1, \\ \max\left\{1, r\sqrt{\lambda(\Xi^{-1}(Nr^{-1}))}\right\} & \text{for } r \geq 1. \end{cases} \tag{2.22}$$

Example 2.8. Let $\delta > 1$, $\Xi(t)$, $\eta(t)$ and $\lambda(t)$ be given in Example 2.3. Then we see that $a \simeq \lambda \in L^1(\mathbb{R}_+)$, (2.20) and

$$F(r) \simeq \begin{cases} 1 & \text{for } \delta \geq -2\alpha, \\ r^{1+\frac{\delta}{2\alpha}} & \text{for } \delta < -2\alpha \end{cases} \quad (r \to \infty). \tag{2.23}$$

Therefore, by Theorem 2.7 we have the following estimates;

$$\frac{1}{2}a(t)^2\|\nabla u(t,\cdot)\|^2 \lesssim \begin{cases} (1+t)^2\lambda(t)^2 E(0) & \text{for } \delta \geq -2\alpha, \\ (1+t)^2\lambda(t)^2\left(\|\nabla u_0(\cdot)\|^2_{H^{1+\frac{\delta}{2\alpha}}} + \|u_1(\cdot)\|^2_{H^{1+\frac{\delta}{2\alpha}}}\right) & \text{for } \delta < -2\alpha. \end{cases} \tag{2.24}$$

Noting $\lim_{t\to\infty}(1+t)\lambda(t) = 0$, (2.24) gives a better estimate for the elastic energy than the estimate from (GEC').

Example 2.9. Let $\Xi(t)$, $\eta(t)$ and $\lambda(t)$ be given as follows:

$$\Xi(t) = (1+t)^{-\kappa}\exp(-(1+t)^\nu),$$

$$\eta(t) = (1+t)^{-\beta}$$

and

$$\lambda(t) = \exp(-(1+t)^\nu),$$

with $\nu > 1$ and $\kappa > \nu - 1$. Then we have

$$\frac{\eta(t)\Xi(t)}{\lambda(t)} = (1+t)^{-\beta-\kappa}$$

and

$$\int_0^t \lambda(s)\left(\frac{\eta(s)}{\lambda(s)}\right)^m ds = \int_0^t \exp\left((m-1)(1+s)^\nu\right)(1+s)^{-m\beta}\,ds$$
$$= \int_0^t \left(\frac{\exp\left((m-1)(1+s)^\nu\right)(1+s)^{-\nu+1}}{m-1}\right)'(1+s)^{-m\beta}\,ds$$
$$+ \frac{-\nu+1}{m-1}\int_0^t \exp\left((m-1)(1+s)^\nu\right)(1+s)^{-\nu-m\beta}\,ds$$
$$=\frac{\exp\left((m-1)(1+t)^\nu\right)(1+t)^{-m\beta-\nu+1}}{m-1} - \frac{\exp(m-1)}{m-1}$$
$$+ \frac{m\beta-\nu+1}{m-1}\int_0^t \exp\left((m-1)(1+s)^\nu\right)(1+s)^{-\nu-m\beta}\,ds$$
$$=\frac{\Xi(t)^{-m+1}(1+t)^{-\kappa(m-1)-m\beta-\nu+1}}{m-1} - \frac{\exp(m-1)}{m-1}$$
$$+ \frac{m\beta-\nu+1}{m-1}\int_0^t \lambda(s)\left(\frac{\eta(s)}{\lambda(s)}\right)^m (1+s)^{-\nu}\,ds,$$

it follows that

$$\Xi(t)^{m-1}\int_0^t \lambda(s)\left(\frac{\eta(s)}{\lambda(s)}\right)^m ds \lesssim 1 + (1+t)^{-\kappa(m-1)-m\beta-\nu+1}.$$

Thus (2.5) and (2.6) are valid for

$$\beta = -\kappa + \frac{\kappa-\nu+1}{m},$$

and we have

$$F(r) \simeq r^{\frac{1}{2}}(\log r)^{\frac{\kappa}{2\nu}} \quad (r\to\infty).$$

The relation between the previous results Theorem 1.5, 1.6 and Theorem 2.1, 2.4 (Example 2.3 with $\delta = 0$) is represented as the following tables and diagrams:

Table 1: $\alpha \geq 0$

	$\beta < \alpha$	$\alpha \leq \beta < \beta_m$	$\beta_m \leq \beta < 1$	$\beta = 1$	$1 < \beta$
(GEC)	No (iii)	?	Yes (iii)	Yes (ii)	Yes (i)

(i) Proposition 1.4.
(ii) Theorem 1.5 [14] (α=1).
(iii) Theorem 1.6 [10] ($0 \leq \alpha < 1$, $m \geq 2$).

Table 2: $\alpha < 0$

	$\beta < \frac{\alpha\nu}{\nu-1}$	$\frac{\alpha\nu}{\nu-1} \leq \beta \leq \tilde{\beta}_{m,0}$	$\tilde{\beta}_{m,0} < \beta < \alpha$	$\alpha \leq \beta < \beta_m$	$\beta_m \leq \beta$
(GEC)	No (iv)	No (iv)	No (iv)	?	Yes (iv)
(BE)	No (v)	?	Yes (v)	Yes (v)	Yes (iv)(v)

	$\beta < \frac{\alpha\nu}{\nu-1}$	$\frac{\alpha\nu}{\nu-1} \leq \beta < \alpha$	$\alpha \leq \beta \leq \tilde{\beta}_{m,0}$	$\tilde{\beta}_{m,0} < \beta < \beta_m$	$\beta_m \leq \beta$
(GEC)	No (iv)	No (iv)	?	?	Yes (iv)
(BE)	No (v)	?	?	Yes (v)	Yes (iv)(v)

(iv) Theorem 2.1.
(v) Theorem 2.4.

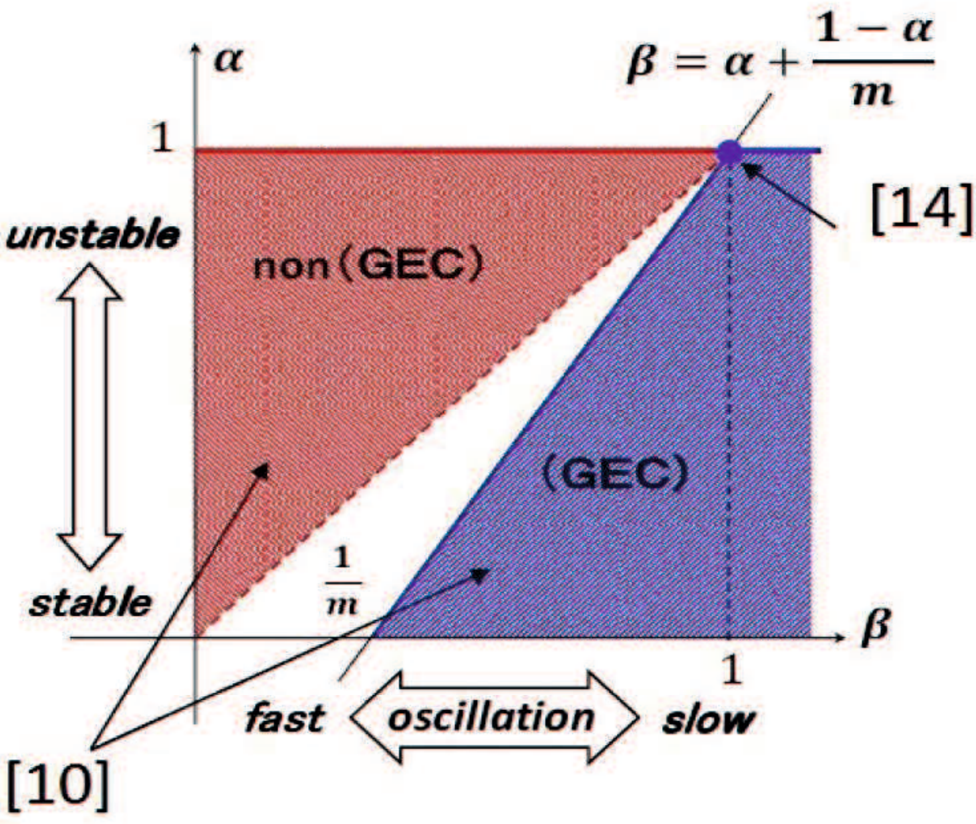


Figure 1: Result from Theorem 1.5 and 1.6

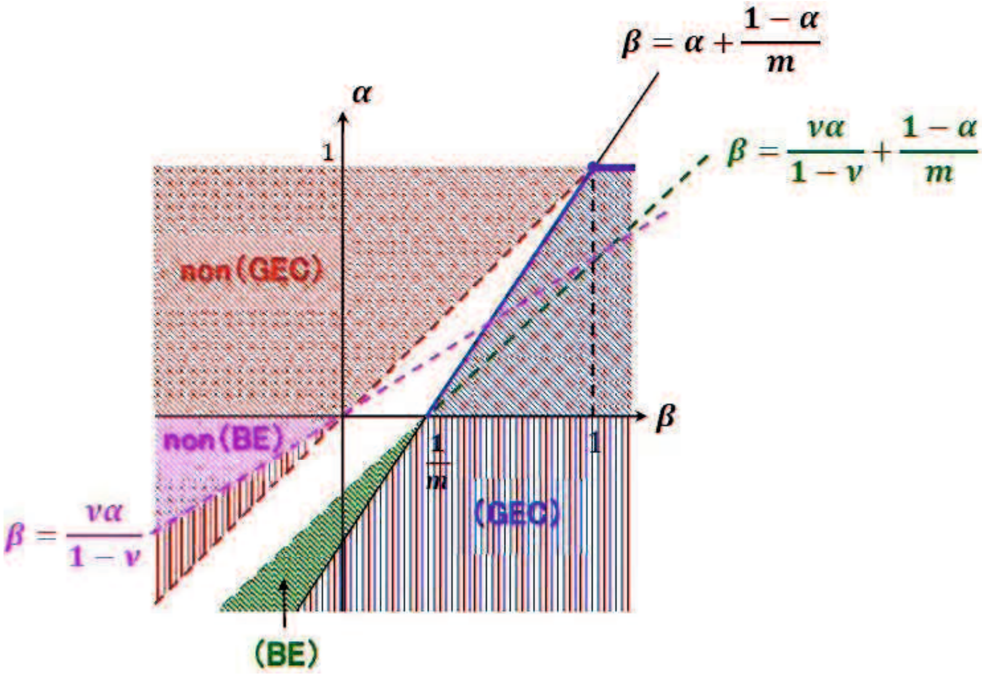


Figure 2: Result from Theorem 1.5, 1.6 and Example 2.3

3 Proofs of Theorems 2.1 and 2.4

3.1 Proof of Theorem 2.1

Denoting $v = v(t,\xi) := \hat{u}(t,\xi)$, (1.1) is reduced to the following problem:

$$\begin{cases} (\partial_t^2 + a(t)^2|\xi|^2)\, v = 0, \quad (t,\xi) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ (v(0,\xi), \partial_t v(0,\xi)) = (\hat{u}_0(\xi), \hat{u}_1(\xi)), \quad \xi \in \mathbb{R}^n. \end{cases} \tag{3.1}$$

We define the energy density functions $\mathcal{E}(t,\xi)$ and $\mathcal{E}_0(t,\xi)$ by

$$\mathcal{E}(t,\xi) := \frac{1}{2}\left(a(t)^2|\xi|^2|v(t,\xi)|^2 + |\partial_t v(t,\xi)|^2\right) \tag{3.2}$$

and

$$\mathcal{E}_0(t,\xi) := \frac{1}{2}\left(\lambda(t)^2|\xi|^2|v(t,\xi)|^2 + |\partial_t v(t,\xi)|^2\right).$$

Then, by (2.1) there exist positive constants a_0 and a_1 such that

$$a_0\lambda(t) \le a(t) \le a_1\lambda(t),$$

and thus

$$a_0^2\mathcal{E}_0(t,\xi) \le \mathcal{E}(t,\xi) \le a_1^2\mathcal{E}_0(t,\xi). \tag{3.3}$$

Noting (2.1), we have the following estimates:

$$\begin{aligned} \partial_t\mathcal{E}_0(t,\xi) =& \lambda'(t)\lambda(t)|\xi|^2|v(t,\xi)|^2 + \frac{(\lambda(t)^2 - a(t)^2)\,|\xi|}{\lambda(t)}\Re\{\lambda(t)|\xi|v(t,\xi)\overline{v_t(t,\xi)}\} \\ \le& (1+a_1)|a(t)-\lambda(t)||\xi|\mathcal{E}_0(t,\xi) \end{aligned}$$

and

$$\partial_t\mathcal{E}_0(t,\xi) \ge \left(\frac{2\lambda'(t)}{\lambda(t)} - (1+a_1)|a(t)-\lambda(t)||\xi|\right)\mathcal{E}_0(t,\xi).$$

Therefore, by Gronwall's inequality and (2.4) we have

$$\left(\frac{\lambda(t)}{\lambda(\tau_0)}\right)^2 \exp\left(-(1+a_1)\Xi(\tau_0)|\xi|\right)\mathcal{E}_0(\tau_0,\xi) \le \mathcal{E}_0(t,\xi) \le \exp\left((1+a_1)\Xi(\tau_0)|\xi|\right)\mathcal{E}_0(\tau_0,\xi) \tag{3.4}$$

for any $0 \le \tau_0 < t$.

For a large constant N to be chosen later, let us define t_ξ by

$$t_\xi := \min\left\{t \in \mathbb{R}_+\,;\, \Xi(t)|\xi| \le N\right\}. \tag{3.5}$$

Moreover, for a positive constant ρ, which provides the estimate

$$\int_{\mathbb{R}^n} \exp\left(2\rho|\xi|^{\frac{1}{\nu}}\right)\mathcal{E}(0,\xi)\, d\xi < \infty$$

for $(u_0,u_1) \in \gamma_1^\nu \times \gamma_0^\nu$, we define $\tilde{t}_\xi$ by

$$\tilde{t}_\xi := \min\left\{t \in \mathbb{R}_+\,;\, \Xi(t)|\xi| \le N + \frac{\rho}{1+a_1}|\xi|^{\frac{1}{\nu}}\right\}. \tag{3.6}$$

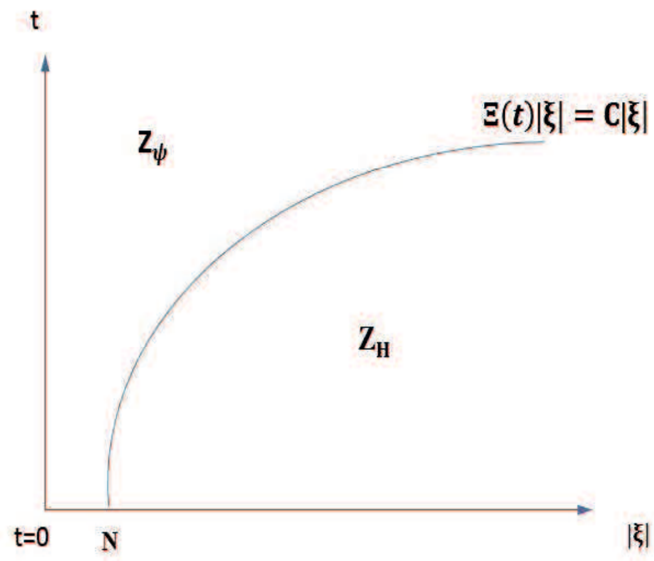


Figure 3: Sketch of the decomposition of the phase space Z_Ψ and Z_H

Then we define the *pseudo-differential zones* Z_Ψ, and $\tilde{Z}_\Psi$ by

$$Z_\Psi := \{(t,\xi) \in \mathbb{R}_+ \times \mathbb{R}^n \; ; \; t \geq t_\xi\}, \tag{3.7}$$

and

$$\tilde{Z}_\Psi := \left\{(t,\xi) \in \mathbb{R}_+ \times \mathbb{R}^n \; ; \; t \geq \tilde{t}_\xi\right\} \tag{3.8}$$

respectively. Thus we have the following lemma by (3.3) and (3.4):

Lemma 3.1. *The following estimates are established:*

$$\frac{1}{K_2}\frac{\lambda(t)^2}{\lambda(t_\xi)^2}\mathcal{E}(t_\xi,\xi) \leq \mathcal{E}(t,\xi) \leq K_2\mathcal{E}(t_\xi,\xi) \quad \text{in} \quad Z_\Psi \tag{3.9}$$

and

$$\mathcal{E}(t,\xi) \leq K_2 \exp\left(\rho|\xi|^{\frac{1}{\nu}}\right)\mathcal{E}(\tilde{t}_\xi,\xi) \quad \text{in} \quad \tilde{Z}_\Psi, \tag{3.10}$$

where $K_2 = a_0^{-2}a_1^2 e^{N(1+a_1)}$.

We define the *hyperbolic zones* Z_H, and $\tilde{Z}_H$ satisfying $Z_H \cup Z_\Psi = \tilde{Z}_H \cup \tilde{Z}_\Psi = \mathbb{R}_+ \times \mathbb{R}^n_\xi$ by

$$Z_H := \{(t,\xi) \in \mathbb{R}_+ \times \mathbb{R}^n \; ; \; t \leq t_\xi\}, \tag{3.11}$$

and

$$\tilde{Z}_H := \left\{(t,\xi) \in \mathbb{R}_+ \times \mathbb{R}^n \; ; \; t \leq \tilde{t}_\xi\right\} \tag{3.12}$$

respectively.

Let $(t,\xi) \in Z_H$. The equation of (3.1) is reduced to the following first order system:

$$\partial_t V_1 = A_1 V_1, \tag{3.13}$$

where

$$V_1 = \begin{pmatrix} v_t + \mathrm{i}a(t)|\xi|v \\ v_t - \mathrm{i}a(t)|\xi|v \end{pmatrix}, \quad A_1 = \begin{pmatrix} \phi_1 & \overline{b_1} \\ b_1 & \overline{\phi_1} \end{pmatrix},$$

$$b_1 = \overline{b_1} = -\frac{a'(t)}{2a(t)} \quad \text{and} \quad \phi_1 = \frac{a'(t)}{2a(t)} + \mathrm{i}a(t)|\xi|. \tag{3.14}$$

We denote

$$\phi_{1\Re} = \Re\{\phi_1\} = \frac{d}{dt}\log\sqrt{a(t)}, \quad \phi_{1\Im} = \Im\{\phi_1\} = a(t)|\xi|,$$

$$\lambda_1 = \phi_{1\Re} + \mathrm{i}\sqrt{\phi_{1\Im}^2 - |b_1|^2}$$

and

$$\theta_1 = \frac{\lambda_1 - \phi_1}{\overline{b_1}} = -\mathrm{i}\frac{\phi_{1\Im}}{\overline{b_1}}\left(1 - \sqrt{1 - \frac{|b_1|^2}{\phi_{1\Im}^2}}\right).$$

Here we note that $\{\lambda_1, \overline{\lambda_1}\}$ and $\{{}^t(1,\theta_1), {}^t(\overline{\theta_1}, 1)\}$ are the eigenvalues of A_1 and their corresponding eigenvectors, respectively. Therefore, if $|\theta_1| < 1$, then A_1 is diagonalized by the diagonalizer $\Theta_1 := ({}^t(1,\theta_1)\ {}^t(\overline{\theta_1}, 1))$ as follows:

$$\Theta_1^{-1}A_1\Theta_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \overline{\lambda_1} \end{pmatrix}.$$

We define $V_2 := \Theta_1^{-1}V_1$. Noting $\Theta_1(\partial_t\Theta_1^{-1}) = -(\partial_t\Theta_1)\Theta_1^{-1}$, V_2 is a solution to the following system:

$$\partial_t V_2 = A_2 V_2, \tag{3.15}$$

where

$$A_2 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \overline{\lambda_1} \end{pmatrix} - \Theta_1^{-1}(\partial_t\Theta_1) = \begin{pmatrix} \phi_2 & \overline{b_2} \\ b_2 & \overline{\phi_2} \end{pmatrix},$$

$$b_2 = -\frac{(\theta_1)_t}{1 - |\theta_1|^2}, \quad \phi_2 = \phi_{2\Re} + \mathrm{i}\phi_{2\Im},$$

$$\phi_{2\Re} = \phi_{1\Re} - \partial_t \log\sqrt{1 - |\theta_1|^2}$$

and

$$\phi_{2\Im} = \sqrt{\phi_{1\Im}^2 - |b_1|^2} - \Im\left\{\overline{\theta_1}b_2\right\}.$$

Generally, we have the following lemma:

Lemma 3.2. *Let V_k be a solution to the following system:*

$$\partial_t V_k = A_k V_k, \quad A_k = \begin{pmatrix} \phi_k & \overline{b_k} \\ b_k & \overline{\phi_k} \end{pmatrix},$$

and the matrix Θ_k *be defined by*

$$\Theta_k = \begin{pmatrix} 1 & \overline{\theta_k} \\ \theta_k & 1 \end{pmatrix}, \quad \theta_k = -\mathrm{i}\frac{\phi_{k\Im}}{\overline{b_k}}\left(1 - \sqrt{1 - \frac{|b_k|^2}{\phi_{k\Im}^2}}\right),$$

where $\phi_{k\Re} = \Re\{\phi_k\}$ *and* $\phi_{k\Im} = \Im\{\phi_k\}$*. If* $|\theta_k| < 1$*, then* $V_{k+1} = \Theta_k^{-1} V_k$ *is a solution to the following system:*

$$\partial_t V_{k+1} = A_{k+1} V_{k+1},$$

where

$$A_{k+1} = \begin{pmatrix} \phi_{k+1} & \overline{b_{k+1}} \\ b_{k+1} & \overline{\phi_{k+1}} \end{pmatrix}, \quad b_{k+1} = -\frac{(\theta_k)_t}{1 - |\theta_k|^2}$$

and

$$\phi_{k+1} = \phi_k - \partial_t \log\sqrt{1 - |\theta_k|^2} + \mathrm{i}\left(-\phi_{k\Im} + \sqrt{\phi_{k\Im}^2 - |b_k|^2} - \Im\{\overline{\theta_k} b_{k+1}\}\right). \tag{3.16}$$

Proof. The proof is straightforward. □

Denoting $V_m = {}^t(V_{m,1}, V_{m,2})$, by Lemma 3.2 we have

$$\begin{aligned}\partial_t |V_m|^2 =& 2\Re\left(A_m V_m, V_m\right)_{\mathbb{C}^2} = 2\phi_{m\Re}|V_m|^2 + 4\Re\{b_m V_{m,1}\overline{V_{m,2}}\} \\ \le& 2\left(\phi_{m\Re} + |b_m|\right)|V_m|^2\end{aligned}$$

and

$$\partial_t |V_m|^2 \ge 2\left(\phi_{m\Re} - |b_m|\right)|V_m|^2,$$

it follows that

$$|V_m(t,\xi)|^2 \begin{cases} \le \exp\left(2\int_0^t \left(\phi_{m\Re}(s,\xi) + |b_m(s,\xi)|\right) ds\right)|V_m(0,\xi)|^2, \\ \ge \exp\left(2\int_0^t \left(\phi_{m\Re}(s,\xi) - |b_m(s,\xi)|\right) ds\right)|V_m(0,\xi)|^2 \end{cases} \tag{3.17}$$

in Z_H and $\tilde{Z}_H$.

Let us introduce some symbol classes restricted in Z_H in order to show $|\theta_k| < 1$ and estimate the right hand sides of (3.17). For integers $p \ge 0$, q and r the symbol class $S^{(p)}\{q, r\}$ is the set of functions satisfying

$$\left|\partial_t^k f(t,\xi)\right| \le C_k \left(\lambda(t)|\xi|\right)^q \eta(t)^{r+k} \ \mathit{in} \ \ Z_H \tag{3.18}$$

for $k = 0, \dots, p$. Then the following properties of the symbol classes are established:

Lemma 3.3. *(S1) If* $f \in S^{(p)}\{q, r\}$ *for* $p \ge 1$*, then* $\partial_t f \in S^{(p-1)}\{q, r+1\}$*.*

(S2) If $f \in S^{(p)}\{q, r\}$*, then* $f \in S^{(p)}\{q+1, r-1\}$*.*

(S3) If $f \in S^{(p)}\{-q, q\}$ *for* $q \ge 1$*, then* $N^q|f| \le C_0 K_0^q$*.*

(S4) If $f_j \in S^{(p_j)}\{q, r\}$ $(j = 1, 2)$*, then* $f_1 + f_2 \in S^{(\min\{p_1, p_2\})}\{q, r\}$*.*

(S5) If $f_j \in S^{(p_j)}\{q_j, r_j\}$ $(j=1,2)$, then $f_1 f_2 \in S^{(\min\{p_1,p_2\})}\{q_1+q_2, r_1+r_2\}$.

Proof. (S1) and (S4) are trivial from the definition of the symbol class. Noting (2.5) and the estimate $\Xi(t)|\xi| \geq N$ in Z_H, (S2) and (S3) are also trivial. (S5) is immediately proved by Leibniz rule as follows:

$$\left|\partial_t^k (f_1 f_2)\right| \leq \sum_{j=0}^{k} \binom{k}{j} \left|\partial_t^j f_1\right| \left|\partial_t^{k-j} f_2\right| \lesssim (\lambda(t)|\xi|)^{q_1+q_2}\, \eta(t)^{r_1+r_2+k}$$

for any $k = 0, \ldots, \min\{p_1, p_2\}$. □

Moreover, we have the following properties:

Lemma 3.4. *The following properties are established:*

(S6) If $f \in S^{(p)}\{-q, q\}$ for $q \geq 1$, then there exists $N_0 > 0$ and $g \in S^{(p)}\{-q, q\}$ such that $2(1-\sqrt{1-f}) = f(1+g)$ for any $N \geq N_0$.

(S7) If $f_1 \in S^{(p_1)}\{q_1, r_1\}$ and $f_2 \in S^{(p_2)}\{-q_2, q_2\}$ for $q_2 \geq 1$, then there exists $N_0 > 0$ such that $f_1/(1-f_2) \in S^{(\min\{p_1,p_2\})}\{q_1, r_1\}$ for any $N \geq N_0$.

Proof. Let us prove (S6). By (S3) and choosing N_0 large, we can suppose that $|f| < 1$ for any $N \geq N_0$. Then, by Taylor expansion we have

$$\sqrt{1-f} = 1 + \binom{\frac{1}{2}}{1}(-f) + \sum_{j=2}^{\infty} \binom{\frac{1}{2}}{j}(-f)^j = 1 - \frac{f}{2} - f \sum_{l=1}^{\infty} \binom{\frac{1}{2}}{l+1}(-f)^l,$$

thus g is given by

$$g = 2\sum_{l=1}^{\infty} \binom{\frac{1}{2}}{l+1}(-f)^l.$$

Let us estimate $\partial_t^k f^l$ focusing in the dependence of l. We define

$$\zeta = \zeta(t,\xi) := \left(\frac{\eta(t)}{\lambda(t)|\xi|}\right)^q$$

and note the following representation due to Faádi Bruno's formula:

$$\partial_t^k f^l = k! \sum_{h=1}^{k} \frac{l!}{(l-h)!} f^{l-h} \sum_{h \in \Lambda_{h,k}} \prod_{j=1}^{k} \frac{(\partial_t^j f)^{h_j}}{h_j! j!^{h_j}}.$$

where $\Lambda_{h,k} := \{h = (h_1, \cdots, h_k) \in (\mathbb{N} \cup \{0\})^k\,;\, h_1 + \cdots + h_k = h,\ 1h_1 + \cdots + kh_k = k\}$. Noting the following estimate with a positive constant C_q:

$$\prod_{j=1}^{k} \left|\partial_t^j f\right|^{h_j} \leq \prod_{j=1}^{k} \left(C_q \zeta \eta^j\right)^{h_j} = \left(C_q \zeta\right)^h \eta^k,$$

we have

$$\begin{aligned}\left|\partial_t^k f^l\right| \leq & k!\sum_{h=1}^{k} l^h (C_p\zeta)^{l-h} \sum_{h\in\Lambda_{h,k}} \frac{(C_q\zeta)^h \eta^k}{h_j! j!^{h_j}}. \\ \leq & \eta^k (C_p\zeta)^l l^k k! \sum_{h=1}^{k} \sum_{h\in\Lambda_{h,k}} 1 \\ \leq & \tilde{C}_p \eta^k (C_p\zeta)^l l^k,\end{aligned}$$

where $\tilde{C}_p = \max_{1\leq k\leq p} k! \sum_{h=1}^{k} \sum_{h\in\Lambda_{h,k}} 1$. By choosing N_0 large enough, we have $(C_p\zeta)^{l-1} l^k \leq (1/2)^l$ for any $N \geq N_0$. Consequently, noting $|\binom{\frac{1}{2}}{l+1}| \leq 1/2$, we have

$$\begin{aligned}\left|\partial_t^k g\right| \leq & 2\sum_{l=1}^{\infty} \binom{\frac{1}{2}}{l+1} \left|\partial_t^k f^l\right| \leq \tilde{C}_p C_p \zeta \eta^k \left(1 + \sum_{l=1}^{\infty} \left(\frac{1}{2}\right)^l\right) \\ \leq & 2\tilde{C}_p C_p (\lambda(t)|\xi|)^{-q} \eta(t)^{q+k}\end{aligned}$$

for any $k = 0, \cdots, p$. Thus the proof of (S6) is concluded. (S7) can be proved on the analogy of the proof of (S7) to the expansion

$$\frac{f_1}{1-f_2} = f_1 \sum_{l=0}^{\infty} f_2^l$$

for $|f_2| < 1$ since $N \gg 1$. □

By using the properties (S1)-(S6) we shall prove the following lemma:

Lemma 3.5. *There exists a positive constant N such that $|\theta_k| \leq 1/2$ for $k = 1, \ldots m$ and $b_m \in S^{(0)}\{-m+1, m\}$.*

Proof. Firstly we show the following estimates:

$$\left|\left(\frac{1}{a(t)}\right)^{(k+1)}\right| \lesssim \frac{\eta(t)^{k+1}}{\lambda(t)} \quad (k = 0, \ldots, m-1), \tag{3.19}$$

where $(1/a)^{(j)} = \frac{d^j}{dt^j}(1/a)$. If $k = 0$, then (3.19) is trivial. Suppose that (3.19) is valid for $k = 1, \ldots, j$. Noting the representations:

$$0 = \frac{d^{j+1}}{dt^{j+1}}\left(a \cdot \frac{1}{a}\right) = a\left(\frac{1}{a}\right)^{(j+1)} + \sum_{l=1}^{j+1} \binom{j+1}{l} a^{(l)} \left(\frac{1}{a}\right)^{(j-l+1)},$$

we have

$$\left|\left(\frac{1}{a}\right)^{(j+1)}\right| \leq \frac{1}{a}\sum_{l=1}^{j+1} \binom{j+1}{l} \left|a^{(l)}\right| \left|\left(\frac{1}{a}\right)^{(j-l+1)}\right| \lesssim \frac{\eta^{j+1}}{\lambda},$$

thus (3.19) is valid for any $k = 0, \ldots, m-1$. Therefore, for $f \in S^{(p)}\{q, r\}$ we have

$$\left|\partial_t^k \frac{f}{\phi_{1\Im}}\right| \leq \frac{|f^{(k)}|}{a|\xi|} + |\xi|^{-1} \sum_{j=1}^{k} \binom{k}{j} \left|f^{(k-j)}\right| \left|\left(\frac{1}{a}\right)^{(j)}\right| \lesssim (\lambda|\xi|)^{q-1} \eta^{r+k}.$$

Thus we have the following property:

(S8) If $f \in S^{(p)}\{q,r\}$, then $f/\phi_{1\Im} \in S^{(p)}\{q-1,r\}$.

By applying (S6) for $f = |b_1|^2/\phi_{1\Im}^2 \in S^{(m-1)}\{-2,2\}$, which follows from (S5) and (S8), there exists $g_1 \in S^{(m-1)}\{-2,2\}$ such that

$$\theta_1 = -\mathrm{i}\frac{\phi_{1\Im}}{\overline{b_1}}\left(1-\sqrt{1-\frac{|b_1|^2}{\phi_{1\Im}^2}}\right) = -\frac{b_1}{2\phi_{1\Im}}(1+g_1) \in S^{(m-1)}\{-1,1\}. \tag{3.20}$$

Therefore, by (S3) we have $|\theta_1| \le 1/2$ for any sufficiently large N. Moreover, by (S1) and (S7) we have

$$b_2 \in S^{(m-2)}\{-1,2\}. \tag{3.21}$$

By (3.20), (3.21), (S4)-(S6) and (S8) we have

$$1-\sqrt{1-\frac{|b_1|^2}{\phi_{1\Im}^2}}+\frac{\Im\left\{\overline{\theta_1}b_2\right\}}{\phi_{1\Im}} \in S^{(m-2)}\{-2,2\}.$$

Therefore, by (3.21), (S5), (S7) and (S8) we have

$$\frac{b_2}{\phi_{2\Im}} = \frac{b_2}{\phi_{1\Im}}\frac{1}{1-\left(1-\sqrt{1-\frac{|b_1|^2}{\phi_{1\Im}^2}}+\frac{\Im\{\overline{\theta_1}b_2\}}{\phi_{1\Im}}\right)} \in S^{(m-2)}\{-2,2\}, \tag{3.22}$$

and thus obtain

$$\theta_2 \in S^{(m-2)}\{-2,2\} \tag{3.23}$$

by using the same arguments to show (3.20). Moreover, by (S3) we have $|\theta_2| \le 1/2$ for any sufficiently large N. On the analogy of the arguments above, we can prove $b_k \in S^{(m-k)}\{-k+1,k\}$, $\theta_k \in S^{(m-k)}\{-k,k\}$ and $|\theta_k| \le 1/2$ for $k=3,\dots,m$. □

The properties (S1)-(S8) except for (S3) are valid in $\tilde{Z}_H$ under the assumption (2.7), and the estimate of (S3) is changed into $N^{q\nu/(\nu-1)}|f| \lesssim 1$.

By (2.6), (2.8), Lemma 3.5 and the representation (3.16), there exists $N \gg 1$ and C_m such that

$$\begin{aligned}\int_0^t 2|b_m(s,\xi)|\,ds \le& 2C_m|\xi|^{-m+1}\int_0^t \lambda(s)\left(\frac{\eta(s)}{\lambda(s)}\right)^m ds\\ \le& 2C_m\left(N^{-1}\Xi(t)\right)^{m-1}\int_0^t \lambda(s)\left(\frac{\eta(s)}{\lambda(s)}\right)^m ds\\ \le& 2C_mK_1N^{-m+1}\end{aligned}$$

and

$$\begin{aligned}\int_0^t 2\phi_{m\Re}(s,\xi)\,ds =& \int_0^t \partial_s\left(\log a(s)-\sum_{k=1}^{m-1}\log\left(1-|\theta_k(s,\xi)|^2\right)\right)ds\\ =& \log\frac{a(t)\prod_{k=1}^{m-1}(1-|\theta_k(0,\xi)|^2)}{a(0)\prod_{k=1}^{m-1}(1-|\theta_k(t,\xi)|^2)}\\ &\begin{cases}\le \log\left(\left(\frac{4}{3}\right)^{m-1}\frac{a_1}{a_0}\frac{\lambda(t)}{\lambda(0)}\right)\\ \ge \log\left(\left(\frac{3}{4}\right)^{m-1}\frac{a_0}{a_1}\frac{\lambda(t)}{\lambda(0)}\right)\end{cases}\end{aligned}$$

in Z_H. Therefore, by (3.17) we have

$$|V_m(t,\xi)|^2 \begin{cases} \le \left(\frac{4}{3}\right)^{m-1} \frac{a_1}{a_0} \exp\left(2C_m K_1 N^{-m+1}\right) \frac{\lambda(t)}{\lambda(0)} |V_m(0,\xi)|^2, \\ \ge \left(\frac{3}{4}\right)^{m-1} \frac{a_0}{a_1} \exp\left(-2C_m K_1 N^{-m+1}\right) \frac{\lambda(t)}{\lambda(0)} |V_m(0,\xi)|^2. \end{cases}$$

By Lemmas 3.2 and 3.5, we have

$$\begin{aligned} |V_k|^2 = |\Theta_k V_{k+1}|^2 &= \left(1+|\theta_k|^2\right)|V_{k+1}|^2 + 4\Re\{\theta_k V_{k+1,1}\overline{V_{k+1,2}}\} \\ &\begin{cases} \le (1+|\theta_k|)^2 |V_{k+1}|^2 \le \frac{9}{4}|V_{k+1}|^2, \\ \ge (1-|\theta_k|)^2 |V_{k+1}|^2 \ge \frac{1}{4}|V_{k+1}|^2, \end{cases} \end{aligned}$$

it follows that

$$\left(\frac{1}{4}\right)^{m-1} |V_m|^2 \le |V_1|^2 \le \left(\frac{9}{4}\right)^{m-1} |V_m|^2.$$

uniformly in Z_H. Consequently, noting

$$|V_1(t,\xi)|^2 = 2\left(a(t)^2|\xi|^2|v(t,\xi)|^2 + |v_t(t,\xi)|^2\right) = 4\mathcal{E}(t,\xi),$$

we have

$$\begin{aligned} \mathcal{E}(t,\xi) \le& \frac{1}{4}\left(\frac{9}{4}\right)^{m-1} |V_m(t,\xi)|^2 \\ \le& \frac{3^{m-1}a_1}{4a_0} \exp\left(2C_m K_1 N^{-m+1}\right) \frac{\lambda(t)}{\lambda(0)} |V_m(0,\xi)|^2 \\ \le& \frac{12^{m-1}a_1 \exp\left(2C_m K_1 N^{-m+1}\right)}{4a_0} \frac{\lambda(t)}{\lambda(0)} |V_1(0,\xi)|^2 \\ =& \frac{12^{m-1}a_1 \exp\left(2C_m K_1 N^{-m+1}\right)}{a_0} \frac{\lambda(t)}{\lambda(0)} \mathcal{E}(0,\xi). \end{aligned}$$

By the same way, we have

$$\mathcal{E}(t,\xi) \ge \frac{a_0}{12^{m-1}a_1 \exp\left(2C_m K_1 N^{-m+1}\right)} \frac{\lambda(t)}{\lambda(0)} \mathcal{E}(0,\xi).$$

Let us consider the estimate in $\tilde{Z}_H$. We define $\tilde{\tau}$ by

$$\tilde{\tau} := \max\left\{t > 0\,;\; 2C_m \tilde{K}_1 (1+a_1)^{\frac{\nu m}{\nu-1}-1} \rho^{-\frac{\nu m}{\nu-1}} \Xi(t)^{-m\left(\kappa-\frac{\nu}{\nu-1}\right)}\right\}.$$

By (2.1)-(2.3) we have

$$\partial_t \mathcal{E}(t,\xi) \le \frac{2\alpha_1 \eta(t)}{a_0} \mathcal{E}(t,\xi) \le \frac{2\alpha_1 \tilde{\eta}_1}{a_0} \mathcal{E}(t,\xi)$$

for $0 \le t \le \tilde{\tau}$, where $\tilde{\eta}_1 = \max_{0 \le t \le \tilde{\tau}}\{\eta(t)\}$, it follows that

$$\mathcal{E}(t,\xi) \le \exp\left(\frac{2\alpha_1 \tilde{\eta}_1}{a_0}\right) \mathcal{E}(0,\xi). \tag{3.24}$$

Let $\tilde{\tau} \le t \le \tilde{t}_\xi$. Noting the estimate $|\xi| \ge (\rho^{-1}(1+a_1)\Xi(t))^{-\nu/(\nu-1)}$ and (2.8), we have

$$\begin{aligned}
\int_{\tilde{\tau}}^t 2|b_m(s,\xi)|\,ds \le & 2C_m|\xi|^{-m+1}\int_0^t \lambda(s)\left(\frac{\eta(s)}{\lambda(s)}\right)^m ds \\
\le & 2C_m\tilde{K}_1\Xi(t)^{-\kappa m+1}|\xi|^{-m+1} \\
\le & 2C_m\tilde{K}_1\left(\frac{1+a_1}{\rho}\right)^{\frac{\nu m}{\nu-1}-1}\Xi(t)^{-m\left(\kappa-\frac{\nu}{\nu-1}\right)}|\xi|^{\frac{1}{\nu}} \\
\le & \rho|\xi|^{\frac{1}{\nu}}
\end{aligned}$$

by (3.24). Then, by the same way in the estimates of Z_H we have

$$\begin{aligned}
\mathcal{E}(t,\xi) \le & \frac{1}{4}\left(\frac{9}{4}\right)^{m-1}|V_m(t,\xi)|^2 \\
\le & \frac{3^{m-1}a_1\lambda(t)}{4a_0\lambda(\tilde{\tau})}\exp\left(\rho|\xi|^{\frac{1}{\nu}}\right)|V_m(\tilde{\tau},\xi)|^2 \\
\le & \frac{12^{m-1}a_1}{a_0}\exp\left(\rho|\xi|^{\frac{1}{\nu}}\right)\mathcal{E}(\tilde{\tau},\xi) \\
\le & \frac{12^{m-1}a_1}{a_0}\exp\left(\frac{2\alpha_1\tilde{\eta}_1}{a_0}\right)\exp\left(\rho|\xi|^{\frac{1}{\nu}}\right)\mathcal{E}(0,\xi).
\end{aligned}$$

Summarizing the estimates above we have the following lemma:

Lemma 3.6. *The following estimates are established:*

$$K_3^{-1}\frac{\lambda(t)}{\lambda(0)}\mathcal{E}(0,\xi) \le \mathcal{E}(t,\xi) \le K_3\frac{\lambda(t)}{\lambda(0)}\mathcal{E}(0,\xi) \quad in \quad Z_H \tag{3.25}$$

and

$$\mathcal{E}(t,\xi) \le \tilde{K}_3\exp\left(\rho|\xi|^{\frac{1}{\nu}}\right)\mathcal{E}(0,\xi) \quad in \quad \tilde{Z}_H, \tag{3.26}$$

where $K_3 = 12^{m-1}a_0^{-1}a_1\exp\left(2C_mK_1N^{-m+1}\right)$ *and* $\tilde{K}_3 = 12^{m-1}a_0^{-1}a_1\exp(2a_0^{-1}\alpha_1\tilde{\eta}_1)$.

Proof of Theorem 2.1. If $(t,\xi)\in Z_H$, then by Lemma 3.6 we have

$$\frac{1}{K_3\lambda(0)^2}\lambda(t)^2\mathcal{E}(0,\xi) \le \mathcal{E}(t,\xi) \le K_3\mathcal{E}(0,\xi).$$

If $(t,\xi)\in\{(t,\xi)\,;\,|\xi|\le N/\Xi(0)\}\subset Z_\Psi$, and thus $t_\xi=0$ by (3.5), then by Lemma 3.1 we have

$$\frac{1}{K_2\lambda(0)^2}\lambda(t)^2\mathcal{E}(0,\xi) \le \mathcal{E}(t,\xi) \le K_2\mathcal{E}(0,\xi).$$

By Lemma 3.6 we have

$$\frac{1}{K_3}\frac{\lambda(t_\xi)}{\lambda(0)}\mathcal{E}(0,\xi) \le \mathcal{E}(t_\xi,\xi) \le K_3\frac{\lambda(t_\xi)}{\lambda(0)}\mathcal{E}(0,\xi). \tag{3.27}$$

Therefore, if $(t,\xi)\in Z_\Psi\cap\{(t,\xi)\,;\,|\xi|\ge N/\Xi(0)\}$, then by Lemma 3.1 and (3.27) we have

$$\mathcal{E}(t,\xi) \le K_2K_3\frac{\lambda(t_\xi)}{\lambda(0)}\mathcal{E}(0,\xi) \le K_2K_3\mathcal{E}(0,\xi)$$

and

$$\mathcal{E}(t,\xi) \geq \frac{1}{K_2K_3}\frac{\lambda(t)^2}{\lambda(0)\lambda(t_\xi)}\mathcal{E}(0,\xi) \geq \frac{\lambda(t)^2}{K_2K_3\lambda(0)^2}\mathcal{E}(0,\xi).$$

Summarizing the estimates above and Parseval's theorem, we have

$$\frac{\lambda(t)^2}{K_2K_3\lambda(0)^2}E(0) \leq E(t) \leq K_2K_3E(0).$$

By the same way we conclude (BE) as follows:

$$E(t) \leq K_2\tilde{K}_3 \int_{\mathbb{R}^n} \exp\left(2\rho|\xi|^{\frac{1}{\nu}}\right)\mathcal{E}(0,\xi)\,d\xi < \infty$$

since (3.2) holds. Thus the proof of Theorem 2.1 is concluded. □

3.2 Proof of Theorem 2.4

We shall prove Theorem 2.4 by constructing concrete examples of the series of the coefficients $\{a_j(t)\}$ and the initial data $\{(u_{j,0},u_{j,1})\}$ making use of the ideas in [4, 10].

Let $\varphi \in C^\infty(\mathbb{R})$ be a 2π-periodic function satisfying

$$\varphi \geq 0, \;\; \varphi(\tau) \equiv 0 \;\text{ near }\; \tau = 0 \;\text{ and }\; \int_0^{2\pi} \varphi(\tau)\cos^2\tau d\tau = \pi. \tag{3.28}$$

We define the 2π-periodic function $\psi(\tau)$ by

$$\psi(\tau) = \psi(\tau;\varepsilon) := 1 + 4\varepsilon\varphi(\tau)\sin(2\tau) - 2\varepsilon\varphi'(\tau)\cos^2\tau - 4\varepsilon^2\varphi(\tau)^2\cos^4\tau, \tag{3.29}$$

where ε is a positive constant providing

$$\frac{1}{2} \leq \sqrt{\psi(\tau;\varepsilon)} \leq \frac{3}{2}. \tag{3.30}$$

Then we have the following lemma:

Lemma 3.7. *The solution to the initial value problem*

$$w'' + \psi(\tau;\varepsilon)w = 0, \;\; w(0) = 1, \;\; w'(0) = 0 \tag{3.31}$$

is represented by

$$w(\tau) = w(\tau;\varepsilon) = \exp\left(2\varepsilon\int_0^\tau \varphi(s)\cos^2 s\,ds\right)\cos\tau. \tag{3.32}$$

Proof. The proof is straightforward. □

For $\alpha,\beta,\delta \in \mathbb{R}$ satisfying $\alpha < 0$, $\beta < 0$, $\delta \geq 0$, (2.11) and (2.12) we define t_j by

$$t_j := (2\pi j)^{\frac{1}{\alpha-\beta+\delta}}. \tag{3.33}$$

Here we note that $\{t_j\}_{j=0}^{\infty}$ is a strictly increasing sequence satisfying $\lim_{j\to\infty} t_j = \infty$. Moreover, for $j = 1, 2, \dots$ we define λ_j, ξ_j and ρ_j by

$$\lambda_j := t_j^{-\delta}, \quad \xi_j := t_j^{-\beta+\delta} \quad \text{and} \quad \rho_j := t_j^{\alpha+\delta}.$$

For $\chi \in C^\infty(\mathbb{R})$ satisfying $\chi' \le 0$, $\chi^{(k)}(\tau) \equiv 0$ near $\tau \in \{0,1\}$ for any $k \in \mathbb{N}$, $\chi(\tau) = 1$ for $\tau \le 0$ and $\chi(\tau) = 0$ for $\tau \ge 1$, we define $\Lambda \in C^\infty(\mathbb{R}_+)$ by

$$\Lambda(t) := \lambda_{l+1} + (\lambda_l - \lambda_{l+1})\chi\left(\frac{t - t_l}{t_{l+1} - t_l}\right) \quad \text{for} \quad t \in [t_l, t_{l+1}) \quad \text{and} \quad l = 0, 1, \dots, \tag{3.34}$$

where $\lambda_0 = \lambda_1$. Then we see that $\Lambda'(t) \le 0$, $\Lambda^{(k)}(t) = 0$ for $t \in [0, t_1)$ and

$$\left|\Lambda^{(k)}(t)\right| \le \max_{0\le\tau\le1} \left\{\left|\chi^{(k)}(\tau)\right|\right\} \frac{\lambda_l - \lambda_{l+1}}{(t_{l+1} - t_l)^k} \le C_k t_l^{-\alpha+\beta-2\delta-(1-\alpha+\beta-\delta)k} \le C_k t_l^{-\delta-\beta k}$$

for $t \in [t_l, t_{l+1})$, $l = 1, 2, \dots$. Let us define $a_j(t)$ and $\lambda(t)$ by

$$a_j(t) := \begin{cases} \Lambda(t), & t \in [0, t_j), \\ \lambda_j \sqrt{\psi\left(\lambda_j \xi_j (t - t_j)\right)}, & t \in [t_j, t_j + \rho_j), \\ \lambda_j, & t \in [t_j + \rho_j, \infty), \end{cases} \tag{3.35}$$

and

$$\lambda(t) := \begin{cases} \Lambda(t), & t \in [0, t_j), \\ \lambda_j, & t \in [t_j, \infty). \end{cases} \tag{3.36}$$

Then we see that $a_j(t) \in C^\infty(\mathbb{R}_+)$ and (2.1) are valid. Moreover, noting the estimates

$$\left|a_j^{(k)}(t)\right| \le C_k \lambda_j (\lambda_j \xi_j)^k = C_k t_j^{-\delta-\beta k}$$

for $t \in [t_j, t_j + \rho_j)$, we have

$$\left|a_j^{(k)}(t)\right| \le \begin{cases} 0, & t \in [0, t_1), \\ C_k t_l^{-\delta-\beta k}, & t \in [t_l, t_{l+1}), \quad l = 1, \dots, j-1, \\ C_k t_j^{-\delta-\beta k}, & t \in [t_j, t_j + \rho_j), \\ 0, & t \in [t_j + \rho_j, \infty) \end{cases}$$

for $k = 1, 2, \dots$. It follows that $a(t) = a_j(t)$ satisfy (2.4) and (2.3) uniformly with respect to j for

$$\Xi(t) = \begin{cases} \int_{t_j}^{t_j+\rho_j} |a_j(s) - \lambda_j|\, ds \simeq \lambda_j \rho_j = t_j^{\alpha}, & t \in [0, t_j + \rho_j), \\ 0, & t \in [t_j + \rho_j, \infty), \end{cases}$$

and

$$\eta(t) = \begin{cases} 0, & t \in [0, t_1), \\ t_l^{-\beta}, & t \in [t_l, t_{l+1}), \quad l = 1, \dots, j-1, \\ t_j^{-\beta}, & t \in [t_j, t_j + \rho_j), \\ 0, & t \in [t_j + \rho_j, \infty). \end{cases}$$

Now we consider the following Cauchy problems:

$$\begin{cases} \partial_t^2 u_j - a_j(t)^2 \partial_x^2 u_j = 0, & (t,x) \in (0,\infty) \times \mathbb{R}, \\ (u_j(0,x), (\partial_t u_j)(0,x)) = (u_{j,0}(x), u_{j,1}(x)), & x \in \mathbb{R}, \end{cases} \tag{3.37}$$

and the corresponding energy of (3.37)

$$E_j(t) = E_j(t; u_{j,0}, u_{j,1}) := \frac{1}{2} \int_{\mathbb{R}} \left(a_j(t)^2 |\partial_x u_j(t,x)|^2 + |\partial_t u_j(t,x)|^2 \right) dx \tag{3.38}$$

for $j = 1, 2, \ldots$. Then we shall prove the following proposition, which implies the first part of the conclusion of Theorem 2.4:

Proposition 3.8. *Suppose that (2.11) and (2.12) are valid. There exists a series of initial data* $\{(u_{j,0}, u_{j,1})\}_{j=1}^{\infty}$ *such that*

$$E_j(0) \le 1 \quad \text{and} \quad \lim_{j\to\infty} \sup_{t>0} \{E_j(t)\} = \infty. \tag{3.39}$$

Remark 3.9. If $\alpha - \beta + \delta < 0$, then (2.5) and (2.6) are valid for $m \ge (\alpha + \delta - 1)/(\alpha - \beta + \delta)$ uniformly with respect to j. Therefore, there exists a positive constant C independent of j such that $E_j(t_j + \rho_j) \le C E_j(0) \le C$ by Theorem 2.1. On the other hand, if $\alpha - \beta + \delta > 0$, that is, (2.12) holds, then we have

$$\frac{\eta(t_j)\Xi(t_j)}{\lambda(t_j)} \simeq t_j^{\alpha-\beta+\delta} \to \infty \quad (j \to \infty)$$

and

$$\Xi(t_j + \rho_j)^{m-1} \int_0^{t_j+\rho_j} \lambda(s) \left(\frac{\eta(s)}{\lambda(s)} \right)^m ds \simeq t_j^{m(\alpha-\beta+\delta)} \to \infty \quad (j \to \infty),$$

that is, (2.5) and (2.6) do not hold. Therefore, Proposition 3.8 implies that the conditions (2.5) and (2.6) cannot be removed.

Proof of Proposition 3.8. Noting $a_j'(t) = 0$ on $[t_j + \rho_j, \infty)$, we have $E_j(t) = E_j(t_j + \rho_j)$ for $t \in [t_j + \rho_j, \infty)$. Moreover, we have

$$E_j'(t) = \Lambda'(t)\Lambda(t) \int_{\mathbb{R}} |\partial_x u_j(t,x)|^2 \, dx \ge \frac{2\Lambda'(t)}{\Lambda(t)} E_j(t)$$

for $t \in (0, t_j]$, it follows that

$$E_j(t) \ge \frac{\Lambda(t)^2}{\Lambda(0)^2} E_j(0) \ge \lambda_j^2 E_j(0). \tag{3.40}$$

We shall find a series of initial data $\{(u_{j,0}, u_{j,1})\}_{j=1}^{\infty}$ providing $\lim_{j\to\infty} E_j(t_j + \rho_j) = \infty$. Let $\{\zeta_j\}_{j=1}^{\infty}$ be a positive sequence, which will be defined in (3.50). By Lemma 3.7, the solution of

$$y_j'' + a_j(t)^2 \xi_j^2 y_j = 0, \quad y_j(t_j) = \frac{\zeta_j}{\lambda_j}, \quad y_j'(t_j) = 0$$

is represented by

$$y_j(t) = \frac{\zeta_j}{\lambda_j} \exp\left(2\varepsilon \int_0^{\lambda_j \xi_j (t-t_j)} \varphi(s) \cos^2 s \, ds\right) \cos(\lambda_j \xi_j (t-t_j)). \tag{3.41}$$

By (3.33) we have $\lambda_j \xi_j \rho_j = 2\pi j$; hence we have

$$y_j(t_j + \rho_j) = \max_{t \in [t_j, t_j+\rho_j)} \{|y_j(t)|\} = \frac{\zeta_j}{\lambda_j} \exp(2\pi\varepsilon j). \tag{3.42}$$

For a positive monotone decreasing sequence $\{\sigma_j\}_{j=1}^{\infty}$ satisfying $\sigma_j \le \xi_j$, which will be defined in (3.50), we define $X_j(\xi)$ by

$$X_j(\xi) := \begin{cases} \zeta_j, & \xi \in \left[\xi_j - \frac{\sigma_j}{2}, \xi_j + \frac{\sigma_j}{2}\right], \\ 0, & \xi \notin \left[\xi_j - \frac{\sigma_j}{2}, \xi_j + \frac{\sigma_j}{2}\right]. \end{cases}$$

Here we note that $\int_{\mathbb{R}} X_j(\xi)^2 \, d\xi = \sigma_j \zeta_j^2$. Denoting $\hat{u}_j(t,\xi) = v_j(t,\xi)$, the equation of (3.37) is reduced to

$$\partial_t^2 v_j + a_j(t)^2 \xi^2 v_j = 0. \tag{3.43}$$

Then we set the initial data of (3.37) by

$$(u_{j,0}(x), u_{j,1}(x)) = (\check{v}_j(0,x), (\partial_t \check{v}_j)(0,x)), \tag{3.44}$$

where $\check{v}$ denotes the inverse partial Fourier transformation, and $v_j(0,x)$ is the solution of (3.43) at $t = 0$ with the initial data at $t = t_j$:

$$(v_j(t_j,\xi), (\partial_t v_j)(t_j,\xi)) = \left(\frac{y_j(t_j)}{\zeta_j} X_j(\xi), 0\right). \tag{3.45}$$

We define $z_j(t,\xi)$ and $\mathcal{Z}_j(t,\xi)$ by

$$z_j(t,\xi) := v_j(t,\xi) - \frac{y_j(t)}{\zeta_j} X_j(\xi) \tag{3.46}$$

and

$$\mathcal{Z}_j(t,\xi) := \frac{1}{2}\left(a_j(t)^2 \xi^2 |z_j(t,\xi)|^2 + |\partial_t z_j(t,\xi)|^2\right). \tag{3.47}$$

Noting (3.42),

$$\partial_t^2 z_j(t,\xi) = -a_j(t)^2 \xi^2 z_j(t,\xi) - a_j(t)^2 \left(\xi^2 - \xi_j^2\right) \frac{y_j(t)}{\zeta_j} X_j(\xi) \tag{3.48}$$

and the inequalities

$$\left|\xi^2 - \xi_j^2\right|^2 X_j(\xi)^2 \le \left(\left(\xi_j + \frac{\sigma_j}{2}\right)^2 - \xi_j^2\right)^2 X_j(\xi)^2 \le \frac{25}{16} \sigma_j^2 \xi_j^2 X_j(\xi)^2, \tag{3.49}$$

we have

$$
\begin{aligned}
\partial_t \mathcal{Z}_j =& a_j' a_j \xi^2 |z_j|^2 - \Re\left\{ \overline{\partial_t z_j} \frac{a_j^2\left(\xi^2-\xi_j^2\right) y_j}{\zeta_j} X_j \right\} \\
\leq& \frac{2C_1\lambda_j^2\xi_j}{a_j}\left(\frac{1}{2}a_j^2\xi^2|z_j|^2+\frac{1}{2}|\partial_t z_j|^2\right)+\frac{a_j}{2C_1\lambda_j^2\xi_j}\frac{1}{2}\left(\frac{a_j^2\left(\xi^2-\xi_j^2\right)y_j}{\zeta_j}X_j\right)^2 \\
\leq& 4C_1\lambda_j\xi_j\mathcal{Z}_j+\frac{25a_j^5\sigma_j^2\xi_j y_j^2}{64C_1\lambda_j^2\zeta_j^2}X_j^2 \\
\leq& 4C_1\lambda_j\xi_j\mathcal{Z}_j+\tilde{C}_1\lambda_j\sigma_j^2\xi_j\exp\left(4\pi\varepsilon j\right)X_j^2
\end{aligned}
$$

by (3.30), where $\tilde{C}_1 = 6075/2048C_1$. Thus, noting $\mathcal{Z}_j(t_j,\xi)=0$, we have

$$
\mathcal{Z}_j(t_j+\rho_j,\xi) \leq \frac{\tilde{C}_1\sigma_j^2}{4C_1}\exp\left(4\pi\varepsilon j+8\pi C_1 j\right)X_j(\xi)^2.
$$

We set σ_j and ζ_j by

$$
\sigma_j := \sqrt{\frac{C_1}{\tilde{C}_1}\frac{\xi_j}{4}}\exp\left(-4\pi C_1 j\right) \quad \text{and} \quad \zeta_j := \sqrt{\frac{8\lambda_j^2}{9\xi_j^2\sigma_j}}. \tag{3.50}
$$

Noting the inequalities (3.40) and $|f+g|^2 \geq |f|^2/2 - 2|g|^2$, we have $\sigma_j \leq \xi_j$,

$$
\begin{aligned}
E_j(t_j+\rho_j) \geq& \frac{1}{2}\lambda_j^2\int_{\mathbb{R}}\xi^2|v_j(t_j+\rho_j,\xi)|^2\,d\xi \\
=& \frac{1}{2}\lambda_j^2\int_{\mathbb{R}}\xi^2\left|\frac{y_j(t_j+\rho_j)}{\zeta_j}X_j(\xi)+z_j(t_j+\rho_j,\xi)\right|^2\,d\xi \\
\geq& \frac{\lambda_j^2 y_j(t_j+\rho_j)^2}{4\zeta_j^2}\int_{\mathbb{R}}\xi^2X_j(\xi)^2\,d\xi-2\int_{\mathbb{R}}\mathcal{Z}_j(t_j+\rho_j,\xi)\,d\xi \\
\geq& \left(\frac{\xi_j^2\exp\left(4\pi\varepsilon j\right)}{16}-\frac{\tilde{C}_1\sigma_j^2}{2C_1}\exp\left(4\pi\varepsilon j+8\pi C_1 j\right)\right)\int_{\mathbb{R}}X_j(\xi)^2\,d\xi \\
=& \frac{\xi_j^2\sigma_j\zeta_j^2\exp\left(4\pi\varepsilon j\right)}{32}=\frac{\lambda_j^2\exp\left(4\pi\varepsilon j\right)}{36} \\
\to& \infty \quad (j\to\infty)
\end{aligned}
$$

and

$$
E_j(0) \leq \frac{1}{\lambda_j^2}E_j(t_j) = \frac{1}{2\lambda_j^2}\int_{\mathbb{R}}\xi^2X_j(\xi)^2 \leq \frac{9\xi_j^2\sigma_j\zeta_j^2}{8\lambda_j^2} = 1.
$$

Thus the proof of Proposition 3.8 is concluded. □

On the analogy of the proof of Proposition 3.8, we can prove the following proposition:

Proposition 3.10. *Let $\nu > 1$. Suppose that (2.15) is valid. There exist a positive constant ρ and a series of initial data $\{(u_{j,0},u_{j,1})\}_{j=1}^{\infty}$ such that*

$$
\int_{\mathbb{R}}\exp\left(2\rho|\xi|^{\frac{1}{\nu}}\right)\left(a_j(0)^2|\partial_x u_{j,0}(x)|^2+|u_{j,1}(x)|^2\right)\,dx \leq 1 \tag{3.51}
$$

and

$$\lim_{j\to\infty}\sup_{t>0}\{E_j(t)\} = \infty. \tag{3.52}$$

Proof. Thanks to $\alpha < 0$, we note that (2.12) is valid since (2.15) holds. Let $v_j(t,\xi)$ be a solution to (3.43), and denote

$$\mathcal{E}_j(t,\xi) := \frac{1}{2}\left(a_j(t)^2|\xi|^2|v_j(t,\xi)|^2 + |\partial_t v_j(t,\xi)|^2\right).$$

We set the initial data of (3.37) by (3.44) with

$$(v_j(t_j,\xi), (\partial_t v_j)(t_j,\xi)) = \left(\frac{y_j(t_j)}{\zeta_j}X_j(\xi)\exp\left(-\rho|\xi|^{\frac{1}{\nu}}\right), 0\right).$$

We may proceed as we did to derive (3.40), obtaining

$$\mathcal{E}_j(t,\xi)\begin{cases} \geq \frac{1}{\lambda_j^2}\mathcal{E}_j(0,\xi) & t\in(0,t_j), \\ = \mathcal{E}_j(t_j+\rho_j,\xi) & t\in[t_j+\rho_j,\infty). \end{cases}$$

We define $\tilde{z}_j(t,\xi)$ and $\tilde{\mathcal{Z}}_j(t,\xi)$ by

$$\tilde{z}_j(t,\xi) := v_j(t,\xi) - \frac{y_j(t)}{\zeta_j}X_j(\xi)\exp\left(-\rho|\xi|^{\frac{1}{\nu}}\right) \tag{3.53}$$

and

$$\tilde{\mathcal{Z}}_j(t,\xi) := \frac{1}{2}\left(a_j(t)^2\xi^2|\tilde{z}_j(t,\xi)|^2 + |\partial_t\tilde{z}_j(t,\xi)|^2\right). \tag{3.54}$$

Then we have

$$\partial_t\tilde{\mathcal{Z}}_j \leq 4C_1\lambda_j\xi_j\tilde{\mathcal{Z}}_j + \tilde{C}_1\lambda_j\sigma_j^2\xi_j\exp(4\pi\varepsilon j)\,X_j(\xi)^2\exp\left(-2\rho|\xi|^{\frac{1}{\nu}}\right),$$

it follows that

$$\begin{aligned} E_j(t_j+\rho_j) \geq& \frac{\xi_j^2\exp(4\pi\varepsilon j)}{32}\int_{\mathbb{R}} X_j(\xi)^2\exp\left(-2\rho|\xi|^{\frac{1}{\nu}}\right)d\xi \\ \geq& \frac{\xi_j^2\sigma_j\zeta_j^2}{32}\exp\left(4\pi\varepsilon j - 2\left(\frac{3}{2}\right)^{\frac{1}{\nu}}\rho\xi_j^{\frac{1}{\nu}}\right) \\ =& \frac{\lambda_j^2}{36}\exp\left(4\pi\varepsilon j - 2\left(\frac{3}{2}\right)^{\frac{1}{\nu}}(2\pi)^{\frac{-\beta+\delta}{\nu(\alpha-\beta+\delta)}}\rho j^{\frac{-\beta+\delta}{\nu(\alpha-\beta+\delta)}}\right) \\ \to& \infty \quad (j\to\infty) \end{aligned}$$

since (2.15) holds. Moreover, we have

$$\int_{\mathbb{R}}\exp\left(2\rho|\xi|^{\frac{1}{\nu}}\right)\mathcal{E}_j(0,\xi)\,dx \leq \frac{1}{\lambda_j^2}\int_{\mathbb{R}}\exp\left(2\rho|\xi|^{\frac{1}{\nu}}\right)\mathcal{E}_j(t_j,\xi)\,dx = \frac{1}{2\lambda_j^2}\int_{\mathbb{R}}\xi^2X_j(\xi)^2\,dx \leq 1.$$

Thus the proof of the proposition is concluded. □

4 Proof of Theorem 2.7

4.1 Zones

We define

$$A(t) := \int_t^\infty a(\tau)d\tau,$$

and suppose that $A(0) = \Xi(0)$ without loss of generality. For a sufficiently large constant N to be chosen later, we split the extended phase space into three zones, *the pseudo defferential zone* $Z_\psi(N)$, *the stabilized zone* $Z_{st}(N)$ and *the hyperbolic zone* $Z_H(N)$ as follows:

$$Z_\psi(N) := \{(t,\xi) \in \mathbb{R}_+ \times \mathbb{R}^n \, ; \, A(t)|\xi| \le N\},$$

$$Z_{st}(N) := \{(t,\xi) \in \mathbb{R}_+ \times \mathbb{R}^n \, ; \, \Xi(t)|\xi| \le N \le A(t)|\xi|\}$$

and

$$Z_H(N) := \{(t,\xi) \in \mathbb{R}_+ \times \mathbb{R}^n \, ; \, \Xi(t)|\xi| \ge N\}.$$

We also denote by τ_ξ, and t_ξ the separating hypersurfaces between $Z_\psi(N)$, and $Z_{st}(N)$, and between $Z_{st}(N)$ and $Z_H(N)$ respectively. That is

$$A(\tau_\xi)|\xi| = \Xi(t_\xi)|\xi| = N.$$

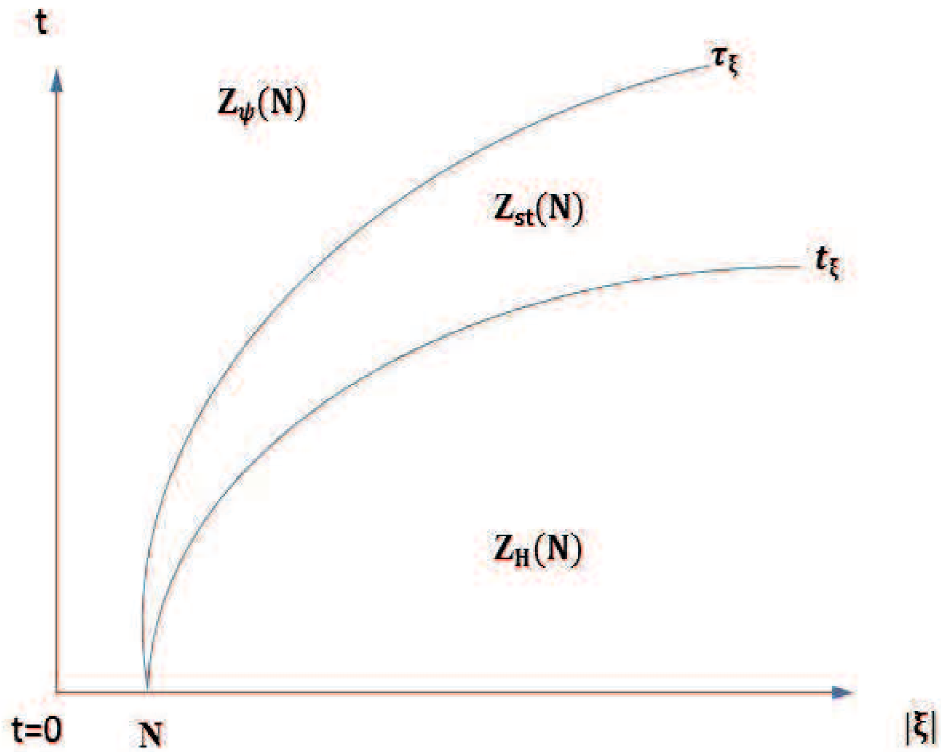


Figure 4: Sketch of the decomposition of the phase space

4.2 Estimate in $Z_\psi(N)$

Let us define $\theta(t)$ by

$$\theta(t) := \frac{a(t)}{A(t)}. \tag{4.1}$$

For $t \geq \tau_\xi$, that is $(t,\xi) \in Z_\psi(N)$ we put

$$V_0(t,\xi) = \begin{pmatrix} \mathrm{i}\theta(t)v(t,\xi) \\ v_t(t,\xi) \end{pmatrix},$$

so that

$$\partial_t V_0 = A_0(t,\xi)V_0, \quad A_0 = \begin{pmatrix} \frac{\theta'(t)}{\theta(t)} & \mathrm{i}\theta(t) \\ \frac{\mathrm{i}|\xi|^2 a(t)^2}{\theta(t)} & 0 \end{pmatrix}. \tag{4.2}$$

Let us consider the fundamental solution $E = E(t,s,\xi)$ to (4.2), that is , the solution of

$$\partial_t E = A_0(t,\xi)E, \quad E(s,s,\xi) = I, \tag{4.3}$$

with $\tau_\xi \leq s \leq t$, where I is the identity matrix. If we put $E = (E_{ij})_{ij=1,2}$, thanks to (4.2) we obtain, for $j = 1,2$, the following integral equations:

$$E_{1j}(t,s,\xi) = \frac{\theta(t)}{\theta(s)}\left(\delta_{1j} + \mathrm{i}\int_s^t \theta(s)E_{2j}(\tau,s,\xi)d\tau\right) \tag{4.4}$$

and

$$E_{2j}(t,s,\xi) = \delta_{2j} + \mathrm{i}|\xi|^2\int_s^t \frac{a(\tau)^2}{\theta(\tau)}E_{1j}(\tau,s,\xi)d\tau. \tag{4.5}$$

By (4.4), (4.5) and integrating by parts we get

$$\begin{aligned} E_{2j}(t,s,\xi) =& \delta_{2j} + \mathrm{i}|\xi|^2\int_s^t \frac{a(\tau)^2}{\theta(s)}\left(\delta_{1j} + \mathrm{i}\int_s^\tau \theta(s)E_{2j}(\sigma,s,\xi)d\sigma\right)d\tau \\ =& \delta_{2j} + \mathrm{i}|\xi|^2\delta_{1j}\int_s^t \frac{a(\tau)^2}{\theta(s)}d\tau - |\xi|^2\int_s^t E_{2j}(\tau,s,\xi)\left(\int_\tau^t a(\sigma)^2 d\sigma\right)d\tau. \end{aligned}$$

By using (2.1), (2.2) and (2.4) we have

$$\int_s^t a(\tau)^2 d\tau \simeq \int_s^t \lambda(\tau)a(\tau)d\tau \leq \lambda(s)\int_s^t a(\tau)d\tau \lesssim a(s)A(s)$$

and

$$\int_s^t \frac{a(\tau)^2}{\theta(s)}d\tau = A(s)\int_s^t \frac{a(\tau)^2}{a(s)}d\tau \lesssim A(s)^2.$$

Taking into account the inequalities

$$\begin{aligned} |E_{2j}(t,s,\xi)| &\lesssim 1 + |\xi|^2 A(s)\int_s^t a(\tau)\frac{a(\tau)}{a(s)}d\tau + |\xi|^2\int_s^t a(\tau)A(\tau)E_{2j}(\tau,s,\xi)d\tau \\ &\lesssim 1 + |\xi|^2 A(s)\int_s^t a(\tau)|E_{2j}(\tau,s,\xi)|d\tau, \end{aligned}$$

by Gronwall's inequality, there exists a positive constant C such that

$$|E_{2j}(t,s,\xi)| \lesssim \exp\left(C(1+|\xi|^2 A(s)\int_s^t a(\tau)d\tau)\right) \le \exp(C(1+N^2)) \lesssim 1 \tag{4.6}$$

for $j=1,2$ uniformly in $Z_\psi(N)$. Therefore, by (4.4) and (4.6) we conclude the estimate

$$|E_{1j}(t,s,\xi)| \lesssim \frac{\theta(t)}{\theta(s)}\left(\delta_{1j}+(1+t)\theta(s)\right) \tag{4.7}$$

for $j=1,2$ uniformly in $Z_\psi(N)$. Summarizing all the consideration above implies the following estimates:

Lemma 4.1. *In $Z_\psi(N)$ the following estimates are established:*

$$|v(t,\xi)| \lesssim \begin{cases} (1+t)\left(|v(0,\xi)|+|v_t(0,\xi)|\right) & \textit{for } |\xi| \le N/A(0), \\ (1+t)(\theta(\tau_\xi)|v(\tau_\xi,\xi)|+|v_t(\tau_\xi,\xi)|) & \textit{for } |\xi| \ge N/A(0). \end{cases} \tag{4.8}$$

Proof. Noting the representation $V_0(t,\xi)=E(t,s,\xi)V_0(s,\xi)$, we have

$$\begin{pmatrix} \mathrm{i}\theta(t)v(t,\xi) \\ v_t(t,\xi) \end{pmatrix} = \begin{pmatrix} E_{11}(t,s,\xi)\mathrm{i}\theta(s)v(s,\xi)+E_{12}(t,s,\xi)v_t(s,\xi) \\ E_{21}(t,s,\xi)\mathrm{i}\theta(s)v(s,\xi)+E_{22}(t,s,\xi)v_t(s,\xi) \end{pmatrix}. \tag{4.9}$$

For $|\xi| \le N/A(0)$, (4.8) trivially follows by using (4.7) to (4.9). For $|\xi| \ge N/A(0)$, by (2.19) and using (4.7) to (4.9) with $s=\tau_\xi$ we have

$$\begin{aligned} |v(t,\xi)| \lesssim& \left(\frac{1}{\theta(\tau_\xi)}+(1+t)\right)\theta(\tau_\xi)|v(\tau_\xi,\xi)|+(1+t)|v_t(\tau_\xi,\xi)| \\ \lesssim& (1+t)(a(t_\xi)|\xi||v(\tau_\xi,\xi)|+|v_t(\tau_\xi,\xi)|). \end{aligned}$$

□

4.3 Estimate in $Z_H(N)$

By Lemma 3.6 we immediately have the following estimate in $Z_H(N)$:

Lemma 4.2. *There exists a positive constant N such that the following estimates are established in $\in Z_H(N)$:*

$$\lambda(t)|\xi||v(t,\xi)|+|v_t(t,\xi)| \lesssim \sqrt{\lambda(t)}\left(|\xi||v(0,\xi)|+|v_t(0,\xi)|\right).$$

4.4 Estimate in $Z_{st}(N)$

For any $t_\xi \le s \le t \le \tau_\xi$, we put

$$V_1(t,\xi) = \begin{pmatrix} i\lambda(t)|\xi|v(t,\xi) \\ v_t(t,\xi) \end{pmatrix},$$

so that

$$\partial_t V_1(t,\xi) = A_1(t,\xi)V_1(t,\xi), \quad A_1 = \begin{pmatrix} \frac{\lambda'(t)}{\lambda(t)} & i\lambda(t)|\xi| \\ \frac{i|\xi|a(t)^2}{i\lambda(t)} & 0 \end{pmatrix}. \tag{4.10}$$

Let M_1 be given by

$$M_1 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

which is a diagonalizer of the principal part of A_1. If we put $W = M_1^{-1} V_1$, then we get

$$\partial_t W = \tilde{A}_1(t,\xi) W, \quad \tilde{A}_1 = \begin{pmatrix} \phi_1 & b_1 \\ \overline{b_1} & \overline{\phi_1}, \end{pmatrix}, \tag{4.11}$$

where

$$\phi_1 = \frac{\lambda'(t)}{2\lambda(t)} + \frac{\mathrm{i}|\xi|(a^2(t)+\lambda^2(t))}{2\lambda(t)}, \quad b_1 = -\frac{\lambda'(t)}{2\lambda(t)} + \frac{\mathrm{i}|\xi|(a^2(t)-\lambda^2(t))}{2\lambda(t)}.$$

Then we have

$$\begin{aligned}
\partial_t |W|^2 =& 2\Re\left(W, \partial_t W\right)_{\mathbb{C}^2} = 2\Re\left(\phi_1\right)|W|^2 + 4\Re\left(b_1 w_1 \overline{w_2}\right) \\
=& \frac{\lambda'(t)}{\lambda(t)}|W|^2 + 4\Re\left(b_1 w_1 \overline{w_2}\right) \\
\le& \frac{\lambda'(t)}{\lambda(t)}|W|^2 + 2|b_1||W|^2 \le \left(\frac{\lambda'(t)}{\lambda(t)} + \left|\frac{\lambda'(t)}{\lambda(t)}\right| + \frac{|\xi|\,|a^2(t)-\lambda^2(t)|}{\lambda(t)}\right)|W|^2 \\
=& \frac{|\xi|\,|a^2(t)-\lambda^2(t)|}{\lambda(t)}|W|^2 \lesssim |\xi||a(t)-\lambda(t)||W|^2.
\end{aligned}$$

By Gronwall's inequality we have

$$\begin{aligned}
|W(t,\xi)|^2 \le& \exp\left(C\int_{t_\xi}^{t} |a(\tau)-\lambda(\tau)|d\tau|\xi|\right)|W(t_\xi,\xi)|^2 \\
\le& \exp\left(C\int_{t_\xi}^{\infty} |a(\tau)-\lambda(\tau)|d\tau|\xi|\right)|W(t_\xi,\xi)|^2 \\
\le& \exp\left(C\Xi(t_\xi)|\xi|\right)|W(t_\xi,\xi)|^2 \\
=& \exp\left(CN\right)|W_1(t_\xi,\xi)|^2.
\end{aligned}$$

Therefore, noting that

$$|W(t,\xi)| = |M_1^{-1}V_1(t,\xi)| \simeq |V_1(t,\xi)| \simeq \lambda(t)|\xi||v(t,\xi)| + |v_t(t,\xi)|$$

we have the following lemma:

Lemma 4.3. *In $(t,\xi) \in Z_{st}(N)$ the following estimate is establised:*

$$\lambda(t)|\xi||v(t,\xi)| + |v_t(t,\xi)| \lesssim \lambda(t_\xi)|\xi||v(t_\xi,\xi)| + |v_t(t_\xi,\xi)|.$$

4.5 Conclusion

Noting $\lambda(t_\xi) = \lambda(\Xi^{-1}(N|\xi|^{-1}))$, the proof of Theorem 2.7 is concluded if the following estimate is established in all zones:

$$a(t)|\xi||v(t,\xi)| \lesssim (1+t)\lambda(t)\max\left\{1, \sqrt{\lambda(t_\xi)}|\xi|\right\}\left(|\xi||v(0,\xi)| + |v_t(0,\xi)|\right). \tag{4.12}$$

Indeed, by applying Parseval theorem we have

$$\begin{aligned}
a(t)^2\|\nabla u(t,\cdot)\|^2 =& a(t)^2\int_{\mathbb{R}^n}|\xi|^2|v(t,\xi)|^2\,d\xi\\
\lesssim&(1+t)^2\lambda(t)^2\int_{\mathbb{R}^n}F(|\xi|)^2\left(|\xi|^2|v(0,\xi)|^2+|v_t(0,\xi)|^2\right)\,d\xi\\
=&(1+t)^2\lambda(t)^2\left(\|\nabla F(|\nabla|)u_0(\cdot)\|^2+\|u_1(\cdot)\|^2\right).
\end{aligned}$$

If $|\xi|\le N/A(0)$, that is, $(t,\xi)\in Z_\psi(N)\cap\ \{|\xi|\le N/A(0)\}$, then by Lemma 4.1 we have

$$a(t)|\xi||v(t,\xi)|\lesssim(1+t)\lambda(t)|\xi|\left(|v(0,\xi)|+|v_t(0,\xi)|\right).$$

Let $(t,\xi)\in Z_H(N)$. If $\sqrt{\lambda(t)}\lesssim(1+t)\lambda(t)$, then by Lemma 4.3 we have

$$\begin{aligned}
a(t)|\xi||v(t,\xi)| \lesssim&\lambda(t)|\xi||v(t,\xi)|+|v_t(t,\xi)|\\
\lesssim&\sqrt{\lambda(t)}\left(|\xi||v(0,\xi)|+|v_t(0,\xi)|\right)\\
\lesssim&(1+t)\lambda(t)\left(|\xi||v(0,\xi)|+|v_t(0,\xi)|\right).
\end{aligned}$$

If $(1+t)\lambda(t)$ is monotone decreasing, then by (2.19) we have

$$\begin{aligned}
\sqrt{\lambda(t)}\le&\frac{(1+t)\lambda(t)}{(1+t_\xi)\sqrt{\lambda(t_\xi)}}=\frac{(1+t)\lambda(t)\sqrt{\lambda(t_\xi)}}{(1+t_\xi)\lambda(t_\xi)}\\
\lesssim&\frac{(1+t)\lambda(t)\sqrt{\lambda(t_\xi)}}{A(t_\xi)}=N^{-1}(1+t)\lambda(t)\sqrt{\lambda(t_\xi)}|\xi|.
\end{aligned}$$

It follows that

$$a(t)|\xi||v(t,\xi)|\lesssim(1+t)\lambda(t)\sqrt{\lambda(t_\xi)}|\xi|\left(|\xi||v(0,\xi)|+|v_t(0,\xi)|\right).$$

If $(t,\xi)\in Z_{st}(N)$, that is, $t_\xi\le t\le\tau_\xi$, then by (2.19), Lemma 4.3, Lemma 4.2 with $t=t_\xi$, and noting that

$$\begin{aligned}
\sqrt{\lambda(t_\xi)}\lesssim&(1+t)\lambda(t)\frac{\sqrt{\lambda(t_\xi)}}{A(t)}\le(1+t)\lambda(t)\frac{\sqrt{\lambda(t_\xi)}}{A(\tau_\xi)}\\
=&N^{-1}(1+t)\lambda(t)\sqrt{\lambda(t_\xi)}|\xi|,
\end{aligned}$$

we have

$$\begin{aligned}
a(t)|\xi||v(t,\xi)|\lesssim&\lambda(t_\xi)|\xi||v(t_\xi,\xi)|+|v_t(t_\xi,\xi)|\\
\lesssim&\sqrt{\lambda(t_\xi)}\left(|\xi||v(0,\xi)|+|v_t(0,\xi)|\right)\\
\lesssim&(1+t)\lambda(t)\sqrt{\lambda(t_\xi)}|\xi|\left(|\xi||v(0,\xi)|+|v_t(0,\xi)|\right).
\end{aligned}$$

If $(t,\xi) \in Z_\psi \cap \{|\xi| \geq N/A(0)\}$, then by Lemma 4.1, Lemma 4.3 with $t = \tau_\xi$, Lemma 4.2 with $t = t_\xi$ and noting $\tau_\xi \leq t$ we have

$$\begin{aligned}
a(t)|\xi||v(t,\xi)| \lesssim &(1+t)\lambda(t)|\xi| \left(\theta(\tau_\xi)|v(\tau_\xi,\xi)| + |v_t(\tau_\xi,\xi)|\right) \\
= &(1+t)\lambda(t)|\xi| \left(\frac{a(\tau_\xi)}{A(\tau_\xi)}|v(\tau_\xi,\xi)| + |v_t(\tau_\xi,\xi)|\right) \\
\simeq &(1+t)\lambda(t)|\xi| \left(\lambda(\tau_\xi)|\xi||v(\tau_\xi,\xi)| + |v_t(\tau_\xi,\xi)|\right) \\
\lesssim &(1+t)\lambda(t)|\xi| \left(\lambda(t_\xi)|\xi||v(t_\xi,\xi)| + |v_t(t_\xi,\xi)|\right) \\
\lesssim &(1+t)\lambda(t)\sqrt{\lambda(t_\xi)}|\xi| \left(|\xi||v(0,\xi)| + |v_t(0,\xi)|\right).
\end{aligned}$$

Thus the estimate (4.12) is vaild for all zones, and the proof of Theorem 2.7 is concluded. ☐

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