Mathematics and Sudoku Ⅵ

Dedicated to the memory of Professor Sibe Mardešić

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We discuss on the worldwide famous Sudoku by using mathematical approach. This paper is the 6th paper in our series, so we use the same notations and terminologies in [1] –[5] without any descriptions.

10. Classification of intersectable systems III.

This section is a continuous section of previous sections 8 and 9. We consider some basic relations among some types in the section 8.

Proposition 50. (Type 15) For each intersectable 9−system $\boldsymbol{\omega}=(S,T)$ of Type 15, we have that $T_{\omega}=1$ in $\boldsymbol{STRF}(f,f_0)$ for each $f\in\boldsymbol{SOL}(f_0)$.

Proof. Since $\boldsymbol{\omega}=(S,T)$ is Type 15 we can put $S=\{s_1,s_2,s_3,s_4,s_5,s_6,s_7,s_8,s_9\}$ and $T=\{t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8,t_9\}$ such that

(1) all s_i, $1\leqq i\leqq 9$, are rows,

(2) all t_j, $1\leqq j\leqq 9$, are columns,

(3) $s_i\cap s_j=\phi$ for $i\neq j$, $1\leqq i\leqq 9$, $1\leqq j\leqq 9$,

(4) $t_i\cap t_j=\phi$ for $i\neq j$, $1\leqq i\leqq 9$, $1\leqq j\leqq 9$,

(5) $s_i\cap t_j\neq\phi$ for i,j, $1\leqq i\leqq 9$, $1\leqq j\leqq 9$,

(6) $s=\cup\{s_i : i=1,2,\ldots,9\}$, $t=\cup\{t_j : j=1,2,\ldots,9\}$.

Take any sudoku matrix $\boldsymbol{K}=(K_\alpha)_{\alpha\in J_1\times J_2}\in SMTX(f,f_0)$ and put $\boldsymbol{K}'=T_{\omega}(\boldsymbol{K})=T(S,T)(\boldsymbol{K})$. We have

(7) $$K'_\alpha=\begin{cases} K_\alpha & for\ \alpha\in(J_1\times J_2-s\cup t)\cup(s\cap t) \\ K_\alpha\cap K_{s-t} & for\ \alpha\in t-s \\ K_\alpha\cap K_{t-s} & for\ \alpha\in s-t \end{cases}.$$

By (1)–(6) we can easily show that

(8) $s=J_1\times J_2=t$.

By (8) we have

(9) $s\cup t=J_1\times J_2=s\cap t$,

(10) $(J_1\times J_2-s\cup t)\cup(s\cap t)=J_1\times J_2$,

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(11) $s-t=\phi$, $t-s=\phi$.
By (7) and (9),(10),(11) we have
(12) $K'_{\alpha}=K_{\alpha}$ for $\alpha\in J_1\times J_2$.
(12) means that $T_{\omega}=1$. Hence we have Proposition 50.

Let X be a subset of J_1 and Y be a subset of J_2 with $|X|=|Y|=n$. We put that $X'=J_1-X$ and $Y'=J_2-Y$. We put that $S=\{s_i:i\in X\}$, $s_i=\{i\}\times J_2$ and $T=\{t_j:j\in Y\}$, $t_j=J_1\times\{j\}$. Also we put that $S'=\{s'_{i'}:i'\in X'\}$, $s'_{i'}=\{i'\}\times J_2$ and $T'=\{t'_{j'}:j'\in Y'\}$, $t'_{j'}=J_1\times\{j'\}$.
Thus $\omega=(S,T)$ and $\omega'=(S',T')$ are intersectable $n-$system and intersectable $(9-n)-$system, respectively. We say that ω' is a dual intersectable system of ω.

Proposition 51. If $\omega'=(S',T')$ is a dual intersectable $(9-n)-$system of an intersectable $n-$system $\omega=(S,T)$, then $T_{\omega}=T_{\omega'}$ in $\boldsymbol{STRF}(f,f_0)$ for each $f\in\boldsymbol{SOL}(f_0)$.
Proof. By definitions we have that
(1) $X\cup X'=J_1$, $X\cap X'=\phi$,
(2) $Y\cup Y'=J_2$, $Y\cap Y'=\phi$,
(3) $|X|=|Y|=n$, $0\leqq n\leqq 9$,
(4) $s=\cup\{s_i:i\in X\}$ and $t=\cup\{t_j:j\in Y\}$,
(5) $s'=\cup\{s'_{i'}:i'\in X'\}$ and $t'=\cup\{t'_{j'}:j'\in Y'\}$.
Take any sudoku matrix $\boldsymbol{K}=(K_{\alpha})_{\alpha\in J_1\times J_2}\in SMTX(f,f_0)$. We put $\boldsymbol{K}^*=T_{\omega}(\boldsymbol{K})=T(S,T)(\boldsymbol{K})$ and $\boldsymbol{K}^{**}=T_{\omega'}(\boldsymbol{K})=T(S',T')(\boldsymbol{K})$. We have

$$(6)\ K^*_{\alpha}=\begin{cases}K_{\alpha} & for\ \alpha\in(J_1\times J_2-s\cup t)\cup(s\cap t)\\ K_{\alpha}\cap K_{s-t} & for\ \alpha\in t-s\\ K_{\alpha}\cap K_{t-s} & for\ \alpha\in s-t\end{cases},$$

$$(7)\ K^{**}_{\alpha}=\begin{cases}K_{\alpha} & for\ \alpha\in(J_1\times J_2-s'\cup t')\cup(s'\cap t')\\ K_{\alpha}\cap K_{s'-t'} & for\ \alpha\in t'-s'\\ K_{\alpha}\cap K_{t'-s'} & for\ \alpha\in s'-t'\end{cases}.$$

By (4) we have
(8) $s=\cup\{s_i=\{i\}\times J_2:i\in X\}=(\cup\{\{i\}:i\in X\})\times J_2=X\times J_2$,
(9) $t=\cup\{t_j=J_1\times\{j\}:j\in Y\}=J_1\times(\cup\{\{j\}:j\in Y\})=J_1\times Y$,
(10) $s\cup t=X\times J_2\cup J_1\times Y$.
By (1),(2),(10) we have

$$\begin{aligned}J_1\times J_2-s\cup t&=(X\cup X')\times(Y\cup Y')-X\times J_2\cup J_1\times Y\\ &=X\times Y\cup X\times Y'\cup X'\times Y\cup X'\times Y'-X\times(Y\cup Y')\cup(X\cup X')\times Y\end{aligned}$$

$= X\times Y\cup X\times Y'\cup X'\times Y\cup X'\times Y'-X\times Y\cup X\times Y'\cup X'\times Y$

$= X'\times Y'$, i.e.,

(11) $J_1\times J_2-s\cup t=X'\times Y'$.

By (8),(9) we have

(12) $s\cap t=X\times Y$.

By (11),(12) we have

(13) $(J_1\times J_2-s\cup t)\cup(s\cap t)=X\times Y\cup X'\times Y'$.

By (1),(2),(8),(9),(12) we have

(14) $s-t=s-s\cap t=X\times J_2-X\times Y=X\times(J_2-Y)=X\times Y'$,

(15) $t-s=t-s\cap t=J_1\times Y-X\times Y=(J_1-X)\times Y=X'\times Y$.

By (5) we have

(16) $s'=\cup\{s'_{i'}=\{i'\}\times J_2 : i'\in X'\}=(\cup\{\{i'\}: i'\in X'\})\times J_2=X'\times J_2$,

(17) $t'=\cup\{t'_{j'}=J_1\times\{j'\}: j'\in Y'\}=J_1\times(\cup\{\{j'\}: j'\in Y'\})=J_1\times Y'$,

(18) $s'\cup t'=X'\times J_2\cup J_1\times Y'$.

By (1),(2),(18) we have

$J_1\times J_2-s'\cup t'=(X\cup X')\times(Y\cup Y')-X'\times J_2\cup J_1\times Y'$

$= X\times Y\cup X\times Y'\cup X'\times Y\cup X'\times Y'-X'\times(Y\cup Y')\cup(X\cup X')\times Y'$

$= X\times Y\cup X\times Y'\cup X'\times Y\cup X'\times Y'-X'\times Y\cup X'\times Y'\cup X\times Y'$

$= X\times Y$, i.e.,

(19) $J_1\times J_2-s'\cup t'=X\times Y$.

By (16),(17) we have

(20) $s'\cap t'=X'\times Y'$.

By (19),(20) we have

(21) $(J_1\times J_2-s'\cup t')\cup(s'\cap t')=X\times Y\cup X'\times Y'$.

By (1),(2),(16),(17),(20) we have

(22) $s'-t'=s'-s'\cap t'=X'\times J_2-X'\times Y'=X'\times(J_2-Y')=X'\times Y$,

(23) $t'-s'=t'-s'\cap t'=J_1\times Y'-X'\times Y'=(J_1-X')\times Y'=X\times Y'$.

By (13),(21) we have

(24) $(J_1\times J_2-s\cup t)\cup(s\cap t)=(J_1\times J_2-s'\cup t')\cup(s'\cap t')$.

By (14),(15),(22),(23) we have

(25) $s-t=t'-s'$,

(26) $t-s=s'-t'$.

By (7),(24),(25),(26) we have

(27) $$K^{**}_\alpha=\begin{cases}K_\alpha & for\ \alpha\in(J_1\times J_2-s'\cup t')\cup(s'\cap t')=(J_1\times J_2-s\cup t)\cup(s\cap t)\\ K_\alpha\cap K_{s'-t'}=K_\alpha\cap K_{t-s} & for\ \alpha\in t'-s'=s-t\\ K_\alpha\cap K_{t'-s'}=K_\alpha\cap K_{s-t} & for\ \alpha\in s'-t'=t-s\end{cases}.$$

By (6) and (27) we have

(28) $K^{*}_{\alpha}=K^{**}_{\alpha}$ for $\alpha\in J_1\times J_2$.
(28) means that $T_{\omega}=T_{\omega'}$. Hence we complete the proof of Proposition 51.

Corollary 52. (Type 11, Type 12, Type 13,Type 14)
(a) Each intersectable 8−system ω of Type 14 and the its dual intersectable 1−system ω' of Type 1 induce the same sudoku transformation $T_{\omega}=T_{\omega'}$.
(b) Each intersectable 7−system ω of Type 13 and the its dual intersectable 2−system ω' of Type 5 induce the same sudoku transformation $T_{\omega}=T_{\omega'}$.
(c) Each intersectable 6−system ω of Type 12 and the its dual intersectable 3−system ω' of Type 9 induce the same sudoku transformation $T_{\omega}=T_{\omega'}$.
(d) Each intersectable 5−system ω of Type 11 and the its dual intersectable 4−system ω' of Type 10 induce the same sudoku transformation $T_{\omega}=T_{\omega'}$.
This Corollary 52 comes from Proposition 51.

Corollary 53. (Type 14) For each intersectable 8−system $\omega=(S,T)$ of Type 14, there are a row s and a column t such that $T_{\omega}\geqq T_{\omega_2}\circ T_{\omega_1}\cap T_{\omega_4}\circ T_{\omega_3}$ and $\omega_1=(s-s\cap t,\, s)$, $\omega_2=(s\cap t,\, t)$, $\omega_3=(t-s\cap t,\, t)$ and $\omega_4=(s\cap t,\, s)$.
This Corollary 53 comes from Corollary 52 and Proposition 44.

In section 5 we define sudoku transformations T_{ω}, $\omega\in \boldsymbol{BTOOL}$. We recall these definitions: $\boldsymbol{BLK}=rOW\cup cOL\cup bLK$ and $rOW=\{row(i): i\in J_1\}$, $cOL=\{col(j): j\in J_2\}$, $bLK=\{blk(k): k\in J\}$. For each n, $1\leqq n\leqq 9$, we put $\boldsymbol{SFS}(n)=\{(s,b): b\in \boldsymbol{BLK},\, s\subset b \text{ and } |s|=n\}$ and $\boldsymbol{SFS}=\cup_{n=1}^{9}\boldsymbol{SFS}(n)$. Also we put $\boldsymbol{IS}(n)=\{(S,T): (S,T) \text{ is a pair of intersectable } n-\text{system of } \boldsymbol{BLK}\}$ and $\boldsymbol{IS}=\cup_{n=1}^{9}\boldsymbol{IS}(n)$.
We put $\boldsymbol{BTOOL}=\boldsymbol{SFS}\cup \boldsymbol{IS}$.

For each n, $1\leqq n\leqq 9$, we put $\boldsymbol{SFS}(n,rOW)=\{(s,b): b\in rOW,\, s\subset b \text{ and } |s|=n\}$, $\boldsymbol{SFS}(n,cOL)=\{(s,b): b\in cOL,\, s\subset b \text{ and } |s|=n\}$ and $\boldsymbol{SFS}(n,bLK)=\{(s,b): b\in bLK,\, s\subset b \text{ and } |s|=n\}$. Thus we have that $\boldsymbol{SFS}(n)=\boldsymbol{SFS}(n,rOW)\cup \boldsymbol{SFS}(n,cOL)\cup \boldsymbol{SFS}(n,bLK)$.
For each n, $1\leqq n\leqq 9$, we put $\boldsymbol{IG}(n)=\cup\{\ll R,C\gg : R\subset rOW,\, C\subset cOL,\, |R|=|C|=n\}$, $\boldsymbol{IG}=\cup_{n=1}^{9}\boldsymbol{IG}(n)$.

For each type K we put
$\boldsymbol{IS}(K)=\{\omega=(S,T): \omega \text{ is an intersectable system of type } K\}$
We have the following relations among them: $\boldsymbol{IS}=\cup_{n=1}^{15}\boldsymbol{IS}(Type\ n)$,

$IS(1)=IS(Type\ 1)\cup IS(Type\ 2)$, $IS(2)=IS(Type\ 3)\cup IS(Type\ 4)\cup IS(Type\ 5)$, $IS(3)=IS(Type\ 6)\cup IS(Type\ 7)\cup IS(Type\ 8A)\cup IS(Type\ 8B)\cup IS(Type\ 9)$, $IS(4)=IS(Type\ 10)$, $IS(5)=IS(Type\ 11)$, $IS(6)=IS(Type\ 12)$, $IS(7)=IS(Type\ 13)$, $IS(8)=IS(Type\ 14)$ and $IS(9)=IS(Type\ 15)$.

$IG(1)=IS(Type\ 1)$, $IG(2)=IS(Type\ 5)$, $IG(3)=IS(Type\ 9)$, $IG(4)=IS(Type\ 10)$, $IG(5)=IS(Type\ 11)$, $IG(6)=IS(Type\ 12)$, $IG(7)=IS(Type\ 13)$, $IG(8)=IS(Type\ 14)$, $IG(9)=IS(Type\ 15)$.

We use the following names for sudoku transformations, which are popular in Japan. For each $\omega\in SFS(n)$, we say that T_ω is an $n-$koku$-$domei sudoku transformation. For each $\omega\in IG(n)$, we say that T_ω is an $n-$igeta sudoku transformation. For each $\omega\in IS(Type\ 2)$, we say that T_ω is a 1$-$igeta(Type 2) sudoku transformation.

Proposition 54. We put $ETOOL=SFS\cup IS(Type\ 2)\cup IG(2)\cup IG(3)\cup IG(4)$. Then $\{T_\omega:\omega\in BTOOL\}\equiv_s\{T_\omega:\omega\in ETOOL\}$ in $STRF(f,f_0)$ for each $f\in SOL(f_0)$.

Proof. For each $U\subset BTOOL$, we put $T[U]=\{T_\omega:\omega\in U\}$. By Proposition 45 we have that

(1) $T[IS(Type\ 3)]\subset T[IS(Type\ 2)]$.

By Proposition47, Proposition 48, Proposition 49 and Proposition 50 we have that

(2) $T[IS(Type\ 6)]=\{1\}$,

(3) $T[IS(Type\ 7)]\subset T[IS(Type\ 2)]$

(4) $T[IS(Type\ 8A)]=\{1\}$,

(5) $T[IS(Type\ 15)]=\{1\}$.

By Proposition 51 we have that

(6) $T[IS(Type\ 14)]=T[IS(Type\ 1)]$,

(7) $T[IS(Type\ 13)]=T[IS(Type\ 5)]$,

(8) $T[IS(Type\ 12)]=T[IS(Type\ 9)]$,

(9) $T[IS(Type\ 11)]=T[IS(Type\ 10)]$.

By Proposition 43 we have that

(10) $IS=\cup_{n=1}^{15}IS(Type\ n)$.

By (1)$-$(10) we have that

(11) $T[IS]=T\big[\cup_{n=1}^{15}IS(Type\ n)\big]=\cup_{n-1}^{15}T[IS(Type\ n)]$
$=T[IS(Type\ 1)]\cup T[IS(Type\ 2)]\cup T[IS(Type\ 4)]\cup T[IS(Type\ 5)]\cup\{1\}\cup$
$\cup T[IS(Type\ 8B)]\cup T[IS(Type\ 9)]\cup T[IS(Type\ 10)]$.

By definition we have that

(12) $IG(2)=IS(Type\ 5)$, $IG(3)=IS(Type\ 9)$, $IG(4)=IS(Type\ 10)$.

By (11), (12) we have

(13) $T[\boldsymbol{IS}]=T[\boldsymbol{IS}(Type\ 1)]\cup T[\boldsymbol{IS}(Type\ 2)]\cup T[\boldsymbol{IS}(Type\ 4)]\cup T[\boldsymbol{IS}(Type\ 8B)]\cup\{1\}\cup$
$\cup T[\boldsymbol{IG}(2)]\cup T[\boldsymbol{IG}(3)]\cup T[\boldsymbol{IG}(4)]$.

Since $\boldsymbol{BTOOL}=\boldsymbol{SFS}\cup\boldsymbol{IS}\supset\boldsymbol{ETOOL}=\boldsymbol{SFS}\cup\boldsymbol{IS}(Type\ 2)\cup\boldsymbol{IG}(2)\cup\boldsymbol{IG}(3)\cup\boldsymbol{IG}(4)$, by (13) we have that

(14) $T[\boldsymbol{BTOOL}]=T\{\boldsymbol{ETOOL}\}\cup T[\boldsymbol{IS}(Type\ 1)]\cup T[\boldsymbol{IS}(Type\ 4)]\cup T[\boldsymbol{IS}(Type\ 8B)]\cup\{1\}$.

We put $T[\boldsymbol{ETOOL}]^{*}=T[\boldsymbol{ETOOL}]\cup\{T_2\circ T_1:T_1,T_2\in T[\boldsymbol{ETOOL}]\}$. Also we put $T[\boldsymbol{ETOOL}]^{*}_{2\cap}=\{T_1\cap T_2:T_1,T_2\in T[\boldsymbol{ETOOL}]^{*}\}$. By definitions we have

(15) $T[\boldsymbol{ETOOL}]^{*}_{2\cap}\supset T[\boldsymbol{ETOOL}]^{*}\supset T[\boldsymbol{ETOOL}]$.

By Proposition 39 we have that

(16) $T[\boldsymbol{ETOOL}]^{*}_{2\cap}\equiv_s T[\boldsymbol{ETOOL}]^{*}$ in $\boldsymbol{STRF}(f,f_0)$ for each $f\in\boldsymbol{SOL}(f_0)$.

Since $\{T_2\circ T_1:T_1,T_2\in T[\boldsymbol{ETOOL}]\}\circ\leqq T[\boldsymbol{ETOOL}]$, by Proposition 40 we have that

(17) $T[\boldsymbol{ETOOL}]^{*}\equiv_s T[\boldsymbol{ETOOL}]$ in $\boldsymbol{STRF}(f,f_0)$ for each $f\in\boldsymbol{SOL}(f_0)$.

By (16), (17) we have that

(18) $T[\boldsymbol{ETOOL}]^{*}_{2\cap}\equiv_s T[\boldsymbol{ETOOL}]$ in $\boldsymbol{STRF}(f,f_0)$ for each $f\in\boldsymbol{SOL}(f_0)$.

Claim 1. $T[\boldsymbol{BTOOL}]\ll_{DC} T[\boldsymbol{ETOOL}]^{*}_{2\cap}$.

Proof of Claim 1. We define a decreasing map $\boldsymbol{\rho}:T[\boldsymbol{BTOOL}]\to T[\boldsymbol{ETOOL}]^{*}_{2\cap}$ such that

(19) $\boldsymbol{\rho}(T)\leqq T$ for each $T\in T[\boldsymbol{BTOOL}]$.

Take any $T\in T[\boldsymbol{BTOOL}]$. By (14) we have the following 5 cases.

Case 1. $T\in T[\boldsymbol{ETOOL}]$.

In this case, we put $\boldsymbol{\rho}(T)=T$. By (15) it is well defined and $\boldsymbol{\rho}(T)\leqq T$.

Case 2. $T\in T[\boldsymbol{IS}(Type\ 1)]$.

Then we have an intersectable system $\boldsymbol{\omega}=(S,T)$ of Type 1 such that $T=T_{\boldsymbol{\omega}}$. Since $\boldsymbol{\omega}$ is of Type 1, we have a row s and a column t such that $S=\{s\}$, $T=\{t\}$. By Proposition 44, we have that $\boldsymbol{\omega}_1=(s-s\cap t,s)$, $\boldsymbol{\omega}_2=(s\cap t,t)$, $\boldsymbol{\omega}_3=(t-s\cap t,t)$, $\boldsymbol{\omega}_4=(s\cap t,s)$ and

(20) $T_{\boldsymbol{\omega}_2}\circ T_{\boldsymbol{\omega}_1}\cap T_{\boldsymbol{\omega}_4}\circ T_{\boldsymbol{\omega}_3}\leqq T_{\boldsymbol{\omega}}=T$.

Since $\boldsymbol{\omega}_1,\boldsymbol{\omega}_2,\boldsymbol{\omega}_3,\boldsymbol{\omega}_4\in\boldsymbol{SFS}$, then we put $\boldsymbol{\rho}(T)=T_{\boldsymbol{\omega}_2}\circ T_{\boldsymbol{\omega}_1}\cap T_{\boldsymbol{\omega}_4}\circ T_{\boldsymbol{\omega}_3}\in T[\boldsymbol{ETOOL}]^{*}_{2\cap}$.

Then it is well defined and by (20), $\boldsymbol{\rho}(T)\leqq T$.

Case 3. $T\in T[\boldsymbol{IS}(Type\ 4)]$.

Then we have an intersectable system $\boldsymbol{\omega}=(S,T)$ of Type 4 such that $T=T_{\boldsymbol{\omega}}$. By

Proposition 46 we have intersectable systems ω_1 and ω_2 of Type 2 such that

(21) $T_{\omega_2}\circ T_{\omega_1}\cap T_{\omega_1}\circ T_{\omega_2}\leqq T_{\omega}=T$.

Since ω_1 and ω_2 are of Type 2, then we put $\rho(T)=T_{\omega_2}\circ T_{\omega_1}\cap T_{\omega_1}\circ T_{\omega_2}$ $\in T[\boldsymbol{ETOOL}]^*_{2\cap}$. Then it is well defined and by (21), $\rho(T)\leqq T$.

Case 4. $T\in T[\boldsymbol{IS}(Type\ 8B)]$.

Then we have an intersectable system $\omega=(S,T)$ of Type 8B such that $T=T_\omega$. By Proposition 49 we have intersectable systems ω_1 and ω_2 of Type 2 such that

(22) $T_{\omega_1}\cap T_{\omega_2}\leqq T_\omega=T$.

Since ω_1 and ω_2 are of Type 2, then we put $\rho(T)=T_{\omega_1}\cap T_{\omega_2}\in T[\boldsymbol{ETOOL}]^*_{2\cap}$. Then it is well defined and by (22), $\rho(T)\leqq T$.

Case 5. $T\in\{1\}$.

In this case $T=1$. Take any $\omega\in\boldsymbol{ETOOL}$ and put $\rho(T)=T_\omega\in T[\boldsymbol{ETOOL}]^*_{2\cap}$. Then it is well defined. Since T_ω is a sudoku transformation, we have that $\rho(T)=T_\omega\leqq 1=T$.

By using the above cases, we can define a map $\rho: T[\boldsymbol{BTOOL}]\to T[\boldsymbol{ETOOL}]^*_{2\cap}$ with (19). Hence we have Claim 1.

By Claim 1 and Proposition 37 we have that

(23) $STBL^{T[\boldsymbol{ETOOL}]^*_{2\cap}}\leqq STBL^{T[\boldsymbol{BTOOL}]}$ in $\boldsymbol{STRF}(f,f_0)$ for each $f\in\boldsymbol{SOL}(f_0)$.

By (18) we have

(24) $STBL^{T[\boldsymbol{ETOOL}]^*_{2\cap}}=STBL^{T[\boldsymbol{ETOOL}]}$ in $\boldsymbol{STRF}(f,f_0)$ for each $f\in\boldsymbol{SOL}(f_0)$.

By (23) and (24) we have that

(25) $STBL^{T[\boldsymbol{ETOOL}]}\leqq STBL^{T[\boldsymbol{BTOOL}]}$ in $\boldsymbol{STRF}(f,f_0)$ for each $f\in\boldsymbol{SOL}(f_0)$.

Since $\boldsymbol{BTOOL}\supset\boldsymbol{ETOOL}$, then $T[\boldsymbol{BTOOL}]\supset T[\boldsymbol{ETOOL}]$. By Corollary 38 we have

(26) $STBL^{T[\boldsymbol{ETOOL}]}\geqq STBL^{T[\boldsymbol{BTOOL}]}$ in $\boldsymbol{STRF}(f,f_0)$ for each $f\in\boldsymbol{SOL}(f_0)$.

By (25) and (26) we have that

(27) $STBL^{T[\boldsymbol{ETOOL}]}=STBL^{T[\boldsymbol{BTOOL}]}$ in $\boldsymbol{STRF}(f,f_0)$ for each $f\in\boldsymbol{SOL}(f_0)$.

By (27) we have that

(28) $T[\boldsymbol{BTOOL}]\equiv_s T[\boldsymbol{ETOOL}]$ in $\boldsymbol{STRF}(f,f_0)$ for each $f\in\boldsymbol{SOL}(f_0)$.

(28) means Proposition 54. Hence we complete the proof of Proposition 54.

Proposition 55. $T_\omega\circ T_\omega=T_\omega$ for each $\omega\in\boldsymbol{BTOOL}$.

Proof. Take any $\omega \in BTOOL$. Take any sudoku matrix $K = (K_\alpha)_{\alpha \in J_1 \times J_2} \in SMTX(f, f_0)$. We put $K' = T_\omega(K)$, $K'' = T_\omega(K')$ and $K' = (K'_\alpha)_{\alpha \in J_1 \times J_2}$, $K'' = (K''_\alpha)_{\alpha \in J_1 \times J_2}$. Since $BTOOL = SFS \cup IS$, we have the following cases.

Claim 1. $T_\omega \circ T_\omega = T_\omega$ for each $\omega \in SFS$

Since $SFS = \cup_{n=1}^{9} SFS(n)$, then $\omega \in SFS(n)$ for some n. Thus we put $\omega = (s, b)$, $s \subset b$, $b \in BLK$, $|s| = n$.

(1) $K' = T_\omega(K) = \begin{cases} nNSF(s,b)(K) & if \ |K_s| = |s| = n \\ K & if \ |K_s| \neq |s| = n \end{cases}$,

(2) $K'' = T_\omega(K') = \begin{cases} nNSF(s,b)(K') & if \ |K'_s| = |s| = n \\ K' & if \ |K'_s| \neq |s| = n \end{cases}$.

Case 1. $|K_s| = |s| = n$.

We conside case 1. By (1)

(3) $K' = nNSF(s,b)(K)$.

By (3) we have

(4) $K'_\alpha = \begin{cases} K_\alpha & for \ \alpha \in s \\ K_\alpha - K_s & for \ \alpha \in b - s \\ K_\alpha & for \ \alpha \in J_1 \times J_2 - b \end{cases}$.

By (4) we have

(5) $K'_\alpha = K_\alpha$ for each $\alpha \in s$.

By (5) we have that $K'_s = \cup\{K'_\alpha : \alpha \in s\} = \cup\{K_\alpha : \alpha \in s\} = K_s$, i.e.,

(6) $K'_s = K_s$.

By the condition of case 1 and (6) we have that

(7) $|K'_s| = |K_s| = |s| = n$.

By (2) and (7) we have that

(8) $K'' = nNSF(s,b)(K')$.

By (8) we have that

(9) $K''_\alpha = \begin{cases} K'_\alpha & for \ \alpha \in s \\ K'_\alpha - K'_s & for \ \alpha \in b - s \\ K'_\alpha & for \ \alpha \in J_1 \times J_2 - b \end{cases}$.

By (4),(6) and (9) we have that

$$(10)\quad K''_\alpha = \begin{cases} K'_\alpha = K_\alpha & for\ \alpha \in s \\ K'_\alpha - K'_s = (K_\alpha - K_s) - K'_s = K_\alpha - K_s \cup K'_s = K_\alpha - K_s & for\ \alpha \in b-s. \\ K'_\alpha & for\ \alpha \in J_1 \times J_2 - b \end{cases}$$

By (4) and (10) we have that

(11) $K''_\alpha = K'_\alpha$ for $a \in J_1 \times J_2$, i.e.,

(13) $\boldsymbol{K}'' = \boldsymbol{K}'$.

(13) means that Claim 1 holds for case 1.

Case 2. $|K_s| \neq |s| = n$.

In this case by (1) we have that

(13) $\boldsymbol{K}' = \boldsymbol{K}$.

By (13) we have

(14) $K'_s = K_s$.

By the condition of case 2 and (14) we have that $|K'_s| = |K_s| \neq |s| = n$, i.e.,

(15) $|K'_s| \neq |s| = n$.

By (2) and (15) we have that

(16) $\boldsymbol{K}'' = \boldsymbol{K}'$.

(16) means that Claim 1 holds for case 2.

Thus, by case 1 and case 2, we have Claim 1.

Claim 2. $T_\omega \circ T_\omega = T_\omega$ for each $\boldsymbol{\omega} \in \boldsymbol{IS}$

Since $\boldsymbol{\omega} \in \boldsymbol{IS} = \cup_{n=1}^{9} \boldsymbol{IS}(n)$, then $\in \boldsymbol{IS}(n)$ for some n. Thus $\boldsymbol{\omega} = (S,T)$ is an itersectable $n-$system such that $S = \{s_1, s_2, \ldots, s_n\}$, $T = \{t_1, t_2, \ldots, t_n\}$ snd $s = \cup_{i=1}^{n} s_i$, $t = \cup_{j=1}^{n} t_j$,

(17) $T_\omega = T(S,T)$.

By (17) we have

$$(18)\quad K'_\alpha = \begin{cases} K_\alpha & for\ \alpha \in (J_1 \times J_2 - s \cup t) \cup (s \cap t) \\ K_\alpha \cap K_{t-s} & for\ \alpha \in s-t \\ K_\alpha \cap K_{s-t} & for\ \alpha \in t-s \end{cases},$$

$$(19)\quad K''_\alpha = \begin{cases} K'_\alpha & for\ \alpha \in (J_1 \times J_2 - s \cup t) \cup (s \cap t) \\ K'_\alpha \cap K'_{t-s} & for\ \alpha \in s-t \\ K'_\alpha \cap K'_{s-t} & for\ \alpha \in t-s \end{cases}.$$

Case 3. $\alpha \in (J_1 \times J_2 - s \cup t) \cup (s \cap t)$.

In this case, by (19) we have

(20) $K''_\alpha = K'_\alpha$.

Case 4, $\alpha \in s-t$.

In this case, by (18) we have

(21) $K'_\alpha = K_\alpha \cap K_{t-s}$.

By (19) we have

(22) $K''_\alpha = K'_\alpha \cap K'_{t-s}$.

By (21) and (22) we have

$$K''_\alpha = K'_\alpha \cap K'_{t-s} = K'_\alpha \cap (\cup\{K'_\beta : \beta \in t-s\}) = K'_\alpha \cap (\cup\{K_\beta \cap K_{s-t} : \beta \in t-s\})$$
$$= K'_\alpha \cap ((\cup\{K_\beta : \beta \in t-s\}) \cap K_{s-t}) = K'_\alpha \cap (K_{t-s} \cap K_{s-t}) = (K_\alpha \cap K_{t-s}) \cap (K_{t-s} \cap K_{s-t})$$
$$= K_\alpha \cap K_{t-s} \cap K_{s-t} = (K_\alpha \cap K_{s-t}) \cap K_{t-s} = K_\alpha \cap K_{t-s} = K'_\alpha \text{ i.e.,}$$

(23) $K''_\alpha = K'_\alpha$.

Case 5, $\alpha \in t-s$.

In this case, by (18) we have

(24) $K'_\alpha = K_\alpha \cap K_{s-t}$.

By (19) we have

(25) $K''_\alpha = K'_\alpha \cap K'_{s-t}$.

By (24) and (25) we have

$$K''_\alpha = K'_\alpha \cap K'_{s-t} = K'_\alpha \cap (\cup\{K'_\beta : \beta \in s-t\}) = K'_\alpha \cap (\cup\{K_\beta \cap K_{t-s} : \beta \in s-t\})$$
$$= K'_\alpha \cap ((\cup\{K_\beta : \beta \in s-t\}) \cap K_{t-s}) = K'_\alpha \cap (K_{s-t} \cap K_{t-s}) = (K_\alpha \cap K_{s-t}) \cap (K_{s-t} \cap K_{t-s})$$
$$= K_\alpha \cap K_{s-t} \cap K_{t-s} = (K_\alpha \cap K_{t-s}) \cap K_{s-t} = K_\alpha \cap K_{s-t} = K'_\alpha \text{ i.e.,}$$

(25) $K''_\alpha = K'_\alpha$.

By (20),(23),(25) we have

(26) $K''_\alpha = K'_\alpha$ for each $\alpha \in J_1 \times J_2$.

(26) means that $\boldsymbol{K}'' = \boldsymbol{K}'$. Hence we have Proposition 55.

References

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