２次元拡張 Green-Naghdi 方程式のハミルトン構造

Hamiltonian structure for two-dimensional extended Green-Naghdi equations

山口大学大学院理工学研究科　松野　好雅 (Yoshimasa Matsuno)
Division of Applied Mathematical Science
Graduate School of Science and Engineering
Yamaguchi University
E-mail address: matsuno@yamaguchi-u.ac.jp

The two-dimensional Green-Naghdi (GN) shallow-water model for surface gravity waves is extended to incorporate the arbitrary higher-order dispersive effects. The linear dispersion relation for the extended GN system is then explored in detail. As illustrative examples of approximate model equations, we derive a higher-order model which is accurate to the fourth power of the dispersion parameter in the case of a flat bottom topography. Subsequently, the extended GN system presented here is shown to have the same Hamiltonian structure as that of the original GN system. Last, we demonstrate that Zakharov's Hamiltonian formulation of surface gravity waves is equivalent to that of the extended GN system by rewriting the former system in terms of the momentum density instead of the velocity potential at the free surface.

1. Introduction

Recently, we extended the Green-Naghdi (GN) shallow-water model equations to incorporate the arbitrary higher-order dispersive effects while preserving the full nonlinearity (Matsuno (2015)). Here, we extend it to the two-dimensional (2D) system by making use of a novel asymptotic analysis, and show that it has the same Hamiltonian structure as that of the original 2D GN system. We consider the three-dimensional irrotational flow of an incompressible and inviscid fluid of variable depth. The effect of surface tension is neglected since it has no appreciable influence on the current water wave phenomena. It can be, however, incorporated in our formulation without difficulty. The governing equation of the water wave problem is given in terms of the dimensionless variables by

$$\delta^2\nabla^2\phi+\phi_{zz}=0,\quad -1+\beta b<z<\epsilon\eta, \tag{1.1}$$

$$\eta_t+\epsilon\nabla\phi\cdot\nabla\eta=\frac{1}{\delta^2}\phi_z,\quad z=\epsilon\eta, \tag{1.2}$$

$$\phi_t+\frac{\epsilon}{2\delta^2}\left\{\delta^2(\nabla\phi)^2+\phi_z^2\right\}+\eta=0,\quad z=\epsilon\eta, \tag{1.3}$$

$$\beta\delta^2\nabla b\cdot\nabla\phi=\phi_z,\quad z=-1+\beta b, \tag{1.4}$$

subjected to the boundary conditions

$$\lim_{|\boldsymbol{x}|\to\infty}\nabla\phi(\boldsymbol{x},z,t)=\boldsymbol{0},\quad \lim_{|\boldsymbol{x}|\to\infty}\phi_z(\boldsymbol{x},z,t)=0,\quad -1+\beta b<z<\epsilon\eta,\quad \lim_{|\boldsymbol{x}|\to\infty}\eta(\boldsymbol{x},t)=0. \tag{1.5}$$

Here, $\phi=\phi(\boldsymbol{x},z,t)$ is the velocity potential with $\boldsymbol{x}=(x,y)$ being a vector in the horizontal plane and z the vertical coordinate pointing upwards, $\nabla=(\partial/\partial x,\partial/\partial y)$ is the 2D gradient operator, $\eta=\eta(\boldsymbol{x},t)$ is the profile of the free surface, $b=b(\boldsymbol{x})$ specifies the bottom topography, and the subscripts z and t appended to ϕ and η denote partial differentiations.

The dimensional quantities, with tildes, are related to the corresponding dimensionless ones by the relations $\tilde{\boldsymbol{x}}=l\boldsymbol{x},\tilde{z}=h_0z,\tilde{t}=(l/c_0)t,\tilde{\eta}=a\eta$, $\tilde{\phi}=(gla/c_0)\phi$ and $\tilde{b}=b_0b$, where l, h_0, a, and b_0 denote a characteristic wavelength, water depth, wave amplitude and bottom profile, respectively. g is the acceleration due to the gravity, and $c_0=\sqrt{gh_0}$ is the long wave phase velocity. There arise the following three independent dimensionless parameters from the above scalings of the variables:

$$\epsilon=\frac{a}{h_0},\quad \delta=\frac{h_0}{l},\quad \beta=\frac{b_0}{h_0}. \tag{1.6}$$

The nonlinearity parameter ϵ characterizes the magnitude of nonlinearity whereas the dispersion parameter δ characterizes the dispersion or shallowness, and the parameter β measures the variation of the bottom topography. What is meant by "full nonlinearity" is that no restriction is imposed on the magnitude of ϵ. Actually, ϵ is assumed to be of order 1 in our analysis. On the other hand, we impose the condition $\delta \ll 1$ for the dispersion parameter which features the shallow water model equations.

In §2, we reformulate the water wave problem posed by equations. (1.1)-(1.5) in terms of the total depth of fluid h and the depth-averaged horizontal velocity $\bar{\boldsymbol{u}}$ which will be defined later. The system of equations thus constructed consists of the exact evolution equation for h and an infinite-order Boussinesq-type equation for $\bar{\boldsymbol{u}}$. By truncating the latter equation at order δ^{2n}, we obtain the extended GN equations which are accurate to δ^{2n}, where n is an arbitrary positive integer. We call it the δ^{2n} model hereafter. The lowest-order approximation $n = 1$ yields the GN equations. We then derive the linear dispersion relation for the extended GN system, and investigate its characteristics in detail. In §3, we derive, as illustrative examples, various approximate model equations which include the 2D δ^4 model with a flat bottom topography and the 2D δ^2 model (or the GN model) with an uneven bottom topography. The 1D δ^6 model with a flat bottom topography is briefly described. In §4, we show that the extended GN equations can be formulated as a Hamiltonian form by introducing an appropriate Lie-Poisson bracket as well as the momentum density in place of $\bar{\boldsymbol{u}}$, and they have the same Hamiltonian structure as that of the GN equations. In §5, we demonstrate that the extended GN equations are equivalent to Zakharov's equations of motion for surface gravity waves. Finally, §6 is devoted to conclusion.

2. Derivation of the extended Green-Naghdi equations

2.1. *Extended GN system*

The GN model is a system of equations for the total depth of fluid h and the depth-averaged (or mean) horizontal velocity $\bar{\boldsymbol{u}} = (\bar{u}, \bar{v})$. The latter variable is defined by

$$\bar{\boldsymbol{u}} = \frac{1}{h}\int_{-1+\beta b}^{\epsilon\eta} \nabla\phi(\boldsymbol{x}, z, t)\mathrm{d}z, \quad h = 1 + \epsilon\eta - \beta b. \tag{2.1}$$

The horizontal component $\boldsymbol{u} = (u, v)$ and verical component w of the surface velocity are given respectively by

$$\boldsymbol{u}(\boldsymbol{x}, t) = \nabla\phi(\boldsymbol{x}, z, t)|_{z=\epsilon\eta}, \quad w(\boldsymbol{x}, t) = \phi_z(\boldsymbol{x}, z, t)|_{z=\epsilon\eta}. \tag{2.2}$$

First, we derive the equation for h. It follows from (1.1), (1.4) and (2.1) that

$$w = \delta^2\{-\nabla\cdot(h\bar{\boldsymbol{u}}) + \epsilon\boldsymbol{u}\cdot\nabla\eta\}. \tag{2.3}$$

Substituting (2.3) into (1.2), we obtain the evolution equation for $h = h(\boldsymbol{x}, t)$

$$h_t + \epsilon\nabla\cdot(h\bar{\boldsymbol{u}}) = 0. \tag{2.4}$$

It is important that (2.4) is an *exact* equation without any approximation.

The equation for $\bar{\boldsymbol{u}}$ can be derived from the equation for $\boldsymbol{u}$. A direct computation yields

$$\nabla\left(\phi_t|_{z=\epsilon\eta}\right) = \boldsymbol{u}_t + \epsilon w_t\nabla\eta - \epsilon\eta_t\nabla w. \tag{2.5}$$

We apply the gradient operator to (1.3) and use (2.5) together with the definition of $\boldsymbol{u}$ and w. This leads to

$$\boldsymbol{u}_t + \epsilon w_t\nabla\eta + \frac{\epsilon}{2}\nabla\boldsymbol{u}^2 + \epsilon\left(-\eta_t + \frac{1}{\delta^2}w\right)\nabla w + \nabla\eta = \boldsymbol{0}. \tag{2.6}$$

It follows by eliminating the term $\nabla\cdot(h\bar{\boldsymbol{u}})$ from (2.3) and (2.4) that $-\eta_t + \frac{1}{\delta^2}w = \epsilon\boldsymbol{u}\cdot\nabla\eta$. If we substitute this expression into the fourth term on the left-hand side of (2.6), we arrive at the evolution equation for $\boldsymbol{u}$:

$$\boldsymbol{u}_t + \epsilon w_t\nabla\eta + \frac{\epsilon}{2}\nabla\boldsymbol{u}^2 + \epsilon^2(\boldsymbol{u}\cdot\nabla\eta)\nabla w + \nabla\eta = \boldsymbol{0}. \tag{2.7}$$

Now, we introduce the new quantity $\boldsymbol{V}$ by

$$\boldsymbol{V} = \boldsymbol{u} + \epsilon w \nabla \eta. \tag{2.8}$$

It then turns out from (2.7) that the evolution equation for $\boldsymbol{V}$ can be written in the form

$$\boldsymbol{V}_t - \epsilon w \nabla \eta_t + \frac{\epsilon}{2} \nabla \boldsymbol{u}^2 + \epsilon^2 (\boldsymbol{u} \cdot \nabla \eta) \nabla w + \nabla \eta = \boldsymbol{0}. \tag{2.9}$$

Last, we substitute the relations

$$-w \nabla \eta_t = \epsilon w \nabla (\boldsymbol{u} \cdot \nabla \eta) - \frac{1}{2\delta^2} \nabla w^2, \quad \epsilon (\boldsymbol{u} \cdot \nabla \eta) w = \boldsymbol{u} \cdot \boldsymbol{V} - \boldsymbol{u}^2, \tag{2.10}$$

which follow from (1.2) and (2.8), respectively, into the corresponding terms in (2.9) to obtain an alternative form of the evolution equation for $\boldsymbol{V}$:

$$\boldsymbol{V}_t + \epsilon \nabla \left(\boldsymbol{u} \cdot \boldsymbol{V} - \frac{1}{2} \boldsymbol{u}^2 - \frac{1}{2\delta^2} w^2 + \frac{\eta}{\epsilon} \right) = \boldsymbol{0}. \tag{2.11}$$

Equation (2.11) represents an exact conservation law for the vector $\boldsymbol{V}$. To interpret the physical meaning of $\boldsymbol{V}$, we introduce the velocity potential evaluated at the free surface

$$\psi(\boldsymbol{x}, t) = \phi(\boldsymbol{x}, \epsilon \eta, t). \tag{2.12}$$

In view of the definition (2.2) of the surface velocity, the gradient of ψ is found to be

$$\nabla \psi = (\nabla \phi + \epsilon \phi_z \nabla \eta)|_{z=\epsilon\eta} = \boldsymbol{u} + \epsilon w \nabla \eta. \tag{2.13}$$

It immediately follows from (2.12) and (2.13) that

$$\boldsymbol{V} = \nabla \psi, \tag{2.14}$$

implying that $\boldsymbol{V}$ is equal to the 2D gradient of the velocity potential evaluated at the free surface, and it lies in the (x, y) plane.

The system of equations (2.4) and (2.7) (or (2.11)) is a consequence deduced from the basic Euler system (1.1)-(1.4). The extended GN equations are obtained if one can express the variables $\boldsymbol{u}, w$ in equation (2.7) in terms of h and $\bar{\boldsymbol{u}}$. As will be shown below, this is always possible. Consequently, the evolution equation for $\bar{\boldsymbol{u}}$ can be recast in the form of an infinite-order Boussinesq-type equation

$$\bar{\boldsymbol{u}}_t = \sum_{n=0}^{\infty} \delta^{2n} \boldsymbol{K}_n, \tag{2.15}$$

where $\boldsymbol{K}_n \in \mathbb{R}^2$ are vector functions of h and $\nabla \cdot \bar{\boldsymbol{u}}, \nabla \cdot \bar{\boldsymbol{u}}_t$ as well as the spatial derivatives of these variables. If one truncates the right-hand side of equation (2.15) at order δ^{2n}, then equation (2.15) yields the evolution equation for $\bar{\boldsymbol{u}}$ which is accurate to δ^{2n}. The special case $n = 1$ coupled with equation (2.4) reduces to the original GN equations. In accordance with this fact, we call the system of equations (2.4) and (2.7) (or (2.11), (2.15)) with h and $\bar{\boldsymbol{u}}$ being the dependent variables the extended GN system.

2.2. *Expressions of the velocities $\boldsymbol{u}, w$ and $\boldsymbol{V}$ in terms of h and $\bar{\boldsymbol{u}}$*

2.2.1. *Flat bottom topography*

Under the assumption $\delta^2 \ll 1$ which is relevant to the shallow water models, the solution of equation (1.1) subjected to the boubdary condition (1.4) with $b = 0$ can be written explicitly in the form of an infinite series

$$\phi(\boldsymbol{x}, z, t) = \sum_{n=0}^{\infty} \frac{(-1)^n \delta^{2n}}{(2n)!} (z+1)^{2n} \nabla^{2n} f, \tag{2.16}$$

where $f = f(\boldsymbol{x}, t)$ is the velocity potential at the fluid bottom. We substitute this expression into (2.1) and perform the integration with respect to z to obtain

$$\bar{\boldsymbol{u}} = \nabla f + \sum_{n=1}^{\infty} \frac{(-1)^n \delta^{2n} h^{2n}}{(2n+1)!} \nabla\nabla^{2n} f, \quad h = 1 + \epsilon\eta. \tag{2.17}$$

Using the formula $\nabla^2 f = \nabla \cdot (\nabla f)$, we can rewrite (2.17) in an alternative form

$$\nabla f = \bar{\boldsymbol{u}} - \sum_{n=1}^{\infty} \frac{(-1)^n \delta^{2n} h^{2n}}{(2n+1)!} \nabla\nabla^{2(n-1)}(\nabla\cdot\nabla f). \tag{2.18}$$

To derive the expansion of ∇f in terms of h and $\bar{\boldsymbol{u}}$, we look for the solution in the form of an infinite series in δ^2

$$\nabla f = \bar{\boldsymbol{u}} + \sum_{n=1}^{\infty} (-1)^n \delta^{2n} \boldsymbol{F}_n, \tag{2.19}$$

where $\boldsymbol{F}_n \in \mathbb{R}^2$ are unknown vector functions to be determined below. Substituting this expression into (2.18) and comparing the coefficients of δ^{2n} $(n = 1, 2, ...)$ on both sides, we obtain $\boldsymbol{F}_n$, first three of which read

$$\boldsymbol{F}_1 = -\frac{h^2}{6} \nabla(\nabla\cdot\bar{\boldsymbol{u}}), \quad \boldsymbol{F}_2 = -\frac{h^4}{120}\nabla\nabla^2(\nabla\cdot\bar{\boldsymbol{u}}) + \frac{h^2}{36}\nabla\nabla\cdot\{h^2\nabla(\nabla\cdot\bar{\boldsymbol{u}})\},$$
$$\boldsymbol{F}_3 = -\frac{h^6}{5040}\nabla\nabla^4(\nabla\cdot\bar{\boldsymbol{u}}) - \frac{h^2}{6}\nabla(\nabla\cdot\boldsymbol{F}_2) - \frac{h^4}{120}\nabla\nabla^2(\nabla\cdot\boldsymbol{F}_1). \tag{2.20}$$

The series expansions of $\boldsymbol{u}, w$ and $\boldsymbol{V}$ can be derived simply by substituting (2.18) with $\boldsymbol{F}_n$ from (2.20) into (2.2)and (2.8), respectively. We write them up to order δ^4 for later use:

$$\boldsymbol{u} = \bar{\boldsymbol{u}} - \frac{\delta^2}{3} h^2\nabla(\nabla\cdot\bar{\boldsymbol{u}}) + \delta^4 \left[-\frac{1}{18}h^2\nabla\nabla\cdot\{h^2\nabla(\nabla\cdot\bar{\boldsymbol{u}})\} + \frac{1}{30}h^4\nabla\nabla^2(\nabla\cdot\bar{\boldsymbol{u}})\right] + O(\delta^6), \tag{2.21}$$

$$w = -\delta^2 h\nabla\cdot\bar{\boldsymbol{u}} - \frac{\delta^4}{3} h^2\nabla h\cdot\nabla(\nabla\cdot\bar{\boldsymbol{u}}) + O(\delta^6), \tag{2.22}$$

$$\boldsymbol{V} = \bar{\boldsymbol{u}} - \frac{\delta^2}{3h} \nabla(h^3\nabla\cdot\bar{\boldsymbol{u}}) - \frac{\delta^4}{45h} \nabla[\nabla\cdot\{h^5\nabla(\nabla\cdot\bar{\boldsymbol{u}})\}] + O(\delta^6). \tag{2.23}$$

2.2.2. *Uneven bottom topography*

The effect of an uneven bottom topography on the propagation characteristics of water waves is prominent in the coastal zone. Here, we provide the formulas of $\boldsymbol{u}, w$ and $\boldsymbol{V}$ in terms of $h, \bar{\boldsymbol{u}}$ and b. In this case, the solution of the Laplace equation (1.1) subjected to the boundary condition (1.5) can be written in the form

$$\phi(\boldsymbol{x}, z, t) = \sum_{n=0}^{\infty} (z + 1 - \beta b)^n \phi_n(\boldsymbol{x}, t), \tag{2.24}$$

where the orders of unknown functions ϕ_n are to be determined. Performing the similar procedure to that has been done for the flat bottom case, we obtain the approximate expressions of $\boldsymbol{u}$, w and $\boldsymbol{V}$ in terms of $\bar{\boldsymbol{u}}, h$ and b:

$$\boldsymbol{u} = \bar{\boldsymbol{u}} + \delta^2 \left[-\frac{h^2}{3}\nabla(\nabla\cdot\bar{\boldsymbol{u}}) + \frac{\beta}{2}\{h\nabla(\nabla b\cdot\bar{\boldsymbol{u}}) + (h\nabla\cdot\bar{\boldsymbol{u}})\nabla b\}\right] + O(\delta^4), \tag{2.25}$$

$$w = \delta^2(-h\nabla\cdot\bar{\boldsymbol{u}} + \beta\nabla b\cdot\bar{\boldsymbol{u}}) + O(\delta^4). \tag{2.26}$$

$$\boldsymbol{V} = \bar{\boldsymbol{u}} + \frac{\delta^2}{h}\left[-\frac{1}{3}\nabla(h^3\nabla\cdot\bar{\boldsymbol{u}}) + \frac{\beta}{2}\{\nabla(h^2\nabla b\cdot\bar{\boldsymbol{u}}) - h^2\nabla b(\nabla\cdot\bar{\boldsymbol{u}})\} + \beta^2 h\nabla b(\nabla b\cdot\bar{\boldsymbol{u}})\right] + O(\delta^4). \quad (2.27)$$

2.3. *Linear dispersion relation for the extended GN system*

Here, we show that the exact linear dispersion relation for the current water wave problem can be derived from the extended GN system, and discuss its structure. We consider the flat bottom case for simplicity. Linearization of equations (2.4) and (2.7) about the uniform state $h = 1$ and $\bar{\boldsymbol{u}} = \boldsymbol{0}$ yields the system of linear equations for η and $\bar{\boldsymbol{u}}$. Explicitly, $\eta_t + \nabla\cdot\bar{\boldsymbol{u}} = 0, \quad \boldsymbol{u}_t + \nabla\eta = \boldsymbol{0}$. We eliminate the variable η from this system of equations and obtain the linear wave equation for $\bar{\boldsymbol{u}}$

$$\boldsymbol{u}_{tt} - \nabla(\nabla\cdot\bar{\boldsymbol{u}}) = \boldsymbol{0}. \quad (2.28)$$

In the linear approximation, the expression $\boldsymbol{u}$ corresponding to (2.21) can be written in the form

$$\boldsymbol{u} = \bar{\boldsymbol{u}} + \sum_{n=1}^{\infty}(-1)^n\delta^{2n}\left\{\frac{1}{(2n)!} + \sum_{r=0}^{n-1}\frac{\alpha_{n-r}}{(2r)!}\right\}\nabla\nabla^{2(n-1)}(\nabla\cdot\bar{\boldsymbol{u}}), \quad (2.29)$$

where α_n are unknown constants which are determined by the recursion relation

$$\alpha_1 = -\frac{1}{6}, \quad \alpha_n = -\frac{1}{(2n+1)!} - \sum_{r=1}^{n-1}\frac{\alpha_{n-r}}{(2r+1)!}, \quad n \geqslant 2. \quad (2.30)$$

In order to examine the linear dispersion characteristics of equation (2.28) with $\boldsymbol{u}$ from (2.29), we assume the solution of the form $\bar{\boldsymbol{u}} = \bar{\boldsymbol{u}}_0\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$, where $\bar{\boldsymbol{u}}_0$ is a 2D constant vector, $\boldsymbol{k}$ is the 2D wavenumber vector and ω is the angular frequency. We substitute (2.29) into equation (2.28) and find that the linear dispersion relation takes the form

$$\omega^2 = \frac{k^2}{D(k\delta)}, \quad (k = |\boldsymbol{k}|), \quad D(k\delta) = 1 + \sum_{n=1}^{\infty}(k\delta)^{2n}\left\{\frac{1}{(2n)!} + \sum_{r=0}^{n-1}\frac{\alpha_{n-r}}{(2r)!}\right\}. \quad (2.31)$$

Using (2.30), we can derive the relation $D(k\delta) = k\delta \coth k\delta$ which, substituted into (2.31), leads to the linear dispersion relation for the extended GN system

$$\omega^2 = \frac{k}{\delta}\tanh k\delta. \quad (2.32)$$

The above expression coincides perfectly with that derived from the linearized system of equations for the current water wave problem (1.1)-(1.5).

The δ^{2n} GN model incorporates the dispersive terms of order δ^{2n}. Referring to equations (2.4) and (2.15), one can write it in the form

$$h_t + \epsilon\nabla\cdot(h\bar{\boldsymbol{u}}) = 0, \quad \bar{\boldsymbol{u}}_t = \sum_{m=0}^{n}\delta^{2m}\boldsymbol{K}_m. \quad (2.33)$$

To detail the dispersion characteristics of this model, we introduce the function $D_{2n}(\kappa)$ by

$$D_{2n}(\kappa) = 1 + \sum_{r=1}^{n}\frac{(-1)^{r-1}2^{2r}}{(2r)!}B_r\kappa^{2r}, \quad B_r = \frac{2(2r)!}{(2\pi)^{2r}}\sum_{j=1}^{\infty}\frac{1}{j^{2r}}, \quad r \geqslant 1, \quad (2.34)$$

where B_r are Bernoulli's numbers. The linear dispersion relation for the δ^{2n} model (2.33) is represented by

$$\omega^2 = \frac{k^2}{D_{2n}(k\delta)}. \quad (2.35)$$

Using the inequality for the Bernoulli numbers, we can show that D_{2n} with odd n are positive for all $k\delta$. More precisely, they have a lower bound 1. As a result, an estimate $\omega/k \leqslant 1$ for $k\delta \geqslant 0$ follows. Consequently, the δ^{2n} models with odd n have a nice property as long as the linear dispersion characteristic is concerned. Actually, they have smooth dispersion relations without any singularities, and possess an important feature that the exact linear dispersion relation has, i.e., $\omega/k = \sqrt{\tanh k\delta/k\delta} \leqslant 1$ for $k\delta \geqslant 0$. On the other hand, D_{2n} models with even n exhibit single positive zero. For example, the positive zeros of D_4, D_8 and D_{10} are found to be $4.19, 3.63$ and 3.33, respectively. An asymptotic analysis shows that the zero of D_{2n} with even n approaches a constant value π as n tends to infinity. These results imply that ω from (2.35) has a singularity and becomes pure imaginary for values of $k\delta$ exceeding the zero. It turns out that the δ^{2n} models with even n exhibit an unphysical dispersion characteristic which leads to the ill-posedness result for the linearized systems of equations, and may cause instabilities in short wave solutions in practical numerical computations. In accordance with these observations, the δ^{2n} models with odd n may be more tractable as the practical model equations than the δ^{2n} models with even n.

3. Approximate model equations

3.1. *The δ^4 model*

3.1.1. *Derivation of the δ^4 model with a flat bottom topography*

For the purpose of deriving the δ^4 model with a flat bottom topography, we only need the evolution equation for $\bar{\boldsymbol{u}}$ since the equation for h is already at hand, as indicated by equation (2.4). The procedure for obtaining the equation for $\bar{\boldsymbol{u}}$ can be performed straightforwardly. Actually, substituting the expressions (2.21)-(2.23) into equation (2.11) and rearranging terms, we finally arrive at the evolution equation for $\bar{\boldsymbol{u}}$:

$$\bar{\boldsymbol{u}}_t + \epsilon(\bar{\boldsymbol{u}}\cdot\nabla)\bar{\boldsymbol{u}} + \nabla\eta = \delta^2 R_1 + \delta^4 R_2 + O(\delta^6), \tag{3.1a}$$

with

$$R_1 = \frac{1}{3h}\nabla\Big[h^3\{\nabla\cdot\bar{\boldsymbol{u}}_t + \epsilon(\bar{\boldsymbol{u}}\cdot\nabla)(\nabla\cdot\bar{\boldsymbol{u}}) - \epsilon(\nabla\cdot\bar{\boldsymbol{u}})^2\}\Big], \tag{3.1b}$$

$$R_2 = \frac{1}{45h}\nabla\Big[\nabla\cdot\big\{h^5\nabla(\nabla\cdot\bar{\boldsymbol{u}}_t) + \epsilon h^5(\nabla^2(\nabla\cdot\bar{\boldsymbol{u}}))\bar{\boldsymbol{u}} - 5\epsilon h^5(\nabla\cdot\bar{\boldsymbol{u}})\nabla(\nabla\cdot\bar{\boldsymbol{u}}) + \epsilon\nabla h^5\times(\bar{\boldsymbol{u}}\times\nabla(\nabla\cdot\bar{\boldsymbol{u}}))\big\}$$
$$-2\epsilon h^5\{\nabla(\nabla\cdot\bar{\boldsymbol{u}})\}^2\Big] - \frac{\epsilon}{45h}\Big[\nabla\cdot\{h^5\nabla(\nabla\cdot\bar{\boldsymbol{u}})\}\nabla(\nabla\cdot\bar{\boldsymbol{u}}) + \frac{h^5}{2}\nabla\{\nabla(\nabla\cdot\bar{\boldsymbol{u}})\}^2\Big]. \tag{3.1c}$$

Various reductions are possible for the δ^4 model. Indeed, if we neglect the δ^4 terms in equation (3.1), then it reduces to the 2D GN system when coupled with equation (2.4)

$$h_t + \epsilon\nabla\cdot(h\bar{\boldsymbol{u}}) = 0, \quad \bar{\boldsymbol{u}}_t + \epsilon(\bar{\boldsymbol{u}}\cdot\nabla)\bar{\boldsymbol{u}} + \nabla\eta = \frac{\delta^2}{3h}\nabla\left[h^3\{\nabla\cdot\bar{\boldsymbol{u}}_t + \epsilon(\bar{\boldsymbol{u}}\cdot\nabla)(\nabla\cdot\bar{\boldsymbol{u}}) - \epsilon(\nabla\cdot\bar{\boldsymbol{u}})^2\}\right], \tag{3.2}$$

whereas the δ^4 model reduces to the classical 2D Boussinesq system

$$h_t + \epsilon\nabla\cdot(h\bar{\boldsymbol{u}}) = 0, \quad \bar{\boldsymbol{u}}_t + \epsilon(\bar{\boldsymbol{u}}\cdot\nabla)\bar{\boldsymbol{u}} + \nabla\eta = \frac{\delta^2}{3}\nabla(\nabla\cdot\bar{\boldsymbol{u}}_t), \tag{3.3}$$

after neglecting the $\epsilon\delta^2$ and higher-order terms. On the other hand, the 1D forms of equations (2.4) and (3.1) become

$$h_t + \epsilon(h\bar{u})_x = 0, \tag{3.4a}$$

$$\bar{u}_t + \epsilon\bar{u}\bar{u}_x + \eta_x = \frac{\delta^2}{3h}\left\{h^3(\bar{u}_{xt} + \epsilon\bar{u}\bar{u}_{xx} - \epsilon\bar{u}_x^2)\right\}_x$$
$$+\frac{\delta^4}{45h}\left[\left\{h^5(\bar{u}_{xxt} + \epsilon\bar{u}\bar{u}_{xxx} - 5\epsilon\bar{u}_x\bar{u}_{xx})\right\}_x - 3\epsilon h^5\bar{u}_{xx}^2\right]_x + O(\delta^6), \tag{3.4b}$$

which are in agreement with equations (2.5) and (2.21) of Matsuno (2015), respectively.

3.1.2. *Conservation laws*

The δ^4 model derived here exhibits the following four conservation laws:

$$M = \int_{\mathbb{R}^2} (h-1)\mathrm{d}\boldsymbol{x}, \tag{3.5}$$

$$\boldsymbol{P} = \int_{\mathbb{R}^2} h\bar{\boldsymbol{u}}\,\mathrm{d}\boldsymbol{x}, \tag{3.6}$$

$$H = \frac{\epsilon^2}{2}\int_{\mathbb{R}^2}\left[h\bar{\boldsymbol{u}}^2 + \frac{\delta^2}{3}h^3(\nabla\cdot\bar{\boldsymbol{u}})^2 - \frac{\delta^4}{45}h^5\{\nabla(\nabla\cdot\bar{\boldsymbol{u}})\}^2 + \frac{1}{\epsilon^2}(h-1)^2\right]\mathrm{d}\boldsymbol{x}, \tag{3.7}$$

$$\boldsymbol{L} = \epsilon\int_{\mathbb{R}^2}\left[\bar{\boldsymbol{u}} - \frac{\delta^2}{3h}\nabla(h^3\nabla\cdot\bar{\boldsymbol{u}}) - \frac{\delta^4}{45h}\nabla\left[\nabla\cdot\{h^5\nabla(\nabla\cdot\bar{\boldsymbol{u}}\}\right]\right]\mathrm{d}\boldsymbol{x}, \tag{3.8}$$

where we have used the notation $\int_{\mathbb{R}^2} F(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\boldsymbol{x},t)\mathrm{d}x\mathrm{d}y$ for any function F decreasing rapidly at infinity. The factors ϵ^2 and ϵ attached in front of the integral sign in H and $\boldsymbol{L}$, respectively are only for convenience. The quantities $M, \boldsymbol{P}$ and H represent the conservation of the mass, momentum and total energy, respectively, which can be confirmed directly by using equations (2.4) and (3.1). The fourth conservation law $\boldsymbol{L}$ follows from (2.11) and (2.23). The geometrical interpretation of $\boldsymbol{L}$ has been discussed in detail in the 1D case. See **Remark 6** of Matsuno (2015).

3.2. *The GN model with an uneven bottom topography*

In accordance with the method developed in §2, let us derive the GN model which takes into account an uneven bottom topography. Since its derivation is almost parallel to that of the flat bottom case, we describe only the final result. The evolution equation for $\bar{\boldsymbol{u}}$ can be written in the form

$$\left(1 + \frac{\delta^2}{h}\mathcal{L}(h,b)\right)\bar{\boldsymbol{u}}_t + \epsilon(\bar{\boldsymbol{u}}\cdot\nabla)\bar{\boldsymbol{u}} + \nabla\eta = \frac{\epsilon\delta^2}{3h}\nabla\Big[h^3\{(\bar{\boldsymbol{u}}\cdot\nabla)\nabla\cdot\bar{\boldsymbol{u}} - (\nabla\cdot\bar{\boldsymbol{u}})^2\}\Big] + \epsilon\delta^2 Q, \tag{3.9a}$$

with

$$Q = -\frac{\beta}{2h}\Big[\nabla\{h^2\bar{\boldsymbol{u}}\cdot\nabla(\nabla b\cdot\bar{\boldsymbol{u}})\} - h^2\{\bar{\boldsymbol{u}}\cdot\nabla(\nabla\cdot\bar{\boldsymbol{u}}) - (\nabla\cdot\bar{\boldsymbol{u}})^2\}\nabla b\Big] - \beta^2\{(\bar{\boldsymbol{u}}\cdot\nabla)^2 b\}\nabla b, \tag{3.9b}$$

where $\mathcal{L}(h,b)$ is a linear differential operator defined by

$$\mathcal{L}(h,b)\boldsymbol{f} = -\frac{1}{3}\nabla(h^3\nabla\cdot\boldsymbol{f}) + \frac{\beta}{2}\{\nabla(h^2\nabla b\cdot\boldsymbol{f}) - h^2\nabla b(\nabla\cdot\boldsymbol{f})\} + \beta^2 h\nabla b(\nabla b\cdot\boldsymbol{f}), \tag{3.9c}$$

for any vector function $\boldsymbol{f}\in\mathbb{R}^2$. This equation coincides perfectly with that obtained by different methods. See Green & Naghdi (1976), Miles & Salmon (1985), Bazdenkov *et al.* (1987), Lannes & Bonneton (2009) and Lannes (2013).

3.3. *Remark*

As already demonstrated in §2.3, the δ^{2n} models with even n have singularities in their linear dispersion relations, although the dispersion characteristics for small values of the dispersion parameter have been improved considerably when compared with those of the original GN model. The simplest extended GN model which avoids this undesirable behavior in higher wavenumber is the 1D δ^6 model with a flat bottom topography. Its derivation can be made straightforwardly by means of the procedure developed in this section.

The evolution equation for $\bar{u}$ which extends equation (3.4b) to order δ^6 can now be written in the form

$$\bar{u}_t + \epsilon\bar{u}\bar{u}_x + \eta_x = \frac{\delta^2}{3h}\left\{h^3(\bar{u}_{xt} + \epsilon\bar{u}\bar{u}_{xx} - \epsilon\bar{u}_x^2)\right\}_x + \frac{\delta^4}{45h}\left[\left\{h^5(\bar{u}_{xxt} + \epsilon\bar{u}\bar{u}_{xxx} - 5\epsilon\bar{u}_x\bar{u}_{xx})\right\}_x - 3\epsilon h^5\bar{u}_{xx}^2\right]_x$$

$$+\frac{\delta^6}{945h}\Big[\{h^7(2\bar{u}_{xxxxt}+2\epsilon\bar{u}\bar{u}_{xxxxx}-14\epsilon\bar{u}_x\bar{u}_{xxxx}-30\epsilon\bar{u}_{xx}\bar{u}_{xxx})\}_x$$

$$+\{h^6h_x(14\bar{u}_{xxxt}+14\epsilon\bar{u}\bar{u}_{xxxx}-112\epsilon\bar{u}_x\bar{u}_{xxx}+42\epsilon\bar{u}_{xx}^2)\}_x$$

$$+\{h^5(hh_x)_x(7\bar{u}_{xxt}+7\epsilon\bar{u}\bar{u}_{xxx}-63\epsilon\bar{u}_x\bar{u}_{xx})\}_x+\epsilon\{10h^7\bar{u}_{xxx}^2-35h^5(hh_x)_x\bar{u}_{xx}^2\}\Big]_x. \tag{3.10}$$

The linear dispersion relation for the system of equations (3.4a) and (3.10) is then given by

$$\omega^2=\frac{k^2}{1+\frac{1}{3}(k\delta)^2-\frac{1}{45}(k\delta)^4+\frac{2}{945}(k\delta)^6}. \tag{3.11}$$

Obviously, the singularity does not occur in ω for arbitrary values of $k\delta$, as opposed to the δ^4 model. This ensures the well-posedness of the system of linearized equations for the model. Various features of the δ^6 model are worth studying in comparison with those of the δ^4 model, as well as those of the δ^2 (or GN) model.

4. Hamiltonian structure

4.1. *Hamiltonian*

In this section, we show that the 2D extended GN system derived in §2 can be formulated as a Hamiltonian form. First, recall that the basic Euler system of equations (1.1)-(1.4) conserves the total energy (or Hamiltonian) H which is the sum of the kinetic energy K and the potential energy U:

$$H=K+U=\frac{\epsilon^2}{2}\int_{\mathbb{R}^2}\left[\int_{-1+\beta b}^{\epsilon\eta}\left\{(\nabla\phi)^2+\frac{1}{\delta^2}\phi_z^2\right\}\mathrm{d}z\right]\mathrm{d}\boldsymbol{x}+\frac{\epsilon^2}{2}\int_{\mathbb{R}^2}\eta^2\mathrm{d}\boldsymbol{x}. \tag{4.1}$$

Using (1.1) and (1.4), this Hamiltonian can be put into a simple form

$$H=\frac{\epsilon^2}{2}\int_{\mathbb{R}^2}\left[h\bar{\boldsymbol{u}}\cdot\nabla\psi+\frac{1}{\epsilon^2}(h-1+\beta b)^2\right]\mathrm{d}\boldsymbol{x}, \tag{4.2}$$

Inserting the expression of $\nabla\psi(=\boldsymbol{V})$ from (2.27) into (4.2), we obtain a series expansion of H in powers of δ^2

$$H=\epsilon^2\sum_{n=0}^{\infty}\delta^{2n}H_n, \tag{4.3a}$$

with the first two of H_n being given by

$$H_0=\frac{1}{2}\int_{\mathbb{R}^2}\left[h\bar{\boldsymbol{u}}^2+\frac{1}{\epsilon^2}(h-1+\beta b)^2\right]\mathrm{d}\boldsymbol{x},\quad H_1=\frac{1}{6}\int_{\mathbb{R}^2}\left[h^3(\nabla\cdot\bar{\boldsymbol{u}})^2-3\beta h^2(\nabla b\cdot\bar{\boldsymbol{u}})\nabla\cdot\bar{\boldsymbol{u}}+3\beta^2h(\nabla b\cdot\bar{\boldsymbol{u}})^2\right]\mathrm{d}\boldsymbol{x}. \tag{4.3b}$$

4.2. *Momentum density*

In formulating the extended GN system as a Hamiltonian form, it is crucial to introduce the momentum density $\boldsymbol{m}$. It is given by the following relation

$$\epsilon\boldsymbol{m}=\frac{\delta H}{\delta\bar{\boldsymbol{u}}}, \tag{4.4}$$

where the operator $\delta/\delta\bar{\boldsymbol{u}}$ is the variational derivative defined by

$$\frac{\partial}{\partial\epsilon}H(\bar{\boldsymbol{u}}+\epsilon\boldsymbol{w})\big|_{\epsilon=0}=\int_{\mathbb{R}^2}\frac{\delta H}{\delta\bar{\boldsymbol{u}}}\cdot\boldsymbol{w}\,\mathrm{d}\boldsymbol{x}, \tag{4.5}$$

for arbitrary vector function $\boldsymbol{w} \in \mathbb{R}^2$. As seen from (4.3) and its higher-order analog, the integrand of K is quadratic in $\bar{\boldsymbol{u}}$, and hence K obeys the scaling law $K(\epsilon\bar{\boldsymbol{u}}, h, b) = \epsilon^2 K(\bar{\boldsymbol{u}}, h, b)$. This leads, after introducing $\boldsymbol{m}$ from (4.4), to the relation $K = \frac{\epsilon}{2}\int_{\mathbb{R}^2} \boldsymbol{m}\cdot\bar{\boldsymbol{u}}\,\mathrm{d}\boldsymbol{x}$, so that H is expressed compactly as

$$H = \frac{1}{2}\int_{\mathbb{R}^2} [\epsilon\,\boldsymbol{m}\cdot\bar{\boldsymbol{u}} + (h - 1 + \beta b)^2]\,\mathrm{d}\boldsymbol{x}. \tag{4.6}$$

Comparing (4.2) and (4.6), we obtain the key relation which connects the variable $\nabla\psi$ with the momentum density $\boldsymbol{m}$:

$$\boldsymbol{m} = \epsilon h \nabla\psi. \tag{4.7}$$

Note that the kinetic energy obeys the scaling law $K(\epsilon\boldsymbol{m}, h, b) = \epsilon^2 K(\boldsymbol{m}, h, b)$, and hence $K = \frac{1}{2}\int_{\mathbb{R}^2} \delta H/\delta\boldsymbol{m}\cdot\boldsymbol{m}\,\mathrm{d}\boldsymbol{x}$. This expression must be equal to $K = \frac{\epsilon}{2}\int_{\mathbb{R}^2} \boldsymbol{m}\cdot\bar{\boldsymbol{u}}\,\mathrm{d}\boldsymbol{x}$, giving the dual relation to (4.7)

$$\epsilon\bar{\boldsymbol{u}} = \frac{\delta H}{\delta\boldsymbol{m}}. \tag{4.8}$$

4.3. *Evolution equation for the momentum density*

To derive the evolution equation for the momentum density $\boldsymbol{m}$, we first compute the variational derivative of H with respect to h. It is given by

$$\frac{\delta H}{\delta h} = \epsilon^2\left(\frac{1}{2}\boldsymbol{u}^2 + \frac{w^2}{2\delta^2} - \boldsymbol{u}\cdot\bar{\boldsymbol{u}} + hw\nabla\cdot\bar{\boldsymbol{u}} - \beta w\nabla b\cdot\bar{\boldsymbol{u}}\right) + h - 1 + \beta b. \tag{4.9}$$

Now, we proceed to derive the evolution equation for $\boldsymbol{m}$. We start from the evolution equation for $\boldsymbol{V}$ from (2.11). After a few manipulations using (2.4) and (4.7), we obtain

$$\boldsymbol{m}_t + \epsilon\nabla(\bar{\boldsymbol{u}}\cdot\boldsymbol{m}) + \epsilon(\nabla\cdot\bar{\boldsymbol{u}})\boldsymbol{m} + \frac{\epsilon}{h}\{(\nabla h\cdot\bar{\boldsymbol{u}})\boldsymbol{m} - (\bar{\boldsymbol{u}}\cdot\boldsymbol{m})\nabla h\} + h\nabla\left(\frac{\delta H}{\delta h}\right) = \boldsymbol{0}. \tag{4.10}$$

Furthermore, if we divide (4.10) by h and use (2.4), we can write it in the form of local conservation law

$$\left(\frac{\boldsymbol{m}}{h}\right)_t + \nabla\left(\frac{\epsilon\bar{\boldsymbol{u}}\cdot\boldsymbol{m}}{h} + \frac{\delta H}{\delta h}\right) = \boldsymbol{0}. \tag{4.11}$$

4.4. *Hamiltonian formulation*

In this section, we demonstrate that the 2D extended GN system can be formulated as a Hamiltonian system. To this end, we introduce the noncanonical Lie-Poisson bracket between any pair of smooth functional F and G

$$\{F, G\} = -\int_{\mathbb{R}^2}\left[\sum_{i,j=1}^{2}\frac{\delta F}{\delta m_i}(m_j\partial_i + \partial_j m_i)\frac{\delta G}{\delta m_j} + h\,\frac{\delta F}{\delta\boldsymbol{m}}\cdot\nabla\frac{\delta G}{\delta h} + \frac{\delta F}{\delta h}\,\nabla\cdot\left(h\,\frac{\delta G}{\delta\boldsymbol{m}}\right)\right]\mathrm{d}\boldsymbol{x}, \tag{4.12}$$

where we have put $\boldsymbol{m} = (m_1, m_2)$ and $\partial_1 = \partial/\partial x, \partial_2 = \partial/\partial y$. Note that the partial derivatives ∂_i $(i = 1, 2)$ operate on all terms they multiply to the right. Then, our main result is given by the following theorem.

Theorem 1. *The 2D extended GN system (2.4) and (2.11) (or equivalently, (4.10)) can be written in the form of Hamilton's equations*

$$h_t = \{h, H\}, \tag{4.13a}$$

$$m_{i,t} = \{m_i, H\}, \quad (i = 1, 2). \tag{4.13b}$$

We recall that the bracket (4.12) has been introduced by Holm (1988) to formulate the 2D GN equations as a Hamiltonian system. Combining this fact with Theorem 1, we conclude that the extended GN system has the same Hamiltonian structure as that of the GN system. Hence, its truncated version like the δ^{2n} model shares the same property.

5. Relation to Zakharov's Hamiltonian formulation

5.1. *Zakharov's formulation*

Zakharov (1968) (see also Zakharov & Kuznetsov (1997)) showed that the water wave problem (1.1)-(1.5) permits a canonical Hamiltonian formulation. Specifically, the equations of motion for the variables h and $\nabla\psi$ are written in the form

$$h_t = -\frac{1}{\epsilon}\nabla\cdot\frac{\delta H}{\delta\nabla\psi}, \quad \nabla\psi_t = -\frac{1}{\epsilon}\nabla\frac{\delta H}{\delta h}, \tag{5.1}$$

$$\{F, G\} = -\frac{1}{\epsilon}\int_{\mathbb{R}^2}\left[\frac{\delta F}{\delta h}\left(\nabla\cdot\frac{\delta G}{\delta\nabla\psi}\right) - \left(\nabla\cdot\frac{\delta F}{\delta\nabla\psi}\right)\frac{\delta G}{\delta h}\right]\mathrm{d}\boldsymbol{x}, \tag{5.2}$$

$$h_t = \{h, H\}, \quad \nabla\psi_t = \{\nabla\psi, H\}. \tag{5.3}$$

5.2. *Transformation of the Zakharov system to the extended GN system*

Here, we establish the following theorem.

Theorem 2. *Zakharov's system of equations (5.3) is equivalent to the extended GN system (4.13).*

This theorem follows by rewriting the Zakharov system in terms of the variable $\boldsymbol{m}$ in place of $\nabla\psi$ while h remains the common variable for both systems. The proof can be performed by using the relations

$$\left.\frac{\delta F}{\delta h}\right|_{\nabla\psi} = \left.\frac{\delta F}{\delta h}\right|_{\boldsymbol{m}} + \frac{1}{h}\left.\frac{\delta F}{\delta\boldsymbol{m}}\right|_{h}\cdot\boldsymbol{m}, \quad \left.\frac{\delta F}{\delta\nabla\psi}\right|_{h} = \epsilon h\left.\frac{\delta F}{\delta\boldsymbol{m}}\right|_{h}, \tag{5.4}$$

$$\left.\frac{\delta H}{\delta h}\right|_{\nabla\psi} = \left.\frac{\delta H}{\delta h}\right|_{\boldsymbol{m}} + \frac{\epsilon\bar{\boldsymbol{u}}\cdot\boldsymbol{m}}{h}, \quad \left.\frac{\delta H}{\delta\nabla\psi}\right|_{h} = \epsilon^2 h\bar{\boldsymbol{u}}. \tag{5.5}$$

6. Conclusion

In this paper, we have developed a systematic procedure for extending the 2D GN model to include higher-order dispersive effects while preserving full nonlinearity of the original GN model, and presented various model equations for both flat and uneven bottom topographies. A detailed analysis of the linearized system of equations for the extended GN models reveals that the linear dispersion relation for the δ^{2n} model coincides with the exact linear dispersion relation for the water wave problem up to order δ^{2n} for small values of the dispersion parameter. For odd n, the dispersion relations have a nice property in the sense that they exhibit no singularities for all values of the dispersion parameter. It turns out that the corresponding model equations are linearly well-posed. When n is even, however, the dispersion relations were found to exhibit a singularity, indicating the possibility of instabilities in short wave solutions. Although the value of the dispersion parameter at which the singularity occurs is greater than π and hence it is beyond the range of applicability of the extended GN models, they may not be appropriate to use as the basis for practical applications to real water wave phenomena. Hence, in order to verify the validity of the models, the rigorous mathematical justification is necessary for the formal derivation of the models, and it will become an important issue to be pursued in a future work.

We have demonstrated that the extended GN equations have the same Hamiltonian structure as that of the GN equations. In the process, we have introduced the momentum density in place of the depth-averaged horizontal velocity, and found a key relation which connects the momentum density with the gradient of the surface potential. Last, the equivalence of the extended GN system and Zakharov's Hamiltonian system was also proved whereby the key relation mentioned above played the central role.

REFERENCES

Bazdenkov, S.V., Morozov, N.N. & Pogutse, O.P. 1987 Dispersive effects in two-dimensional hydrodynamics. *Sov. Phys. Dokl.* **32**, 262-264.

Green, A.E. & Naghdi, P.M. 1976 A derivation of equations for wave propagation in water of variable depth. *J. Fluid Mech.* **78**, 237-246.

Holm, D.D. 1988 Hamiltonian structure for two-dimensional hydrodynamics with nonlinear dispersion. *Phys. Fluids* **31**, 2371-2373.

Lannes, D. & Bonneton, P. 2009 Derivation of asymptotic two-dimensional time-dependent equations for surface water wave propagation. *Phys. Fluids* **21**, 016601.

Lannes, D. 2013 *Water Waves Problem: Mathematical Analysis and Asymptotics*, Mathematical Surveys and Monographs, vol. 188. American Mathematical Society.

Matsuno, Y. 2015 Hamiltonian formulation of the extended Green-Naghdi equations. *Physica D* **301-302**, 1-7.

Miles, J. & Salmon, R. 1985 Weakly dispersive nonlinear gravity waves *J. Fluid Mech.* **157**, 519-531.

Su, C.H. & Gardner, C.S. 1969 Korteweg-de Vries equation and generalizations. III. Derivation of the Korteweg-de Vries equation and Burgers equation. *J. Math. Phys.* **10**, 536-539.

Zakharov, V.E. 1968 Stability of periodic waves of finite amplitude on the surface of a deep fluid. *J. Appl. Mech. Tech. Phys.* **9**, 190-194.

Zakharov, V.E. & Kuznetsov, E.A. 1997 Hamiltonian formalism for nonlinear waves. *Physics-Uspekhi* **40**, 1087-1116.