Mathematics and Sudoku Ⅳ

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We discuss on the worldwide famouse Sudoku by using mathematical approach. This paper is the 4th paper in our series, so we use the same notations and terminologies in [1],[2]and [3] without any descriptions.

7. Properties of stability.

Let S be a set. We put $2^S = \{A : A \subset S\}$. 2^S is called by the power set of S and sometimes it is denoted by $Pow(S)$.

Let A and B be subsets of $\boldsymbol{STRF}(f, f_0)$. We say that A and B are stable equivalent, in notation $A \equiv_s B$, provided that $STBL^A = STBL^B$ in $\boldsymbol{STRF}(f, f_0)$.

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Proposition 35. (a) The stable relation $\equiv_s$ in $\boldsymbol{STRF}(f, f_0)$ is an equivalence relation.

(b) The stable relation $\equiv_s$ in $\boldsymbol{STRF}(f_0)$ is an equivalence relation.

We can easily show Proposition 35.

Let $\boldsymbol{A}$, $\boldsymbol{B}$ be sets of sudoku transformations and $\boldsymbol{\rho} : \boldsymbol{A} \to \boldsymbol{B}$ be a map. We say that $\boldsymbol{\rho}$ is a decreasing map provided that it satisfies the property:

(DC) $\boldsymbol{\rho}(T) \leqq \boldsymbol{T}$ for each $T \in \boldsymbol{A}$.

We say that $\boldsymbol{A}$ decreases $\boldsymbol{B}$, in notaion $\boldsymbol{A} \ll_{DC} \boldsymbol{B}$, provided that there exists a decreasing map $\boldsymbol{\rho} : \boldsymbol{A} \to \boldsymbol{B}$.

Proposition 36. Let $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ be sets of sudoku transformations. Then we have the followings:

(a) $\boldsymbol{A} \ll_{DC} \boldsymbol{A}$.

(b) If $\boldsymbol{A} \ll_{DC} \boldsymbol{B}$ and $\boldsymbol{B} \ll_{DC} \boldsymbol{C}$, then $\boldsymbol{A} \ll_{DC} \boldsymbol{C}$.

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(c) If $\boldsymbol{A} \ll_{DC} \boldsymbol{B}$, then $\boldsymbol{A} \cup \boldsymbol{C} \ll_{DC} \boldsymbol{B} \cup \boldsymbol{C}$ for any $\boldsymbol{C}$.

(d) If $\boldsymbol{A} \subset \boldsymbol{B}$, then $\boldsymbol{A} \ll_{DC} \boldsymbol{B}$

We can easily show Proposition 36.

Proposition 37. (a) Let $\boldsymbol{A}, \boldsymbol{B} \subset \boldsymbol{STRF}(f,f_0)$. If $\boldsymbol{A} \ll_{DC} \boldsymbol{B}$, then $STBL^{\boldsymbol{B}} \leqq STBL^{\boldsymbol{A}}$ in $\boldsymbol{STRF}(f,f_0)$.

(b) Let $\boldsymbol{A}, \boldsymbol{B} \subset \boldsymbol{STRF}(f_0)$. If $\boldsymbol{A} \ll_{DC} \boldsymbol{B}$, then $STBL^{\boldsymbol{B}} \leqq STBL^{\boldsymbol{A}}$ in $\boldsymbol{STRF}(f_0)$.

Proof. We show (a). Take any $\boldsymbol{K} \in \boldsymbol{STMX}(f,f_0)$. We define $P^{\boldsymbol{A}}, P^{\boldsymbol{B}}$: $\boldsymbol{STMX}(f,f_0) \to \boldsymbol{STMX}(f,f_0)$ as follows: $P^{\boldsymbol{A}}(\boldsymbol{K}) = \cap\{T(\boldsymbol{K}) : T \in \boldsymbol{A}\}$ and $P^{\boldsymbol{B}}(\boldsymbol{K}) = \cap\{S(\boldsymbol{K}) : S \in \boldsymbol{B}\}$.

Since $\boldsymbol{A} \ll_{DC} \boldsymbol{B}$, there is a decreasing map $\boldsymbol{\rho} : \boldsymbol{A} \to \boldsymbol{B}$. Since $\boldsymbol{\rho}$ is a decreasing map, we have

(1) $\boldsymbol{\rho}(T) \leqq T$ for each $T \in \boldsymbol{A}$.

By (1) we have

(2) $\boldsymbol{\rho}(T)(\boldsymbol{K}) \subset T(\boldsymbol{K})$.

We put $\boldsymbol{B}^* = \{\boldsymbol{\rho}(T) : T \in \boldsymbol{A}\} \subset \boldsymbol{B}$. Thus, by (2), we have

(3) $P^{\boldsymbol{B}}(\boldsymbol{K}) = \cap\{S(\boldsymbol{K}) : S \in \boldsymbol{B}\} \subset \cap\{\boldsymbol{\rho}(T)(\boldsymbol{K}) : \boldsymbol{\rho}(T) \in \boldsymbol{B}^*\} \subset \cap\{T(\boldsymbol{K}) : T \in \boldsymbol{A}\} = P^{\boldsymbol{A}}(\boldsymbol{K})$

By (3) $P^{\boldsymbol{A}}(\boldsymbol{K}) \supset P^{\boldsymbol{B}}(\boldsymbol{K})$, i.e.,

(4) $P^{\boldsymbol{A}} \geqq P^{\boldsymbol{B}}$.

Let $\boldsymbol{P}^{\boldsymbol{A}} = (P^{\boldsymbol{A}}, P^{\boldsymbol{A}} \ldots, P^{\boldsymbol{A}}, \ldots.)$ and $\boldsymbol{P}^{\boldsymbol{B}} = (P^{\boldsymbol{B}}, P^{\boldsymbol{B}}, \ldots, P^{\boldsymbol{B}}, \ldots)$ be infinte sequences. By (4) and Proposition 33 we have

(5) $\boldsymbol{P}^{\boldsymbol{A}}(\boldsymbol{K})_\infty \supset \boldsymbol{P}^{\boldsymbol{B}}(\boldsymbol{K})_\infty$.

Since $\boldsymbol{P}^{\boldsymbol{A}}_\infty(\boldsymbol{K}) = \boldsymbol{P}^{\boldsymbol{A}}(\boldsymbol{K})_\infty$ and $\boldsymbol{P}^{\boldsymbol{B}}_\infty(\boldsymbol{K}) = \boldsymbol{P}^{\boldsymbol{B}}(\boldsymbol{K})_\infty$, by (5) and Claim 8 in the proof of Proposition 21 we have that

(6) $STBL^{\boldsymbol{A}}(\boldsymbol{K}) = \boldsymbol{P}^{\boldsymbol{A}}_\infty(\boldsymbol{K}) \supset \boldsymbol{P}^{\boldsymbol{B}}_\infty(\boldsymbol{K}) = STBL^{\boldsymbol{B}}(\boldsymbol{K})$.

By (6) we have

(7) $STBL^{\boldsymbol{A}} \geqq STBL^{\boldsymbol{B}}$.

Hence by (7) we have (a).

By the same way we can show (b). Hence, we have Proposition 37.

Corollary 38. (a) Let $\boldsymbol{A}, \boldsymbol{B} \subset \boldsymbol{STRF}(f,f_0)$. If $\boldsymbol{A} \subset \boldsymbol{B}$, then $STBL^{\boldsymbol{B}} \leqq STBL^{\boldsymbol{A}}$ in $\boldsymbol{STRF}(f,f_0)$.

(b) Let $\boldsymbol{A}, \boldsymbol{B} \subset \boldsymbol{STRF}(f_0)$. If $\boldsymbol{A} \subset \boldsymbol{B}$, then $STBL^{\boldsymbol{B}} \leqq STBL^{\boldsymbol{A}}$ in $\boldsymbol{STRF}(f_0)$.

Let $\boldsymbol{A}$ be a set of sudoku transformations. Thus for each n we put

($n\cap$) $\boldsymbol{A}_{n\cap} = \{S_1 \cap S_2 \cap \ldots \cap S_n : S_1, S_2, \ldots, S_n \in \boldsymbol{A}\}$.

When $S_1 = S_2 = ... = S_n = S$, we have $S_1 \cap S_2 \cap ... \cap S_n = S$. Thus $A_{n\cap} \supset A$.

Proposition 39. (a) Let $A \subset \boldsymbol{STRF}(f,f_0)$. Then $A_{n\cap} \equiv_s A$ in $\boldsymbol{STRF}(f,f_0)$ for each n.

(b) Let $A \subset \boldsymbol{STRF}(f_0)$. Then $A_{n\cap} \equiv_s A$ in $\boldsymbol{STRF}(f_0)$ for each n.

Proof. We show (a). Let $B = A_{n\cap}$. Take any $K \in \boldsymbol{STMX}(f,f_0)$. We define P^A, P^B: $\boldsymbol{STMX}(f,f_0) \to \boldsymbol{STMX}(f,f_0)$ as follows: $P^A(K) = \cap\{T(K) : T \in A\}$ and $P^B(K) = \cap\{S(K) : S \in B\}$. By the definition of $(n\cap)$ we can easily show

(1) $P^A(K) = P^B(K)$.

Let $\boldsymbol{P}^A = (P^A, P^A ..., P^A, ...)$ and $\boldsymbol{P}^B = (P^B, P^B, ..., P^B, ...)$ be infinte sequences. By (1) we have

(2) $\boldsymbol{P}^A(K)_\infty = \boldsymbol{P}^B(K)_\infty$.

Since $\boldsymbol{P}^A_\infty(K) = \boldsymbol{P}^A(K)_\infty$ and $\boldsymbol{P}^B_\infty(K) = \boldsymbol{P}^B(K)_\infty$, by (2) and Claim 8 in the proof of Proposition 21 we have that

(3) $STBL^A(K) = \boldsymbol{P}^A_\infty(K) = \boldsymbol{P}^B_\infty(K) = STBL^B(K)$.

By (3) we have

(4) $STBL^A = STBL^B$.

Hence by (4) we have (a).

By the same way we can show (b). Hence, we have Proposition 39.

Let A, B be sets of sudoku transformations. We say that A is dominated by B, in notatin $A \circ\leqq B$, provided that it satisfies the condiiton:

(DT) for each $S \in A$ there exist finite $T_1, T_2, ..., T_n \in B$ such that $S = T_1 \circ T_2 \circ ... \circ T_n$.

Proposition 40. (a) Let $A, B \subset \boldsymbol{STRF}(f,f_0)$. If $A \circ\leqq B$, then $A \cup B \equiv_s B$ in $\boldsymbol{STRF}(f,f_0)$.

(b) Let $A, B \subset \boldsymbol{STRF}(f_0)$. If $A \circ\leqq B$, then $A \cup B \equiv_s B$ in $\boldsymbol{STRF}(f_0)$.

Proof. We show (a). By Proposition 26 we have finite sequences $\boldsymbol{T} = (T_1, T_2, ..., T_n)$ in B and $\boldsymbol{S} = (S_1, S_2, ..., S_m)$ in $A \cup B$ such that for each $K \in \boldsymbol{STMX}(f,f_0)$

(1) $(T_n \circ T_{n-1} \circ ... \circ T_1)(K)$ is $B-$stable,

(2) $(S_m \circ S_{m-1} \circ ... \circ S_1)(K)$ is $A \cup B-$stable.

By Claim 8 in the proof of Proposition 21, we have

(3) $STBL^B(K) = (T_n \circ T_{n-1} \circ ... \circ T_1)(K)$,

(4) $STBL^{A\cup B}(K) = (S_m \circ S_{m-1} \circ ... \circ S_1)(K)$.

Since $B \subset A \cup B$, by (2) we have

(5) $(S_m \circ S_{m-1} \circ ... \circ S_1)(K)$ is $B-$stable.

Since S is a sequence in $A \cup B$, then we have

(6) $S_i \in A \cup B$ for $i, 1 \leqq i \leqq m$.

Since $A \circ \leqq B$, we have the condition (DT). By (6) some $S_i \in A$ can be the composition of finite elements in B. Thus we have a finite sequence $T' = (T'_1, T'_2, \dots T'_k)$ in B such that

(7) $S_m \circ S_{m-1} \circ \dots \circ S_1 = T'_k \circ T'_{k-1} \circ \dots \circ T'_1$.

By (5) and (7) we have

(8) $(T'_k \circ T'_{k-1} \circ \dots \circ T'_1)(K) = (S_m \circ S_{m-1} \circ \dots \circ S_1)(K)$ is $B-$stable.

Since T and T' are finite sequences in B and they satisfy (1) and (8), by Proposition 31 we have

(9) $(T_n \circ T_{n-1} \circ \dots \circ T_1)(K) = (T'_k \circ T'_{k-1} \circ \dots \circ T'_1)(K)$.

By (8) and (9) we have

(10) $(T_n \circ T_{n-1} \circ \dots \circ T_1)(K) = (S_m \circ S_{m-1} \circ \dots \circ S_1)(K)$.

By (3),(4) and (10) we have

(11) $STBL^B(K) = STBL^{A \cup B}(K)$.

(11) means that

(12) $STBL^B = STBL^{A \cup B}$.

Thus, by (12) we have

(13) $A \cup B \equiv_s B$ in $STRF(f, f_0)$.

Hence, we have (a).

By the same way we can show (b). We complete the proof of Proposition 40.

Proposition 41. (a) Let $A, B, C \subset STRF(f, f_0)$. If $A \circ \leqq B$ and $C \ll_{DC} A$, then $A \cup B \cup C \equiv_s A \cup B \equiv_s B \cup C \equiv_s B$ in $STRF(f, f_0)$.

(b) Let $A, B, C \subset STRF(f_0)$. If $A \circ \leqq B$ and $C \ll_{DC} A$, then $A \cup B \cup C \equiv_s A \cup B \equiv_s B \cup C \equiv_s B$ in $STRF(f_0)$.

Proof. We show (a). Since $A \circ \leqq B$, by Proposition 40 we have

(1) $A \cup B \equiv_s B$

By (1) we have

(2) $STBL^{A \cup B} = STBL^B$.

Since $A \circ \leqq B$, we can easily show that $A \circ \leqq B \cup C$. Thus by Proposition 40 we have

(3) $A \cup B \cup C \equiv_s B \cup C$.

Since $C \ll_{DC} A$, we can easily show that $B \cup C \ll_{DC} A \cup B$. Thus by Proposition 37 we have

(4) $STBL^{A \cup B} \leqq STBL^{B \cup C}$.

By (2) and (4) we have

(5) $STBL^{B} \leqq STBL^{B \cup C}$.

Since $\boldsymbol{B} \subset \boldsymbol{B} \cup \boldsymbol{C}$, by Corollary 38 we have

(6) $STBL^{B \cup C} \leqq STBL^{B}$.

By (5) and (6) we have

(7) $STBL^{B \cup C} = STBL^{B}$.

(7) means that

(8) $\boldsymbol{B} \cup \boldsymbol{C} \equiv_{s} \boldsymbol{B}$.

By (1), (3) and (8) we have the required properties. Thus we have (a).

By the same way we can show (b). Hence we complete the proof of Proposition 41.

8. Classification of intersectable systems I.

In the section 8 we discuss intersectable systems. In this section we classify them and show some basic relations among them.

By Proposition 18 we have the following:

Proposition 42. $\{T_{\omega} : \omega \in \boldsymbol{BTOOL}\} \subset \boldsymbol{STRF}(f_0)$.

Let $S = \{s_1, s_2, \ldots, s_n\}$, $T = \{t_1, t_2, \ldots, t_n\} \subset \boldsymbol{BLK}$ for n, $1 \leqq n \leqq 9$. The pair (S,T) is intersectable n−system provided that it satisfies the following conditions:

(i) $s_i \cap s_j = \phi$, $t_i \cap t_j = \phi$ $1 \leqq i \leqq n, 1 \leqq j \leqq n,$ $i \neq j$

(ii) $s_i \cap t_j \neq \phi$, $1 \leqq i \leqq n, 1 \leqq j \leqq n$

(iii) $s = s_1 \cup s_2 \cup \ldots \cup s_n$, $t = t_1 \cup t_2 \cup \ldots \cup t_n$.

When we classify intersectable systems, we can identify (S,T) and (T,S). Thus we put $\ll A, B \gg = \{(A,B), (B,A)\}$.

We can easily classify intersectable systems by rudimentary techniques as follows:

Proposition 43. All intersectable systems are classified as following types. For $r_i \in \boldsymbol{rOW}$, $c_i \in \boldsymbol{cOW}$, $b_i \in \boldsymbol{bLK}$, $1 \leqq i \leqq 9$,

Type 1: $\ll \{r_1\}, \{c_1\} \gg$

Type 2: $\ll \{b_1\}, \{r_1\} \gg \cup \ll \{b_1\}, \{c_1\} \gg$

Type 3: $\ll \{b_1, b_2\}, \{r_1, r_2\} \gg \cup \ll \{b_1, b_2\}, \{c_1, c_2\} \gg$

Type 4: $\ll \{b_1, r_1\}, \{c_1, c_2\} \gg \cup \ll \{b_1, c_1\}, \{r_1, r_2\} \gg$

Type 5: $\ll\{r_1,r_2\},\{c_1,c_2\}\gg$
Type 6: $\ll\{b_1,b_2,b_3\},\{r_1,r_2,r_3\}\gg\cup\ll\{b_1,b_2,b_3\},\{c_1,c_2,c_3\}\gg$
Type 7: $\ll\{b_1,b_2,r_1\},\{c_1,c_2,c_3\}\gg\cup\ll\{b_1,b_2,c_1\},\{r_1,r_2,r_3\}\gg$
Type 8: $\ll\{b_1,r_1,r_2\},\{c_1,c_2,c_3\}\gg\cup\ll\{b_1,c_1,c_2\},\{r_1,r_2,r_3\}\gg$
Type 9: $\ll\{r_1,r_2,r_3\},\{c_1,c_2,c_3\}\gg$
Type 10: $\ll\{r_1,r_2,r_3,r_4\},\{c_1,c_2,c_3,c_4\}\gg$
Type 11: $\ll\{r_1,r_2,r_3,r_4,r_5\},\{c_1,c_2,c_3,c_4,c_5\}\gg$
Type 12: $\ll\{r_1,r_2,r_3,r_4,r_5,r_6\},\{c_1,c_2,c_3,c_4,c_5,c_6\}\gg$
Type 13: $\ll\{r_1,r_2,r_3,r_4,r_5,r_6,r_7\},\{c_1,c_2,c_3,c_4,c_5,c_6,c_7\}\gg$
Type 14: $\ll\{r_1,r_2,r_3,r_4,r_5,r_6,r_7,r_8\},\{c_1,c_2,c_3,c_4,c_5,c_6,c_7,c_8\}\gg$
Type 15: $\ll\{r_1,r_2,r_3,r_4,r_5,r_6,r_7,r_8,r_9\},\{c_1,c_2,c_3,c_4,c_5,c_6,c_7,c_8,c_9\}\gg$

Proposition 44. (Type 1) Let $\boldsymbol{\omega}=(S,T)$ be an intersectable system as type 1. Let $S=\{s_1\}$, $T=\{t_1\}$ such that s_1 is a row and t_1 is a clolumn. Let $s=s_1$ and $t=t_1$. Let $\boldsymbol{\omega}_1=(s-s\cap t,\ s)$, $\boldsymbol{\omega}_2=(s\cap t,\ t)$, $\boldsymbol{\omega}_3=(t-s\cap t,\ t)$ and $\boldsymbol{\omega}_4=(s\cap t,\ s)$. We have that

(a) $T_{\omega_2}\circ T_{\omega_1}\cap T_{\omega_4}\circ T_{\omega_3}\leqq T_\omega$ in $\boldsymbol{STRF}(f,f_0)$ for each $f\in\boldsymbol{SOL}(f_0)$.

Proof. Take any $f\in\boldsymbol{SOL}(f_0)$ and take any $\boldsymbol{K}=(K_\alpha)_{\alpha\in J_1\times J_2}\in\boldsymbol{STMX}(f,f_0)$. Let $T_\omega(\boldsymbol{K})=T(S,T)(\boldsymbol{K})=\boldsymbol{K}'=(K'_\alpha)_{\alpha\in J_1\times J_2}$. By the definition we have

$$(1)\quad K'_\alpha=\begin{cases}K_\alpha & for\ \alpha\in(J_1\times J_2-s\cup t)\cup(s\cap t)\\ K_\alpha\cap K_{t-s}=K_\alpha-(J_3-K_{t-s}) & for\ \alpha\in s-s\cap t\\ K_\alpha\cap K_{s-t}=K_\alpha-(J_3-K_{s-t}) & for\ \alpha\in t-s\cap t\end{cases}$$

We put $s=\{(i_0,1),(i_0,2),....,(i_0,9)\}$ and $t=\{(1,j_0),(2,j_0),...,(9,j_0)\}$ for some i_0 and j_0. We put $\alpha_0=(i_0,j_0)$, and $s\cap t=\{\alpha_0\}$. Thus

(2) $s-t=s-s\cap t=\{(i_0,j):1\leqq j\leqq 9,\ j\neq j_0\}$

(3) $t-s=t-s\cap t=\{(i,j_0):1\leqq i\leqq 9,\ i\neq i_0\}$.

We recall that f is a sudoku map $f:J_1\times J_2\to J_3$ with the property (SDM). By (SDM) we have

(4) $f\mid s:s\to J_3$ and $f\mid t:t\to J_3$ are bijective.

By (4)

(5) $f(s)=f(t)=J_3$ and then $|f(s)|=|f(t)|=|J_3|=9$.

(6) $|f(s-t)|=|f(t-s)|=8$.

Since $\boldsymbol{K}=(K_\alpha)_{\alpha\in J_1\times J_2}\in\boldsymbol{STMX}(f,f_0)$, we have

(7) $f(\alpha)\in K_\alpha$ for $\alpha\in J_1\times J_2$.

By (7) we have

(8) $f(s-t)=\{f((i_0,j)):1\leqq j\leqq 9,\ j\neq j_0\}\subset\cup\{K_{(i_0,j)}:1\leqq j\leqq 9,\ j\neq j_0\}=K_{s-t}\subset J_3$.

By (6), (8) we have

(9) $8=|f(s-t)|=|\{f((i_0,j)):1\leqq j\leqq 9,\ j\neq j_0\}|\leqq|K_{s-t}|\leqq|J_3|=9$.

By (9) we have

(10) $|K_{s-t}|=8$ or $|K_{s-t}|=9$.

We assume that $|K_{s-t}|=8$. By (6) and (8), we have

(11) $f(s-t)=\{f((i_0,j)):1\leqq j\leqq 9,\ j\neq j_0\}=\cup\{K_{(i_0,j)}:1\leqq j\leqq 9,\ j\neq j_0\}=K_{s-t}$.

By (5) and (11), we have

(12) $J_3-K_{s-t}=f(s)-f(s-t)=f(s\cap t)=\{f((i_0,j_0))\}=\{f(\alpha_0)\}$.

We assume that $|K_{s-t}|=9$. By (5) and (8) we have

(12) $K_{s-t}=J_3=f(s)$.

By (12) we have

(13) $J_3-K_{s-t}=\phi$.

Claim 1. (b) If $J_3-K_{s-t}\neq\phi$, then $f(s-t)=\{f((i_0,j)):1\leqq j\leqq 9,\ j\neq j_0\}=\cup\{K_{(i_0,j)}:1\leqq j\leqq 9,\ j\neq j_0\}=K_{s-t}$ and $J_3-K_{s-t}=f(s\cap t)=\{f((i_0,j_0))\}=\{f(\alpha_0)\}$.

(c) If $J_3-K_{t-s}\neq\phi$, then $f(t-s)=\{f((i,j_0)):1\leqq i\leqq 9,\ i\neq i_0\}=\cup\{K_{(i,j_0)}:1\leqq i\leqq 9,\ i\neq i_0\}=K_{t-s}$ and $J_3-K_{t-s}=f(s\cap t)=\{f((i_0,j_0))\}=\{f(\alpha_0)\}$.

Proof of Claim 1. We show (b). By (10) $|K_{s-t}|=8$ or $|K_{s-t}|=9$. We assume that $|K_{s-t}|=9$. Thus by (13), $J_3-K_{s-t}=\phi$. This contradicts to our assumtion $J_3-K_{s-t}\neq\phi$. Then it must be $|K_{s-t}|=8$. Thus, by (11) and (12) we have the required properties. Hence, we have (b).

By the same way we can show (c). Therefore, we have Claim 1.

Claim 2. $T_{\omega_2}\circ T_{\omega_1}(K)\cap T_{\omega_4}\circ T_{\omega_3}(K)\subset T_{\omega}(K)$

Proof of Claim 2. We have the following cases:

Case 1. $J_3-K_{t-s}=\phi$ and $J_3-K_{s-t}=\phi$.

Case 2. $J_3-K_{t-s}\neq\phi$ and $J_3-K_{s-t}=\phi$.

Case 3. $J_3-K_{t-s}=\phi$ and $J_3-K_{s-t}\neq\phi$.

Case 4. $J_3-K_{t-s}\neq\phi$ and $J_3-K_{s-t}\neq\phi$.

We consider the case 1. Since $J_3-K_{t-s}=\phi$ and $J_3-K_{s-t}=\phi$, by (1) we have that $K'_\alpha=K_\alpha$ for $\alpha\in J_1\times J_2$, thus, we have

(14) $T_\omega(K)=T(S,T)(K)=K'=K.$

By Proposition 42 , $T_{\omega_1}, T_{\omega_2}, T_{\omega_3}, T_{\omega_4}$ are sudoku transformations. By Proposition 19, $T_{\omega_2}\circ T_{\omega_1}$ and $T_{\omega_4}\circ T_{\omega_3}$ are sudoku transformations. By Proposition 19, $T_{\omega_2}\circ T_{\omega_1}\cap T_{\omega_4}\circ T_{\omega_3}$ is a sudoku transformation. Then we have

(15) $(T_{\omega_2}\circ T_{\omega_1}\cap T_{\omega_4}\circ T_{\omega_3})(K)\leqq K.$

By (14) and (15) we have

(16) $(T_{\omega_2}\circ T_{\omega_1}\cap T_{\omega_4}\circ T_{\omega_3})(K)\leqq K=T_\omega(K).$

Hence, in this case 1, Claim 2 does hold.

We consider the case 2. By Claim 1 we have

(17) $J_3-K_{t-s}=f(s\cap t)=\{f(\alpha_0)\}$, $K_{t-s}=f(t-s)$ and $J_3-K_{s-t}=\phi.$

By (1) and (17) we have

(18) $$K'_\alpha=\begin{cases}K_\alpha & for\ \alpha\in(J_1\times J_2-s\cup t)\cup\{\alpha_0\}\\ K_\alpha-f(s\cap t)=K_\alpha-\{f(\alpha_0)\} & for\ \alpha\in s-\{\alpha_0\}\\ K_\alpha & for\ \alpha\in t-\{\alpha_0\}\end{cases}$$

By (6) , (17)

(19) $|K_{t-s}|=|f(t-s)|=8$

(19) means that $t-s=t-s\cap t$ is a naked 8−self−filled set of K and $t-s\cap t\subset t.$

Since $\omega_3=(t-s\cap t,\ t)$, we have

(20) $T_{\omega_3}(K)=8NSF((t-s\cap t,\ t),\ K)=K^*=(K^*_\alpha)_{\alpha\in J_1\times J_2}\in STMX(f,f_0).$

(21) $$K^*_\alpha=\begin{cases}K_\alpha & \alpha\in t-s\cap t\\ K_\alpha-K_{t-s\cap t} & \alpha\in t-(t-s\cap t)=s\cap t\\ K_\alpha & \alpha\in J_1\times J_2-t\end{cases}$$

Since $s\cap t=\{(i_0,j_0)\}=\{\alpha_0\}$, by(17), (20), (21) we have

(22) $f(\alpha_0)\in K^*_{\alpha_0}=K_{\alpha_0}-K_{t-s\cap t}=K_{\alpha_0}-K_{t-s}\subset J_3-K_{t-s}=f(s\cap t)=\{f(\alpha_0)\}$.

By (22) we have

(23) $K^*_{\alpha_0}=K_{\alpha_0}-K_{t-s\cap t}=\{f(\alpha_0)\}$.

By (21), (23) we have

(24) $$K^*_\alpha=\begin{cases}K_\alpha & \alpha\in t-s\cap t=t-\{\alpha_0\}\\ \{f(\alpha_0)\} & \alpha\in t-(t-s\cap t)=s\cap t=\{\alpha_0\}\\ K_\alpha & \alpha\in J_1\times J_2-t\end{cases}$$

Since $K^*_{s\cap t}=K^*_{\alpha_0}$, by (24) we have

(25) $|K^*_{s\cap t}|=|s\cap t|=1.$

(25) means that $s\cap t$ is a naked 1−self−filled set of K^* and $s\cap t\subset s.$

Since $\omega_4=(s\cap t,\, s)$, we have

(26) $T_{\omega_4}(K^*)=1NSF((s\cap t,s),K^*)=K^{**}=(K^{**}_\alpha)_{\alpha\in J_1\times J_2}\in STMX(f,f_0)$,

(27) $K^{**}_\alpha=\begin{cases}K^*_\alpha & for\ \alpha\in s\cap t\\ K^*_\alpha-K^*_{s\cap t} & for\ \alpha\in s-s\cap t\\ K^*_\alpha & for\ \alpha\in J_1\times J_2-s\end{cases}$

By (24) and (27) we have

(28) $K^{**}_\alpha=\begin{cases}\{f(\alpha_0)\} & for\ \alpha=\alpha_0\\ K_\alpha-\{f(\alpha_0)\} & for\ \alpha\in s-\{\alpha_0\}\\ K_\alpha & for\ \alpha\in J_1\times J_2-s\end{cases}$

Thus by (18) and (28) we have

(29) $K^{**}\leqq K'$.

By (29) we have that $(T_{\omega_2}\circ T_{\omega_1})(K)\cap(T_{\omega_4}\circ T_{\omega_3})(K)\leqq(T_{\omega_4}\circ T_{\omega_3})(K)=T_{\omega_4}(T_{\omega_3}(K))=$
$T_{\omega_4}(K^*)=K^{**}\leqq K'=T_\omega(K)$

This means that

(30) $(T_{\omega_2}\circ T_{\omega_1}\cap T_{\omega_4}\circ T_{\omega_3})(K)\leqq T_\omega(K)$.

Hence, in the case 2, Claim 2 does hold.

We consider case 3. By the similar way as case 2 we can show Claim 2 in this case 3. We give a brief sketch and key points.

In this case, by (1) and Claim 1 we have

(31) $K'_\alpha=\begin{cases}K_\alpha & for\ \alpha\in(J_1\times J_2-s\cup t)\cup\{\alpha_0\}\\ K_\alpha-f(s\cap t)=K_\alpha-\{f(\alpha_0)\} & for\ \alpha\in t-\{\alpha_0\}\\ K_\alpha & for\ \alpha\in s-\{\alpha_0\}\end{cases}$

Since $J_3-K_{s-t}\neq\phi$, by the similar way as defining K^* and K^{**}, we can put $K^\#, K^{\#\#}\in STMX(f,f_0)$ as follows:

(32) $T_{\omega_1}(K)=8NSF((s-s\cap t,\, s),\, K)=K^\#=(K^\#_\alpha)_{\alpha\in J_1\times J_2}\in STMX(f,f_0)$.

(33) $K^\#_\alpha=\begin{cases}K_\alpha & \alpha\in s-s\cap t=s-\{\alpha_0\}\\ \{f(\alpha_0)\} & \alpha\in s-(s-s\cap t)=s\cap t=\{\alpha_0\}\\ K_\alpha & \alpha\in J_1\times J_2-s\end{cases}$

(34) $T_{\omega_2}(K^\#)=1NSF((s\cap t,t),K^\#)=K^{\#\#}=(K^{\#\#}_\alpha)_{\alpha\in J_1\times J_2}\in STMX(f,f_0)$,

(35) $K^{\#\#}_\alpha=\begin{cases}\{f(\alpha_0)\} & for\ \alpha=\alpha_0\\ K_\alpha-\{f(\alpha_0)\} & for\ \alpha\in t-\{\alpha_0\}\\ K_\alpha & for\ \alpha\in J_1\times J_2-t\end{cases}$

By (31) and (35) we have

(36) $K^{\#\#}\leqq K'$.

By (36) we have that $(T_{\omega_2}\circ T_{\omega_1})(\boldsymbol{K})\cap(T_{\omega_4}\circ T_{\omega_3})(\boldsymbol{K})\leqq(T_{\omega_2}\circ T_{\omega_1})(\boldsymbol{K})=T_{\omega_2}(T_{\omega_1}(\boldsymbol{K}))=$
$T_{\omega_2}(\boldsymbol{K}^{\#})=\boldsymbol{K}^{\#\#}\leqq\boldsymbol{K}'=T_{\omega}(\boldsymbol{K})$

This means that

(37) $(T_{\omega_2}\circ T_{\omega_1}\cap T_{\omega_4}\circ T_{\omega_3})(\boldsymbol{K})\leqq T_{\omega}(\boldsymbol{K})$.

Hence, in the case 3, Claim 2 does hold.

We consider case 4. In this case, by (1) and Claim 1 we have

(38) $K'_{\alpha}=\begin{cases}K_{\alpha} & for\ \alpha\in(J_1\times J_2-s\cup t)\cup(s\cap t)\\ K_{\alpha}\cap K_{t-s}=K_{\alpha}-\{f(\alpha_0)\} & for\ \alpha\in s-s\cap t\\ K_{\alpha}\cap K_{s-t}=K_{\alpha}-\{f(\alpha_0)\} & for\ \alpha\in t-s\cap t\end{cases}$

Since $J_3-K_{t-s}\fallingdotseq\phi$ and $J_3-K_{s-t}\fallingdotseq\phi$, we have $\boldsymbol{K}^{*}$, $\boldsymbol{K}^{**}$ as in case 2 and $\boldsymbol{K}^{\#}$, $\boldsymbol{K}^{\#\#}$ as in case 3. We put $\boldsymbol{K}^{**}\cap\boldsymbol{K}^{\#\#}=\boldsymbol{K}^{!}=(K^{!}_{\alpha})_{\alpha\in J_1\times J_2}\in\boldsymbol{STMX}(f,f_0)$. By (28) and (35) we

have

(39) $K^{!}_{\alpha}=\begin{cases}\{f(\alpha_0)\} & for\ \alpha=\alpha_0\\ K_{\alpha}-\{f(\alpha_0)\} & for\ \alpha\in s\cup t-\{\alpha_0\}\\ K_{\alpha} & for\ \alpha\in J_1\times J_2-s\cup t\end{cases}$

By (38) and (39) we have

(40) $\boldsymbol{K}^{!}\leqq\boldsymbol{K}'$.

By (40) we have that $(T_{\omega_2}\circ T_{\omega_1})(\boldsymbol{K})\cap(T_{\omega_4}\circ T_{\omega_3})(\boldsymbol{K})=T_{\omega_2}(T_{\omega_1}(\boldsymbol{K}))\cap T_{\omega_4}(T_{\omega_1}(\boldsymbol{K}))=$
$T_{\omega_2}(\boldsymbol{K}^{\#})\cap T_{\omega_4}(\boldsymbol{K}^{*})=\boldsymbol{K}^{\#\#}\cap\boldsymbol{K}^{**}=\boldsymbol{K}^{!}\leqq\boldsymbol{K}'=T_{\omega}(\boldsymbol{K})$. In this case we have Claim 2.

Thus we have Claim 2 and hence we have (a). We complete the proof of Proposition 44.

References

[1] T.Kitamoto and T.Watanabe, Matematics and Sudoku I, Bulletin of the Faculty of Education, Yamaguchi University, pp.193−201, vol.64, PT2, 2014.
[2] T.Kitamoto and T.Watanabe, Matematics and Sudoku II, Bulletin of the Faculty of Education, Yamaguchi University, pp.202−208, vol.64, PT2, 2014.
[3] T.Kitamoto and T.Watanabe, Matematics and Sudoku III, to appear in Bulletin of the Faculty of Education, Yamaguchi University.