Mathematics and Sudoku Ⅲ

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We discuss on the worldwide famous Sudoku by using mathematical approach. This paper is the third paper in our series, so we use the same notations and terminologies in [1] and [2] without any descriptions.

6. Stability in sudoku transformations.

Let $\boldsymbol{K}=(K_\alpha)_{\alpha\in J_1\times J_2}$ and $\boldsymbol{L}=(L)_{\alpha\in J_1\times J_2}$ be sudoku matices assosiated with (f,f_0), i.e., $\boldsymbol{K}, \boldsymbol{L}\in \boldsymbol{STMX}(f,f_0)$. We say that $\boldsymbol{L}$ is smaller than $\boldsymbol{K}$, in notatin $\boldsymbol{L}\leqq\boldsymbol{K}$, provided that $L_\alpha\subset K_\alpha$ for each $\alpha\in J_1\times J_2$. Sometimes $\boldsymbol{L}\leqq\boldsymbol{K}$ is denoted by $\boldsymbol{L}\subset\boldsymbol{K}$.

Let $\boldsymbol{K}=(K_\alpha)_{\alpha\in J_1\times J_2}$ and $\boldsymbol{L}=(L)_{\alpha\in J_1\times J_2}$ be sudoku matices assosiated with f_0, i.e., $\boldsymbol{K}, \boldsymbol{L}\in \boldsymbol{STMX}(f_0)=\cap\{\boldsymbol{STMX}(f,f_0): f\in \boldsymbol{SOL}(f_0)\}$. We say that $\boldsymbol{L}$ is smaller than $\boldsymbol{K}$, in notatin $\boldsymbol{L}\leqq\boldsymbol{K}$, provided that $L_\alpha\subset K_\alpha$ for each $\alpha\in J_1\times J_2$. Sometimes $\boldsymbol{L}\leqq\boldsymbol{K}$ is denoted by $\boldsymbol{L}\subset\boldsymbol{K}$.

We say that a map $T:\boldsymbol{STMX}(f,f_0)\to\boldsymbol{STMX}(f,f_0)$ is a sudoku transformation associated with (f,f_0) provided that it satisfies the following conditions:

(i) $T(\boldsymbol{K})\leqq\boldsymbol{K}$ for each $\boldsymbol{K}\in\boldsymbol{STMX}(f,f_0)$,

(ii) $T(\boldsymbol{K})\geqq T(\boldsymbol{L})$ for each $\boldsymbol{K}, \boldsymbol{L}\in\boldsymbol{STMX}(f,f_0)$ with $\boldsymbol{K}\geqq\boldsymbol{L}$.

We put $\boldsymbol{STRF}(f,f_0)=\{T: T$ is a sudoku transformation associated with $(f,f_0)\}$ and $\boldsymbol{STRF}(f_0)=\cap\{\boldsymbol{STRF}(f,f_0): f\in\boldsymbol{SOL}(f_0)\}$. Each element $T\in\boldsymbol{STRF}(f_0)$ is called as a sudoku transformation associated with f_0 and T is denoted by $T:\boldsymbol{STMX}(f_0)\to\boldsymbol{STMX}(f_0)$.

Let $T, S:\boldsymbol{STMX}(f,f_0)\to\boldsymbol{STMX}(f,f_0)$ be sudoku transformations associated with (f,f_0). We say that T is smaller than S, in notation $T\leqq S$, provided that

(iii) $T(\boldsymbol{K})\leqq S(\boldsymbol{K})$ for each $\boldsymbol{K}\in\boldsymbol{STMX}(f,f_0)$.

Let $T, S:\boldsymbol{STMX}(f_0)\to\boldsymbol{STMX}(f_0)$ be sudoku transformations associated with f_0. We say that T is smaller than S, in notation $T\leqq S$, provided that

(iv) $T(\boldsymbol{K})\leqq S(\boldsymbol{K})$ for each $\boldsymbol{K}\in\boldsymbol{STMX}(f,f_0)$ and for each $f\in\boldsymbol{SOL}(f_0)$.

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Let $T_1, T_2 : \boldsymbol{STMX}(f,f_0) \to \boldsymbol{STMX}(f,f_0)$ be maps. We define a map $T_1 \cap T_2 : \boldsymbol{STMX}(f,f_0) \to \boldsymbol{STMX}(f,f_0)$ as follows:

(v) $(T_1 \cap T_2)(\boldsymbol{K}) = T_1(\boldsymbol{K}) \cap T_2(\boldsymbol{K})$ for each $\boldsymbol{K} \in \boldsymbol{STMX}(f,f_0)$.

Let $T_1, T_2 : \boldsymbol{STMX}(f_0) \to \boldsymbol{STMX}(f_0)$ be maps. We define a map $T_1 \cap T_2 : \boldsymbol{STMX}(f_0) \to \boldsymbol{STMX}(f_0)$ as follows:

(vi) $(T_1 \cap T_2)(\boldsymbol{K}) = T_1(\boldsymbol{K}) \cap T_2(\boldsymbol{K})$ for each $\boldsymbol{K} \in \boldsymbol{STMX}(f,f_0)$ and each $f \in \boldsymbol{SOL}(f_0)$.

We say that the map $T_1 \cap T_2$ is the intersection map of $\{T_1, T_2\}$.

Proposition 19. Let $T_1, T_2, T_3 : \boldsymbol{STMX}(f,f_0) \to \boldsymbol{STMX}(f,f_0)$ be sudoku transformations. Then we have the followings:

(a) The identity map $1 : \boldsymbol{STMX}(f,f_0) \to \boldsymbol{STMX}(f,f_0)$ is a sudoku transformation.

(b) The composition map $T_2 \circ T_1 : \boldsymbol{STMX}(f,f_0) \to \boldsymbol{STMX}(f,f_0)$ is a sudoku transformation such that $1 \circ T_1 = T_1$, $T_1 \circ 1 = T_1$ and $(T_3 \circ T_2) \circ T_1 = T_3 \circ (T_2 \circ T_1)$.

(c) The intersection map $T_1 \cap T_2 : \boldsymbol{STMX}(f,f_0) \to \boldsymbol{STMX}(f,f_0)$ is a sudoku transformation such that $T_1 \cap T_2 \leqq T_1$ and $T_1 \cap T_2 \leqq T_2$.

Proof. Oviously we have (a) by the definition.

We show (b). Take any $\boldsymbol{K} \in \boldsymbol{STMX}(f,f_0)$. Since T_1 is a sudoku transformation, by (i) we have

(1) $T_1(\boldsymbol{K}) \leqq \boldsymbol{K}$.

Since T_2 is a sudoku transformation, by (ii) and (1) we have

(2) $T_2(T_1(\boldsymbol{K})) \leqq T_2(\boldsymbol{K})$.

Since T_2 is a sudoku transformation, by (i) we have

(3) $T_2(\boldsymbol{K}) \leqq \boldsymbol{K}$.

By (1),(2),(3) we have $(T_2 \circ T_1)(\boldsymbol{K}) = T_2(T_1(\boldsymbol{K})) \leqq T_2(\boldsymbol{K}) \leqq \boldsymbol{K}$, i.e.,

(4) $(T_2 \circ T_1)(\boldsymbol{K}) \leqq \boldsymbol{K}$.

Thus (4) means that $T_2 \circ T_1$ has the property (i).

Next, we take each $\boldsymbol{K}, \boldsymbol{L} \in \boldsymbol{STMX}(f,f_0)$ with $\boldsymbol{K} \geqq \boldsymbol{L}$. Since T_1 is a sudoku transformation, by (ii) we have

(5) $T_1(\boldsymbol{K}) \geqq T_1(\boldsymbol{L})$.

Since T_2 is a sudoku transformation, by (ii) and (5) we have

(6) $T_2(T_1(\boldsymbol{K})) \geqq T_2(T_1(\boldsymbol{L}))$.

Since $T_2(T_1(\boldsymbol{K})) = (T_2 \circ T_1)(\boldsymbol{K})$ and $T_2(T_1(\boldsymbol{L})) = (T_2 \circ T_1)(\boldsymbol{L})$, by (6) we have

(7) $(T_2 \circ T_1)(\boldsymbol{K}) \geqq (T_2 \circ T_1)(\boldsymbol{L})$.

Thus, (7) means that $T_2 \circ T_1$ has the property (ii). Therefore, $T_2 \circ T_1$ is a sudoku transformation.

We can easily show that $1\circ T_1=T_1$, $T_1\circ 1=T_1$ and $(T_3\circ T_2)\circ T_1=T_3\circ(T_2\circ T_1)$.

We show (c). Take any $\boldsymbol{K}\in \boldsymbol{STMX}(f,f_0)$. Since T_1, T_2 are sudoku transformations, we have that $T_1(\boldsymbol{K})$, $T_2(\boldsymbol{K})\in \boldsymbol{STMX}(f,f_0)$. By Proposition 1 we have $T_1(\boldsymbol{K})\cap T_2(\boldsymbol{K})\in \boldsymbol{STMX}(f,f_0)$. This means that the intersection map $T_1\cap T_2:\boldsymbol{STMX}(f,f_0)\to \boldsymbol{STMX}(f,f_0)$ is well−defined.

By (1), (3) we have

(8) $(T_1\cap T_2)(\boldsymbol{K})=T_1(\boldsymbol{K})\cap T_2(\boldsymbol{K})\leqq \boldsymbol{K}$.

Thus (8) means that $T_1\cap T_2$ has the property (i).

Take any $\boldsymbol{K}, \boldsymbol{L}\in \boldsymbol{STMX}(f,f_0)$ with $\boldsymbol{K}\geqq \boldsymbol{L}$. Since T_2 is a sudoku transformation, by (ii) we have

(9) $T_2(\boldsymbol{K})\geqq T_2(\boldsymbol{L})$.

By (5) and (9) we have $T_1(\boldsymbol{K})\cap T_2(\boldsymbol{K})\supset T_1(\boldsymbol{L})\cap T_2(\boldsymbol{L})$, i.e.,

(10) $(T_1\cap T_2)(\boldsymbol{K})=T_1(\boldsymbol{K})\cap T_2(\boldsymbol{K})\supset T_1(\boldsymbol{L})\cap T_2(\boldsymbol{L})=(T_1\cap T_2)(\boldsymbol{L})$.

Thus, (10) menas that $T_1\cap T_2$ has the property (ii). Hence $T_1\cap T_2$ is a sudoku transformation.

By definitions we have

(11) $(T_1\cap T_2)(\boldsymbol{K})=T_1(\boldsymbol{K})\cap T_2(\boldsymbol{K})\leqq T_1(\boldsymbol{K})$, and

(12) $(T_1\cap T_2)(\boldsymbol{K})=T_1(\boldsymbol{K})\cap T_2(\boldsymbol{K})\leqq T_2(\boldsymbol{K})$.

Thus, by (11) and (12) we have

(13) $T_1\cap T_2\leqq T_1$ and $T_1\cap T_2\leqq T_2$.

Thus, by (13), $T_1\cap T_2$ has the required properties. Hence, we have Proposition 19.

Proposition 20. Let $T_1, T_2, T_3:\boldsymbol{STMX}(f_0)\to \boldsymbol{STMX}(f_0)$ be sudoku transformations. Then we have the followings:

(a) The identity map $1:\boldsymbol{STMX}(f_0)\to \boldsymbol{STMX}(f_0)$ is a sudoku transformation.

(b) The composition map $T_2\circ T_1:\boldsymbol{STMX}(f_0)\to \boldsymbol{STMX}(f)$ is a sudoku transformation such that $1\circ T_1=T_1$, $T_1\circ 1=T_1$ and $(T_3\circ T_2)\circ T_1=T_3\circ(T_2\circ T_1)$.

(c) The intersection map $T_1\cap T_2:\boldsymbol{STMX}(f_0)\to \boldsymbol{STMX}(f_0)$ is a sudoku transformation such that $T_1\cap T_2\leqq T_1$ and $T_1\cap T_2\leqq T_2$.

We can easily show Proposition 20 by Proposition 19 and definitions.

Let $\boldsymbol{TOOL}\subset \boldsymbol{STRF}(f,f_0)$ be a subset of $\boldsymbol{STRF}(f,f_0)$. We say $\boldsymbol{L}\in \boldsymbol{STMX}(f,f_0)$ is $\boldsymbol{TOOL}$−stable provided that it satisfies the following condition:

(ST) $T(\boldsymbol{L})=\boldsymbol{L}$ for each $T\in \boldsymbol{TOOL}$.

Similarly, let $\boldsymbol{TOOL}\subset \boldsymbol{STRF}(f_0)$ be a subset of $\boldsymbol{STRF}(f_0)$. We say that

$\boldsymbol{L} \in \boldsymbol{STMX}(f_0)$ is $\boldsymbol{TOOL}-$stable provided that it satisfies the following condition:
(ST) $T(\boldsymbol{L}) = \boldsymbol{L}$ for each $T \in \boldsymbol{TOOL} \subset \boldsymbol{STRF}(f, f_0)$ and for each $f \in \boldsymbol{SOL}(f_0)$.

Proposition 21. Let $\boldsymbol{TOOL} \subset \boldsymbol{STRF}(f, f_0)$ be a subset of $\boldsymbol{STRF}(f, f_0)$. Then we have a sudoku transformation $STBL^{TOOL}: \boldsymbol{STMX}(f, f_0) \to \boldsymbol{STMX}(f, f_0)$ with the following properties:
(a) $STBL^{TOOL}(\boldsymbol{K})$ is $\boldsymbol{TOOL}-$stable for each $\boldsymbol{K} \in \boldsymbol{STMX}(f, f_0)$.
(b) If $\boldsymbol{L} \in \boldsymbol{STMX}(f, f_0)$ is $\boldsymbol{TOOL}-$stable, then $STBL^{TOOL}(\boldsymbol{L}) = \boldsymbol{L}$.

Proposition 22. Let $\boldsymbol{TOOL} \subset \boldsymbol{STRF}(f_0)$ be a subset of $\boldsymbol{STRF}(f_0)$. Then we have a sudoku transformation $STBL^{TOOL}: \boldsymbol{STMX}(f_0) \to \boldsymbol{STMX}(f_0)$ with the following properties:
(a) $STBL^{TOOL}(\boldsymbol{K})$ is $\boldsymbol{TOOL}-$stable for each $\boldsymbol{K} \in \boldsymbol{STMX}(f, f_0)$ and for each $f \in \boldsymbol{SOL}(f_0)$.
(b) If $\boldsymbol{L} \in \boldsymbol{STMX}(f_0)$ is $\boldsymbol{TOOL}-$stable, then $STBL^{TOOL}(\boldsymbol{L}) = \boldsymbol{L}$.

When $\boldsymbol{TOOL} = \phi$, we can take the identity map $1: \boldsymbol{SMTX}(f, f_0) \to \boldsymbol{SMTX}(f, f_0)$ and $1: \boldsymbol{SMTX}(f_0) \to \boldsymbol{SMTX}(f_0)$ as $STBL^{\phi}: \boldsymbol{SMTX}(f, f_0) \to \boldsymbol{SMTX}(f, f_0)$ and $STBL^{\phi}: \boldsymbol{SMTX}(f_0) \to \boldsymbol{SMTX}(f_0)$, respectively. Therefore, in the follwing discussin we can assume that $\boldsymbol{TOOL} \neq \phi$.
For our proofs of Proposition 21 and Proposition 22 we need many steps .

Proposition 23. Let V be a finite set. If $V \supset V_1 \supset V_2 \supset \ldots \supset V_i \supset V_{i+1} \supset \ldots$ is a decreasing sequence of sets, then there exists an n_0 such that $V_n = V_{n_0}$ for each $n \geqq n_0$ and hence $V_\infty = \cap_{i=1}^{\infty} V_i = V_{n_0}$.
Proof. If $V = \phi$, we can choose $n_0 = 1$. So in the following discussion we assume that $V \neq \phi$.
We assume that the conclusion does not hold. Thus there exists an increasing sequence of integers such that
(1) $n_1 < n_2 < \ldots < n_i < n_{i+1} < \ldots$.
(2) $V_{n_i} \supsetneq V_{n_{i+1}}$ for each $i \geqq 1$.
By (2) we can take a $p_{n_i} \in V_{n_i} - V_{n_{i+1}}$ for each $i \geqq 1$, and put $P = \{p_{n_i} : i \geqq 1\}$. Thus we have that
(3) $P \subset V$ and
(4) P is infinite.

We show (3). Take any i. since $p_{n_i} \in V_{n_i} \subset V$, then $p_{n_i} \in V$. Thus we have (3).

We show (4). We assume that P is finite. Then there exist $i,j \geqq 1$ such that

(5) $j > i$ and $p_{n_i} = p_{n_j}$.

By (5) and our construction we have that $p_{n_i} \in V_{n_i} - V_{n_i+1}$, $p_{n_j} \in V_{n_j} - V_{n_j+1}$, that is,

(6) $p_{n_i} \not\in V_{n_i+1}$ and

(7) $p_{n_j} \in V_{n_j}$.

By (1) and $j > i$, $n_j \geqq n_i + 1$ and then

(8) $V_{n_i+1} \supset V_{n_j}$.

By (6) and (8)

(9) $p_{n_i} \not\in V_{n_j}$.

Since we have (5), (7) and (9) make a contradiction. Hence, (4) is true.

By (3), we have that $|P| \leqq |V| < \infty$, that is, P is finite. This contadicts to (4). Hence, we have Proposition 23.

Proposition 24. Let K and K_i be sudoku matrices associated with (f, f_0), $i = 1,2,....$ If $K \geqq K_1 \geqq K_2 \geqq ... \geqq K_i \geqq K_{i+1} \geqq ...$ is a decreasing sequence of sudoku matices, then there exists an n_0 such that $K_n = K_{n_0}$ for each $n \geqq n_0$ and hence $K_\infty = \cap_{i=1}^{\infty} K_i = K_{n_0}$.

Proof. Let $K = (K_\alpha)_{\alpha \in J_1 \times J_2}$ and $K_k = (K_{k,\alpha})_{\alpha \in J_1 \times J_2} \in SMTX(f, f_0)$. By the assumption we have

(1) $K_\alpha \supset K_{1,\alpha} \supset ... \supset K_{k,\alpha} \supset K_{k+1,\alpha} \supset ....$ for each $\alpha \in J_1 \times J_2$.

For each $\alpha \in J_1 \times J_2$, since $|K_\alpha| \leqq 9$, by (1) and Proposition 23 there exists an n_α such that

(2) $K_{n,\alpha} = K_{n_\alpha,\alpha}$ for each $n \geqq n_\alpha$.

We put $n_0 = max\{n_\alpha : \alpha \in J_1 \times J_2\}$. By (2) we have that

(3) $K_{n,\alpha} = K_{n_0,\alpha}$ for each $n \geqq n_0$ and each $\alpha \in J_1 \times J_2$.

(3) means that $K_n = K_{n_0}$ for each $n \geqq n_0$. Hence we have Proposition 24.

Proposition 25. Let K and K_i be sudoku matrices associated with f_0. If $K \geqq K_1 \geqq K_2 \geqq ... \geqq K_i \geqq K_{i+1} \geqq ...$ is a decreasing sequence of sudoku matices, then there exists an n_0 such that $K_n = K_{n_0}$ for each $n \geqq n_0$ and hence $K_\infty = \cap_{i=1}^{\infty} K_i = K_{n_0}$.

Proof. Since $STMX(f_0) = \cap\{STMX(f, f_0) : f \in SOL(f_0)\}$ and the assumption, for each $f \in SOL(f_0)$ we have that

(1) $\boldsymbol{K} \geqq \boldsymbol{K}_1 \geqq \boldsymbol{K}_2 \geqq \ldots \geqq \boldsymbol{K}_i \geqq \boldsymbol{K}_{i+1} \geqq \ldots$ is a decreasing sequence in $\boldsymbol{STMX}(f, f_0)$.

By (1) and Proposition 24, there exists an $n_0(f)$ such that

(2) $\boldsymbol{K}_n = \boldsymbol{K}_{n_0(f))}$ in $\boldsymbol{STMX}(f, f_0)$ for each $n \geqq n_0(f)$.

Since $\boldsymbol{SOL}(f_0)$ is finite, we can put $n_0 = max\{n_0(f): f \in \boldsymbol{SOL}(f_0)\}$. By (2) we have

(3) $\boldsymbol{K}_n = \boldsymbol{K}_{n_0}$ in $\boldsymbol{STMX}(f, f_0)$ for each $n \geqq n_0$ and each $f \in \boldsymbol{SOL}(f_0)$.

Thus we have Proposition 25.

Proposition 26. Let $\boldsymbol{TOOL}$ be a non−empty subset of $\boldsymbol{STRF}(f, f_0)$. Then there exists a finite sequence $\boldsymbol{T} = (T_1, T_2, \ldots, T_{n_0})$ in $\boldsymbol{TOOL}$ with the following property:

(a) $(T_{n_0} \circ T_{n_0-1} \circ \ldots \circ T_1)(\boldsymbol{K})$ is $\boldsymbol{TOOL}$−stable for each $\boldsymbol{K} \in \boldsymbol{STMX}(f, f_0)$.

Proposition 27. Let $\boldsymbol{TOOL}$ be a non−empty subset of $\boldsymbol{STRF}(f_0)$. Then there exists a finite sequence $\boldsymbol{T} = (T_1, T_2, \ldots, T_{m_0})$ in $\boldsymbol{TOOL}$ with the following property:

(a) $(T_{n_0} \circ T_{n_0-1} \circ \ldots \circ T_1)(\boldsymbol{K})$ is $\boldsymbol{TOOL}$−stable for each $\boldsymbol{K} \in \boldsymbol{STMX}(f, f_0)$ and for each $f \in \boldsymbol{SOL}(f_0)$.

(b) $(T_{m_0} \circ T_{m_0-1} \circ \ldots \circ T_1)(\boldsymbol{K})$ is $\boldsymbol{TOOL}$−stable for each $\boldsymbol{K} \in \boldsymbol{STMX}(f_0)$.

To prove Proposition 26 and Proposition 27 we need some propositions.

Let $\boldsymbol{TOOL}$ be a non−empty finite set. We take an infinite sequence $\boldsymbol{T} = (T_1, T_2, \ldots, T_i, T_{i+1}, \ldots)$ in $\boldsymbol{TOOL}$, that is, each $T_i \in \boldsymbol{TOOL}$. We say $\boldsymbol{T} = (T_1, T_2, \ldots, T_i, T_{i+1}, \ldots)$ is full in $\boldsymbol{TOOL}$ provided that it satisfies the following full condition:

(FUL) For each n, $n \geqq 1$, $\{T_j : j \geqq n\} = \boldsymbol{TOOL}$.

Proposition 28. Let $\boldsymbol{TOOL}$ be a non−empty set. Then there exists an infinite sequence $\boldsymbol{T} = (T_1, T_2, \ldots, T_i, \ldots)$ in $\boldsymbol{TOOL}$, which is full in $\boldsymbol{TOOL}$.

Proof. Since $\boldsymbol{TOOL}$ is finite. We put

(1) $\boldsymbol{TOOL} = \{S_1, S_2, \ldots, S_m\}$, $m \geqq 1$.

We make an infinite sequence $\boldsymbol{T} = (T_1, T_2, \ldots, T_i, \ldots)$ as follows:

(2) $T_i = S_k$ which $i = um + k$, $0 < k \leqq m$.

Take any integer $s > 0$. Since $s \leqq ms < ms+1 < ms+2 < \ldots. < ms+m$, we have

(3) $\{S_1, S, \ldots, S_m\} = \{T_{ms+1}, T_{ms+2}, \ldots, T_{ms+m}\} \subset \{T_j : j \geqq s\} \subset \{T_j : j = 1, 2, \ldots\} = \{S_1, S_2, \ldots, S_m\}$

By (1) and (3) we have that

(4) $\{T_j : j \geqq s\} = \{S_1, S_2, ..., S_m\} = \boldsymbol{TOOL}$.
Thus, $\boldsymbol{T}$ is full in $\boldsymbol{TOOL}$. Hence we have Proposition 28.

Proposition 29. Let $\boldsymbol{TOOL}$ be a subset of $\boldsymbol{STRF}(f,f_0)$. Let $\boldsymbol{T} = (T_1, T_2, ..., T_i, T_{i+1}, ...)$ be an infinite sequence in $\boldsymbol{TOOL}$. If $\boldsymbol{T}$ is full in $\boldsymbol{TOOL}$, then there exists an n_0 such that
(a) $(T_{n_0} \circ T_{n_0-1} \circ .... \circ T_1)(\boldsymbol{K})$ is $\boldsymbol{TOOL}-$stable for each $\boldsymbol{K} \in \boldsymbol{STMX}(f,f_0)$.

Proposition 30. Let $\boldsymbol{TOOL}$ be subset of $\boldsymbol{STRF}(f_0)$. Let $\boldsymbol{T} = (T_1, T_2, ..., T_i, T_{i+1}, ...)$ is an infinite sequence in $\boldsymbol{TOOL}$. If $\boldsymbol{T}$ is full in $\boldsymbol{TOOL}$, then there exists an m_0 such that
(a) $(T_{m_0} \circ T_{m_0-1} \circ .... \circ T_1)(\boldsymbol{K})$ is $\boldsymbol{TOOL}-$stable for each $\boldsymbol{K} \in \boldsymbol{STMX}(f,f_0)$ and for each $f \in \boldsymbol{SOL}(f_0)$ and
(b) $(T_{m_0} \circ T_{m_0-1} \circ .... \circ T_1)(\boldsymbol{K})$ is $\boldsymbol{TOOL}-$stable for each $\boldsymbol{K} \in \boldsymbol{STMX}(f_0)$

Proof of Proposition 29. Since $\boldsymbol{TOOL}$ is a subset of $\boldsymbol{STRF}(f,f_0)$, then for each k, $T_k : \boldsymbol{STMX}(f,f_0) \to \boldsymbol{STMX}(f,f_0)$ is a sudoku transformation. For each i, j with $j \geqq i \geqq 1$ we put $T_{i,j} = T_j \circ T_{j-1} \circ ... \circ T_i : \boldsymbol{STMX}(f,f_0) \to \boldsymbol{STMX}(f,f_0)$, which is the composition of sudoku transformations $T_k : \boldsymbol{STMX}(f,f_0) \to \boldsymbol{STMX}(f,f_0)$, $i \leqq k \leqq j$.
Take any $\boldsymbol{K} \in \boldsymbol{STMX}(f,f_0)$. We put $\boldsymbol{T}(\boldsymbol{K})_j = T_{1,j}(\boldsymbol{K})$ for each j, $j \geqq 1$ and thus we have a decreasing sequence of sudoku matrices as follows:
(1) $\boldsymbol{K} \supset \boldsymbol{T}(\boldsymbol{K})_1 \supset \boldsymbol{T}(\boldsymbol{K})_2 \supset .... \supset \boldsymbol{T}(\boldsymbol{K})_j \supset T_{j+1}(\boldsymbol{T}(\boldsymbol{K})_j) = \boldsymbol{T}(\boldsymbol{K})_{j+1} \supset ...$
We denote $\boldsymbol{T}(\boldsymbol{K})_\infty = \cap_{j=1}^{\infty} \boldsymbol{T}(\boldsymbol{K})_j$. By Proposition 21 there exists an integer $n(\boldsymbol{T}, \boldsymbol{TOOL}, \boldsymbol{K}, (f,f_0))$ such that
(2) $\boldsymbol{T}(\boldsymbol{K})_j = \boldsymbol{T}(\boldsymbol{K})_{n(\boldsymbol{T}, \boldsymbol{TOOL}, \boldsymbol{K}, (f,f_0))}$ for each $j \geqq n(\boldsymbol{T}, \boldsymbol{TOOL}, \boldsymbol{K}, (f,f_0))$.
By (2) we have that
(3) $\boldsymbol{T}(\boldsymbol{K})_\infty = \boldsymbol{T}(\boldsymbol{K})_{n(\boldsymbol{T}, \boldsymbol{TOOL}, \boldsymbol{K}, (f,f_0))}$.
Since $\boldsymbol{STMX}(f,f_0)$ is finite, we can put
(4) $n_0 = n(\boldsymbol{T}, \boldsymbol{TOOL}, (f,f_0)) = max\{n(\boldsymbol{T}, \boldsymbol{TOOL}, \boldsymbol{K}, (f,f_0)) : \boldsymbol{K} \in \boldsymbol{STMX}(f,f_0)\}$.
Thus by (2) and (3) we have that
(5) $\boldsymbol{T}(\boldsymbol{K})_j = \boldsymbol{T}(\boldsymbol{K})_{n_0}$ for each $j \geqq n_0$ and each $\boldsymbol{K} \in \boldsymbol{STMX}(f,f_0)$,
(6) $\boldsymbol{T}(\boldsymbol{K})_\infty = \boldsymbol{T}(\boldsymbol{K})_{n_0}$ for each $\boldsymbol{K} \in \boldsymbol{STMX}(f,f_0)$.
Since $\boldsymbol{T}$ is full in $\boldsymbol{TOOL}$, we have
(7) $\{T_i : i \geqq n_0 + 1\} = \boldsymbol{TOOL}$.
Take any $T \in \boldsymbol{TOOL}$, thus by (7) there exists an i_0 such that

(8) $T=T_{i_0}$ and $i_0 \geqq n_0+1$.

Since $T_{1,i_0}=T_{i_0}\circ T_{i_0-1}\circ\ldots\circ T_{n_0}\circ\ldots\circ T_1=T_{i_0}\circ T_{1,i_0-1}$, we have that

(9) $\quad \boldsymbol{K}\supset\ldots\supset \boldsymbol{T}(\boldsymbol{K})_{n_0}\supset\ldots.\supset \boldsymbol{T}(\boldsymbol{K})_{i_0-1}\supset T_{i_0}(\boldsymbol{T}(\boldsymbol{K})_{i_0-1})=T(\boldsymbol{K})_{i_0}$.

By (8) we have

(10) $i_0, i_0-1 \geqq n_0$

By (5) and (10) we have

(11) $\boldsymbol{T}(\boldsymbol{K})_{i_0}=\boldsymbol{T}(\boldsymbol{K})_{i_0-1}=\boldsymbol{T}(\boldsymbol{K})_{n_0}$

By (8), (9) and (10) we have

(12) $T(\boldsymbol{T}(\boldsymbol{K})_{n_0})=T_{i_0}(\boldsymbol{T}(\boldsymbol{K})_{i_0-1})=T(\boldsymbol{K})_{i_0}=\boldsymbol{T}(\boldsymbol{K})_{n_0}$.

Thus, (12) means that

(13) $\boldsymbol{T}(\boldsymbol{K})_{n_0}$ is $\boldsymbol{TOOL}-$stable for each $\boldsymbol{K}\in \boldsymbol{STMX}(f,f_0)$

Note, by (6) and (13) we have

(14) $\boldsymbol{T}(\boldsymbol{K})_{\infty}$ is $\boldsymbol{TOOL}-$stable for each $\boldsymbol{K}\in \boldsymbol{STMX}(f,f_0)$.

Hence we have Proposition 29.

Proof of Proposition 30. Since $\boldsymbol{TOOL}\subset \boldsymbol{STRF}(f_0)=\cap\{\boldsymbol{STRF}(f,f_0): f\in \boldsymbol{SOL}(f_0)\}$ and $\boldsymbol{STMX}(f_0)=\cap\{\boldsymbol{STMX}(f,f_0): f\in \boldsymbol{SOL}(f_0)\}$, then $\boldsymbol{TOOL}\subset \boldsymbol{STRF}(f,f_0)$ for each $f\in \boldsymbol{SOL}(f_0)$. Thus by Proposition 29 we have an $n(\boldsymbol{T},\boldsymbol{TOOL},(f,f_0))$ for each $f\in \boldsymbol{SOL}(f_0)$. Since $\boldsymbol{SOL}(f_0)$ is finite, we can put

(1) $m_0=m(\boldsymbol{T},\boldsymbol{TOOL},f_0)=max\{n(\boldsymbol{T},\boldsymbol{TOOL},(f,f_0)): f\in \boldsymbol{SOL}(f_0)\}$

By (13) in the proof of Proposition 29, and (1) we can easily show that

(2) $\boldsymbol{T}(\boldsymbol{K})_{m_0}$ is $\boldsymbol{TOOL}-$stable for each $\boldsymbol{K}\in \boldsymbol{STMX}(f,f_0)$ and for each $f\in \boldsymbol{SOL}(f_0)$.

Thus, by (2) we have

(3) $\boldsymbol{T}(\boldsymbol{K})_{m_0}$ is $\boldsymbol{TOOL}-$stable for each $\boldsymbol{K}\in \boldsymbol{STMX}(f_0)$.

Note, by (14) in the proof of Proposition 29 , (2),(3) we have the followings:

(4) $\boldsymbol{T}(\boldsymbol{K})_{\infty}$ is $\boldsymbol{TOOL}-$stable for each $\boldsymbol{K}\in \boldsymbol{STMX}(f,f_0)$ and for each $f\in \boldsymbol{SOL}(f_0)$.

(5) $\boldsymbol{T}(\boldsymbol{K})_{\infty}$ is $\boldsymbol{TOOL}-$stable for each $\boldsymbol{K}\in \boldsymbol{STMX}(f_0)$.

Hence we have Poposition 30.

Proofs of Proposition 26 and Proposition 27.

Since $\boldsymbol{STRF}(f,f_0)$ is a finite set, then $\boldsymbol{TOOL}$ is also finite. Hence, Proposition 26 comes from Proposition 28 and Proposition 29.

Since $\boldsymbol{STRF}(f_0)$ is a finite set, then $\boldsymbol{TOOL}$ is also finite. Hence, Proposition 27 comes from Proposition 28 and Proposition 30. Therefore we complete the proofs of Propositions 26 and 27.

Proposition 31. Let $\boldsymbol{TOOL}$ be a non−empty subset of $\boldsymbol{STRF}(f,f_0)$. Let $\boldsymbol{T} = (T_1,T_2,...,T_{n_0})$ and $\boldsymbol{T'} = (T'_1,T'_2,...,T'_{n_1})$ be finite sequences in $\boldsymbol{TOOL}$. If they satisfy the followings:

(a) $(T_{n_0} \circ T_{n_0-1} \circ .... \circ T_1)(\boldsymbol{K})$ is $\boldsymbol{TOOL}$−stable for each $\boldsymbol{K} \in \boldsymbol{STMX}(f,f_0)$ and

(b) $(T'_{n_1} \circ T'_{n_1-1} \circ .... \circ T'_1)(\boldsymbol{K})$ is $\boldsymbol{TOOL}$−stable for each $\boldsymbol{K} \in \boldsymbol{STMX}(f,f_0)$,

then we hve that

(c) $T_{n_0} \circ T_{n_0-1} \circ ... \circ T_1 = T'_{n_1} \circ T'_{n_1-1} \circ ... \circ T'_1$.

Proposition 32. Let $\boldsymbol{TOOL}$ be a non−empty subset of $\boldsymbol{STRF}(f_0)$. Let $\boldsymbol{T} = (T_1,T_2,...,T_{n_0})$ and $\boldsymbol{T'} = (T'_1,T'_2,...,T'_{n_1})$ be finite sequences in $\boldsymbol{TOOL}$. If they satisfy the followings:

(a) $(T_{n_0} \circ T_{n_0-1} \circ .... \circ T_1)(\boldsymbol{K})$ is $\boldsymbol{TOOL}$−stable for each $\boldsymbol{K} \in \boldsymbol{STMX}(f_0)$ and

(b) $(T'_{n_1} \circ T'_{n_1-1} \circ .... \circ T'_1)(\boldsymbol{K})$ is $\boldsymbol{TOOL}$−stable for each $\boldsymbol{K} \in \boldsymbol{STMX}(f_0)$,

then we hve that

(c) $T_{n_0} \circ T_{n_0-1} \circ ... \circ T_1 = T'_{n_1} \circ T'_{n_1-1} \circ ... \circ T'_1$.

To prove Proposition 31 and Proposition 32 we need some discussions.

Proposition 33. Let $\boldsymbol{TOOL}$ and $\boldsymbol{TOOL'}$ be a non−empty subsets of $\boldsymbol{STRF}(f,f_0)$. Let $\boldsymbol{T} = (T_1, T_2,...,T_i, ...)$ and $\boldsymbol{T'} = (T'_1, T'_2,...,T'_i, ...)$ be infinite sequences in $\boldsymbol{TOOL}$ and in $\boldsymbol{TOOL'}$, respectively. If $T'_i \leqq T_i$ for each i, then we have that

(a) $\boldsymbol{T'}(\boldsymbol{K})_\infty \leqq \boldsymbol{T}(\boldsymbol{K})_\infty$ for each $\boldsymbol{K} \in \boldsymbol{STMX}(f,f_0)$.

Proposition 34. Let $\boldsymbol{TOOL}$ and $\boldsymbol{TOOL'}$ be a non−empty subsets of $\boldsymbol{STRF}(f_0)$. Let $\boldsymbol{T} = (T_1, T_2,...,T_i, ...)$ and $\boldsymbol{T'} = (T'_1, T'_2,...,T'_i, ...)$ be infinite sequences in $\boldsymbol{TOOL}$ and in $\boldsymbol{TOOL'}$, respectively. If $T'_i \leqq T_i$ for each i, then we have that

(a) $\boldsymbol{T'}(\boldsymbol{K})_\infty \leqq \boldsymbol{T}(\boldsymbol{K})_\infty$ for each $\boldsymbol{K} \in \boldsymbol{STMX}(f_0)$.

Proofs of Propositions 33 and 34.

We show Proposition 33. We use the same notations as in the proof of Proposition 29. Take any $\boldsymbol{K} \in \boldsymbol{STMX}(f,f_0)$. We show the following commutative diagram (D):

$$
(D) \quad
\begin{array}{ccccccccccc}
\boldsymbol{K} & \supset & \boldsymbol{T}(\boldsymbol{K})_1 & \supset & \boldsymbol{T}(\boldsymbol{K})_2 & \supset .... \supset & \boldsymbol{T}(\boldsymbol{K})_i & \supset & \boldsymbol{T}(\boldsymbol{K})_{i+1} & \supset ..... \\
 & & \cup & & \cup & & \cup & & \cup & \\
\boldsymbol{K} & \supset & \boldsymbol{T'}(\boldsymbol{K})_1 & \supset & \boldsymbol{T'}(\boldsymbol{K})_2 & \supset .... \supset & \boldsymbol{T'}(\boldsymbol{K})_i & \supset & \boldsymbol{T'}(\boldsymbol{K})_{i+1} & \supset ..... .
\end{array}
$$

To prove (D), we consider the following commutative diagram (Di) for each i,

$$\begin{array}{ccccccccc} & K & \supset & T(K)_1 & \supset & T(K)_2 & \supset \cdots \supset & T(K)_i \\ (Di) & & & \cup & & \cup & & \cup \\ & K & \supset & T'(K)_1 & \supset & T'(K)_2 & \supset \cdots \supset & T'(K)_i \end{array}.$$

First, we show $(D1)$. Since T_1, T_1' are sudoku transformations, we have

(1) $T(K)_1 = T_1(K) \subset K$ and $T'(K)_1 = T_1'(K) \subset K$.

Since $T_1' \leqq T_1$, we have

(2) $T_1'(K) \subset T_1(K)$.

By (1) and (2) we have $(D1)$ as follows:

$$\begin{array}{cccc} & K & \supset & T(K)_1 \\ (D1) & & & \cup \\ & K & \supset & T'(K)_1 \end{array}.$$

Secondly we assume that (Di) holds. We show that $(D(i+1))$ holds. By (Di) we have

(3) $T'(K)_i \subset T(K)_i$.

Since T_{i+1} is a sudoku transformation, by (3) we have

(4) $T_{i+1}(T'(K)_i) \subset T_{i+1}(T(K)_i) = T(K)_{i+1} \subset T(K)_i$.

Since $T_{i+1}' \leqq T_{i+1}$ and T_{i+1}' is a sudoku transformation, we have

(5) $T'(K)_i \supset T'(K)_{i+1} = T_{i+1}'(T'(K)_i) \subset T_{i+1}(T'(K)_i)$.

By (4) and (5) we have

$$\begin{array}{cccc} & T(K)_i & \supset & T(K)_{i+1} \\ (D') & \cup & & \cup \\ & T'(K)_i & \supset & T'(K)_{i+1} \end{array}.$$

By (Di) and (D') we have $(D(i+1))$. Hence by the matematical induction we have the diagram (D).

By the diagram (D) we have

(6) $T'(K)_\infty = \cap_{j=1}^{\infty} T'(K)_j \subset \cap_{j=1}^{\infty} T(K)_j = T(K)_\infty$

(6) implies (a). Hence we have Proposition 33. By the same way we can show Proposition 34.

Proofs of Propositions 21,22,31 and 32.

We show Proposition 31. Since $STRF(f,f_0)$ is finite, $TOOL$ is also finite. Thus we can put

(1) $TOOL = \{S_1, S_2, \ldots, S_m\}$, $m \geqq 1$.

Since each S_i is a sudoku transformation, by Proposition 19 we have

(2) $P = S_1 \cap S_2 \cap \ldots \cap S_m : STMX(f,f_0) \to STMX(f,f_0)$ is a sudoku transformation,

(3) $P \leqq S_i$ for each i, $1 \leqq i \leqq m$.

Let $\boldsymbol{TOOL}^* = \{P\}$ and let $\boldsymbol{P} = (P_1, P_2, ..., P_i, P_{i+1}, ...)$ be the infinite sequence with $P_i = P$ for each i. Since $\boldsymbol{P}$ is $\boldsymbol{TOOL}^*$−full, by the proof of Proposition 29 there exists an integer p_0 such that

(4) $\boldsymbol{P}(\boldsymbol{K})_j = \boldsymbol{P}(\boldsymbol{K})_{p_0}$ for each $j \geqq p_0$ and for each $\boldsymbol{K} \in \boldsymbol{STMX}(f, f_0)$,

(5) $\boldsymbol{P}(\boldsymbol{K})_{p_0}$ is $\boldsymbol{TOOL}^*$−stable for each $\boldsymbol{K} \in \boldsymbol{STMX}(f, f_0)$.

Let p_1 be another integer p_1 such that

(4') $\boldsymbol{P}(\boldsymbol{K})_j = \boldsymbol{P}(\boldsymbol{K})_{p_1}$ for each $j \geqq p_1$ and for each $\boldsymbol{K} \in \boldsymbol{STMX}(f, f_0)$,

(5') $\boldsymbol{P}(\boldsymbol{K})_{p_1}$ is $\boldsymbol{TOOL}^*$−stable for each $\boldsymbol{K} \in \boldsymbol{STMX}(f, f_0)$.

Now, let $p_2 = p_0 + p_1$. By (4) and (4') we have that

(6) $\boldsymbol{P}(\boldsymbol{K})_{p_0} = \boldsymbol{P}(\boldsymbol{K})_{p_2} = \boldsymbol{P}(\boldsymbol{K})_{p_1}$ for each $\boldsymbol{K} \in \boldsymbol{STMX}(f, f_0)$.

Since $\boldsymbol{P}(\boldsymbol{K})_{p_0} = (P_{p_0} \circ P_{p_0 - 1} \circ ... \circ P_1)(\boldsymbol{K}) = (P \circ P \circ ... P)(\boldsymbol{K}) = P^{p_0}(\boldsymbol{K})$ and $\boldsymbol{P}(\boldsymbol{K})_{p_1} = (P_{p_1} \circ P_{p_1 - 1} \circ ... \circ P_1)(\boldsymbol{K}) = (P \circ P \circ ... P)(\boldsymbol{K}) = P^{p_1}(\boldsymbol{K})$, by (6) we have the following:

Claim 1. $P^{p_0} = P^{p_1}$ and $P^{p_0}(\boldsymbol{K}) = \boldsymbol{P}(\boldsymbol{K})_{p_0}$ for each $\boldsymbol{K} \in \boldsymbol{STMX}(f, f_0)$.

We can define a map $\boldsymbol{P}_\infty : \boldsymbol{STMX}(f, f_0) \to \boldsymbol{STMX}(f, f_0)$ as follows:

(7) $\boldsymbol{P}_\infty = P^{p_0}$.

Claim 1 means that $\boldsymbol{P}_\infty$ is well−defined. Since $\boldsymbol{P}(\boldsymbol{K})_\infty = \cap_{j=1}^{\infty} \boldsymbol{P}(\boldsymbol{K})_j$, by (4) we have $\boldsymbol{P}(\boldsymbol{K})_\infty = \boldsymbol{P}(\boldsymbol{K})_{p_0}$. By Claim 1 we have that $\boldsymbol{P}_\infty(\boldsymbol{K}) = P^{p_0}(\boldsymbol{K}) = \boldsymbol{P}(\boldsymbol{K})_{p_0} = \boldsymbol{P}(\boldsymbol{K})_\infty$. Since P is a sudoku transformation, by Proposition 19 $\boldsymbol{P}_\infty = P^{p_0}$ is also a sudoku transformation. Thus we have the following Claim 2.

Claim 2. $\boldsymbol{P}_\infty : \boldsymbol{STMX}(f, f_0) \to \boldsymbol{STMX}(f, f_0)$ is a sudoku transformation and $\boldsymbol{P}_\infty(\boldsymbol{K}) = \boldsymbol{P}(\boldsymbol{K})_\infty = \boldsymbol{P}(\boldsymbol{K})_{p_0}$ for each $\boldsymbol{K} \in \boldsymbol{STMX}(f, f_0)$.

Claim 3. $\boldsymbol{P}_\infty(\boldsymbol{K})$ is $\boldsymbol{TOOL}$−stable for each $\boldsymbol{K} \in \boldsymbol{STMX}(f, f_0)$.

Proof of Claim 3. We assume that Claim 3 does not hold. Thus there exists a $\boldsymbol{K} \in \boldsymbol{STMX}(f, f_0)$ such that $\boldsymbol{P}_\infty(\boldsymbol{K})$ is not $\boldsymbol{TOOL}$−stable. Since $\boldsymbol{P}_\infty(\boldsymbol{K}) = \boldsymbol{P}(\boldsymbol{K})_\infty = \boldsymbol{P}(\boldsymbol{K})_{p_0}$ by Claim 2, there exists a $S_{i_0} \in \boldsymbol{TOOL}$ such that

(8) $S_{i_0}(\boldsymbol{P}(\boldsymbol{K})_{p_0}) \subsetneqq \boldsymbol{P}(\boldsymbol{K})_{p_0}$.

Thus, by (2) and (8), we have

(9) $P\left(P(K)_{p_0}\right)=(S_1\cap S_2\cap\ldots\cap S_m)\left(P(K)_{p_0}\right)\subset S_{i_0}\left(P(K)_{p_0}\right)\subsetneqq P(K)_{p_0}$.

By (5) we have

(10) $P\left(P(K)_{p_0}\right)=P(K)_{p_0}$.

By (9) and (10) we have

(11) $P(K)_{p_0}\subsetneqq P(K)_{p_0}$.

Since (11) is a contradiction, we have Claim 3.

Claim 4. If $K\in STMX(f,f_0)$ is $TOOL-$stable, then $P_\infty(K)=K$.

Proof of Claim 4. Let $K\in STMX(f,f_0)$ be $TOOL-$stable. Thus we have

(12) $S_i(K)=K$ for each i, $1\leqq i\leqq m$.

By (2) and (12) we have that

$P(K)=(S_1\cap S_2\cap\ldots\cap S_m)(K)=S_1(K)\cap S_2(K)\cap\ldots\cap S_m(K)=K$, *i.e.*,

(13) $P(K)=K$.

By Claim 2 and (13) we have that

(14) $P_\infty(K)=P(K)_\infty=P(K)_{p_0}=\left(P_{p_0}\circ P_{p_0-1}\circ\ldots\circ P_1\right)(K)=(P\circ P\circ\ldots.\circ P)(K)=K$.

Hence, by (14), we have Claim 4.

Claim 5. Let K, $L\in STMX(f,f_0)$. If $K\geqq L\geqq P_\infty(K)$ and L is $TOOL-$stable, then $L=P_\infty(K)$.

Proof of Claim 5. Since P_∞ is a sudoku transformation by Claim 2, by the assumption $K\geqq L\geqq P_\infty(K)$ induces that $P_\infty(K)\geqq P_\infty(L)\geqq P_\infty(P_\infty(K))$. Thus we have

(15) $P_\infty(K)\supset P_\infty(L)\supset P_\infty(P_\infty(K))$.

By Claim 3 we have

(16) $P_\infty(K)$ is $TOOL-$stable.

By (16) and Claim 4 we have

(17) $P_\infty(P_\infty(K))=P_\infty(K)$.

By (15) and (17) we have

(18) $P_\infty(K)=P_\infty(L)$.

Since L is $TOOL-$stable by the assumption, by Claim 4 we have

(19) $P_\infty(L)=L$.

Thus, by (18) and (19) we have

(20) $P_\infty(K)=L$.

By (20) we have Claim 5.

By the assumptions of Proposition 31 we have finite sequences $\boldsymbol{T}=(T_1,T_2,...,T_{n_0})$ and $\boldsymbol{T}'=(T_1',T_2',...,T_{n_1}')$ in $\boldsymbol{TOOL}$ with the properties (a) and (b), respectively.

Claim 6. For each $\boldsymbol{K}\in\boldsymbol{STMX}(f,f_0)$, we have

(21) $\boldsymbol{K}\supset(T_{n_0}\circ T_{n_0-1}\circ....\circ T_1)(\boldsymbol{K})\supset\boldsymbol{P}_\infty(\boldsymbol{K})$,

(22) $\boldsymbol{K}\supset(T_{n_1}'\circ T_{n_1-1}'\circ....\circ T_1')(\boldsymbol{K})\supset\boldsymbol{P}_\infty(\boldsymbol{K})$.

Proof of Claim 6. We show (21). We make an infinite sequence $\boldsymbol{T}^*$ $=(T_1^*, T_2^*,...,T_i^*,...)$ as follows:

(23) $T_i^*=T_i$ for i, $1\leqq i\leqq n_0$ and

(24) $T_i^*=S_1$ for each i, $i\geqq n_0+1$.

Since $\boldsymbol{T}$ is a sequence in $\boldsymbol{TOOL}$, by (23) $T_i^*=T_i\in\boldsymbol{TOOL}$ for i, $1\leqq i\leqq n_0$ and by (24) $T_i^*=S_1\in\boldsymbol{TOOL}$ for each i, $i\geqq n_0+1$. Thus we hve

(25) $\boldsymbol{T}^*$ is an infinite sequence in $\boldsymbol{TOOL}$.

By (2) and (25) we have

(26) $P\leqq T_i^*$ for each i.

By (26) and Proposition 33, we have

(27) $\boldsymbol{P}_\infty(\boldsymbol{K})=\boldsymbol{P}(\boldsymbol{K})_\infty\leqq\boldsymbol{T}^*(\boldsymbol{K})_\infty$.

Claim 7. $\boldsymbol{T}^*(\boldsymbol{K})_\infty=\boldsymbol{T}^*(\boldsymbol{K})_{n_0}=(T_{n_0}\circ T_{n_0-1}\circ....\circ T_1)(\boldsymbol{K})$.

Proof of Claim 7. By (23) we have

(28) $\boldsymbol{T}^*(\boldsymbol{K})_{n_0}=(T_{n_0}^*\circ T_{n_0-1}^*\circ....\circ T_1^*)(\boldsymbol{K})=(T_{n_0}\circ T_{n_0-1}\circ...\circ T_1)(\boldsymbol{K})$.

We show that

(29) $\boldsymbol{T}^*(\boldsymbol{K})_j=\boldsymbol{T}^*(\boldsymbol{K})_{n_0}$ for each j, $j\geqq n_0$.

When $j=n_0$, clearly (29) holds. Take any j, $j\geqq n_0+1$. Thus by (24) we have

(30) $\boldsymbol{T}^*(\boldsymbol{K})_j=(T_j^*\circ T_{j-1}^*\circ...\circ T_{n_0+1}^*)(\boldsymbol{T}^*(\boldsymbol{K})_{n_0})=(S_1\circ S_1\circ...\circ S_1)(\boldsymbol{T}^*(\boldsymbol{K})_{n_0})$

By (28) and (a) we have

(31) $\boldsymbol{T}^*(\boldsymbol{K})_{n_0}$ is $\boldsymbol{TOOL}-$stable.

Since $S_1\in\boldsymbol{TOOL}$, by (31) we have

(32) $S_1(\boldsymbol{T}^*(\boldsymbol{K})_{n_0})=\boldsymbol{T}^*(\boldsymbol{K})_{n_0}$.

By (32) we have

(33) $(S_1\circ S_1\circ...\circ S_1)(\boldsymbol{T}^*(\boldsymbol{K})_{n_0})=\boldsymbol{T}^*(\boldsymbol{K})_{n_0}$.

By (30) and (33) we have (29).

By (29) we have

(34) $\boldsymbol{T}^*(\boldsymbol{K})_\infty = \cap_{j=1}^{\infty} \boldsymbol{T}^*(\boldsymbol{K})_j = \boldsymbol{T}^*(\boldsymbol{K})_{n_0}$.

By (34), (28) we have Claim 7.

By (27) and Claim 7 we have

(35) $\boldsymbol{P}_\infty(\boldsymbol{K}) = \boldsymbol{P}(\boldsymbol{K})_\infty \subset \boldsymbol{T}^*(\boldsymbol{K})_\infty = \boldsymbol{T}^*(\boldsymbol{K})_{n_0} = (T_{n_0} \circ T_{n_0-1} \circ \dots \circ T_1)(\boldsymbol{K})$.

Since each T_i , $1 \leqq i \leqq n_0$, is a sudoku transformation, by Proposition 19 $T_{n_0} \circ T_{n_0-1} \circ \dots \circ T_1$ is also a sudoku transformation. Thus, we have

(36) $(T_{n_0} \circ T_{n_0-1} \circ \dots \circ T_1)(\boldsymbol{K}) \subset \boldsymbol{K}$.

By (35) and (36) we have (21).

By the same way we can show (22). Hence we have Claim 6.

By Claim 6 we have

(37) $\boldsymbol{K} \geqq (T_{n_0} \circ T_{n_0-1} \circ \dots \circ T_1)(\boldsymbol{K}) \geqq \boldsymbol{P}_\infty(\boldsymbol{K})$ and

(38) $\boldsymbol{K} \geqq (T'_{n_1} \circ T'_{n_1-1} \circ \dots \circ T'_1)(\boldsymbol{K}) \geqq \boldsymbol{P}_\infty(\boldsymbol{K})$.

Since $(T_{n_0} \circ T_{n_0-1} \circ \dots \circ T_1)(\boldsymbol{K})$ is $\boldsymbol{TOOL}-$stable by (a) in Proposition 31, we have the following by (37) and Claim 5

(39) $(T_{n_0} \circ T_{n_0-1} \circ \dots \circ T_1)(\boldsymbol{K}) = \boldsymbol{P}_\infty(\boldsymbol{K})$.

Since $(T'_{n_1} \circ T'_{n_1-1} \circ \dots \circ T'_1)(\boldsymbol{K})$ is $\boldsymbol{TOOL}-$stable by (b), we have the following by (38) and Claim 5

(40) $(T'_{n_1} \circ T'_{n_1-1} \circ \dots \circ T'_1)(\boldsymbol{K}) = \boldsymbol{P}_\infty(\boldsymbol{K})$.

By (39) and (40) we have

(41) $(T_{n_0} \circ T_{n_0-1} \circ \dots \circ T_1)(\boldsymbol{K}) = \boldsymbol{P}_\infty(\boldsymbol{K}) = (T'_{n_1} \circ T'_{n_1-1} \circ \dots \circ T'_1)(\boldsymbol{K})$

Hence we have, by (41), the following:

Claim 8. $T_{n_0} \circ T_{n_0-1} \circ \dots \circ T_1 = T'_{n_1} \circ T'_{n_1-1} \circ \dots \circ T'_1 = \boldsymbol{P}_\infty$.

Claim 8 is (c) in Proposition 31. Hence we complete the proof of Proposition 31.

By Claim 8 we can define $STBL^{\boldsymbol{TOOL}} = T_{n_0} \circ T_{n_0-1} \circ \dots \circ T_1 = T'_{n_1} \circ T'_{n_1-1} \circ \dots \circ T'_1 = \boldsymbol{P}_\infty$.

By Claims 2,3,4 it has the required properties in Proposition 21.

By the same way we can prove Proposition 32 and Proposition 22.

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