Yanagi and Sekikawa *Journal of Inequalities and Applications* (2015) 2015:381
DOI 10.1186/s13660-015-0895-x

Journal of Inequalities and Applications
a SpringerOpen Journal

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Non-hermitian extensions of Heisenberg type and Schrödinger type uncertainty relations

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Abstract

In quantum mechanics it is well known that the Heisenberg-Schrödinger uncertainty relations hold for two non-commutative observables and density operator. Recently Dou and Du (J. Math. Phys. 54:103508, 2013; Int. J. Theor. Phys. 53:952-958, 2014) obtained several uncertainty relations for two non-commutative non-hermitian observables and density operators. In this paper, we show that their results can be given as corollaries of our non-hermitian extensions of Heisenberg type or Schrödinger type uncertainty relations for the generalized metric adjusted skew information or generalized metric adjusted correlation measures which were obtained in Furuichi and Yanagi (J. Math. Anal. Appl. 388:1147-1156, 2012).

MSC: Primary 15A45; 47A63; secondary 94A17

Keywords: trace inequality; metric adjusted skew information; non-hermitian observable

1 Introduction

Let $M_n(\mathbb{C})$ (resp. $M_{n,sa}(\mathbb{C})$) be the set of all $n \times n$ complex matrices (resp. all $n \times n$ self-adjoint matrices), endowed with the Hilbert-Schmidt scalar product $\langle A, B\rangle = \mathrm{Tr}[A^*B]$. Let $M_{n,+}(\mathbb{C})$ be the set of strictly positive elements of $M_n(\mathbb{C})$ and $M_{n,+,1}(\mathbb{C}) \subset M_{n,+}(\mathbb{C})$ be the set of strictly positive density matrices, that is, $M_{n,+,1}(\mathbb{C}) = \{\rho \in M_n(\mathbb{C}) | \mathrm{Tr}[\rho] = 1, \rho > 0\}$. If it is not otherwise specified, from now on we shall treat the case of faithful states, that is, $\rho > 0$. It is well known that the expectation of an observable $A \in M_{n,sa}(\mathbb{C})$ in a state $\rho \in M_{n,+,1}(\mathbb{C})$ is defined by

$$E_\rho(A) := \mathrm{Tr}(\rho A),$$

and the variance of an observable $A \in M_{n,sa}(\mathbb{C})$ in a state $\rho \in M_{n,+,1}(\mathbb{C})$ is defined by

$$V_\rho(A) = \mathrm{Tr}\bigl[\rho\bigl(A - E_\rho(A)I\bigr)^2\bigr].$$

In order to represent the degree of non-commutativity between $\rho \in M_{n,+,1}(\mathbb{C})$ and $A \in M_{n,sa}(\mathbb{C})$, the Wigner-Yanase skew information $I_\rho(A)$ is defined by

$$I_\rho(A) = \frac{1}{2}\mathrm{Tr}\bigl[\bigl(i\bigl[\rho^{1/2}, A\bigr]\bigr)^2\bigr] = \mathrm{Tr}\bigl[\rho A^2\bigr] - \mathrm{Tr}\bigl[\rho^{1/2}A\rho^{1/2}A\bigr],$$

where $[X,Y] = XY - YX$. Furthermore the Wigner-Yanase-Dyson skew information $I_{\rho,\alpha}(A)$ is defined by

$$I_{\rho,\alpha}(A) = \frac{1}{2}\operatorname{Tr}\big[\big(i\big[\rho^{\alpha},A\big]\big)\big(i\big[\rho^{1-\alpha},A\big]\big)\big] = \operatorname{Tr}\big[\rho A^2\big] - \operatorname{Tr}\big[\rho^{\alpha}A\rho^{1-\alpha}A\big] \quad \big(\alpha \in [0,1]\big).$$

The convexity of $I_{\rho,\alpha}(A)$ with respect to ρ was famously shown by Lieb [4]. The relationship between Wigner-Yanase skew information and the uncertainty relation was given by Luo and Zhang [5] for the first time. Afterward, the relationship between Wigner-Yanase-Dyson skew information and the uncertainty relation was given by Kosaki [6] and Yanagi *et al.* [7]. Furthermore metric adjusted skew information was defined by Hansen [8] which is an extension of Wigner-Yanase-Dyson skew information. The relationship between metric adjusted skew information and the uncertainty relation was given by Yanagi [9] and was generalized in Yanagi *et al.* [10] for generalized metric adjusted skew information and generalized metric adjusted correlation measures. In this paper we give some non-hermitian extensions of Heisenberg type and Schrödinger type uncertainty relations related to generalized quasi-metric adjusted skew information and generalized quasi-metric adjusted correlation measures. As a result we can obtain some results of non-hermitian uncertainty relations given by Dou and Du as corollaries of our results.

2 Dou-Du's non-hermitian uncertainty relations

Definition 1 For $A,B \in M_n(\mathbb{C})$, $\rho \in M_{n,+,1}(\mathbb{C})$, we define the following.

(1) $[A,B]^0 = \frac{1}{2}([A,B] + [A^*,B^*])$, where $[A,B] = AB - BA$.

(2) $\{A,B\}^0 = \frac{1}{2}(\{A,B\} + \{A^*,B^*\})$, where $\{A,B\} = AB + BA$.

(3) $|\operatorname{Var}_{\rho}|(A) = \operatorname{Tr}[\rho A_0^* A_0]$, where $A_0 = A - \operatorname{Tr}[\rho A]I$.

(4) $|\operatorname{Var}_{\rho}^0|(A) = \frac{1}{2}(|\operatorname{Var}_{\rho}|(A) + |\operatorname{Var}_{\rho}|(A^*))$.

Theorem 1 (Dou-Du [2]) *For $A,B \in M_n(\mathbb{C})$, $\rho \in M_{n,+,1}(\mathbb{C})$, we have the following.*

(1) $|\operatorname{Var}_{\rho}^0|(A)|\operatorname{Var}_{\rho}^0|(B) \geq \frac{1}{4}|\operatorname{Tr}[\rho[A,B]]|^2$.

(2) $|\operatorname{Var}_{\rho}^0|(A)|\operatorname{Var}_{\rho}^0|(B) \geq \frac{1}{4}|\operatorname{Tr}[\rho\{A_0,B_0\}]|^2$.

(3) $|\operatorname{Var}_{\rho}^0|(A)|\operatorname{Var}_{\rho}^0|(B) \geq \frac{1}{4}|\operatorname{Tr}[\rho[A,B]^0]|^2 + \frac{1}{4}|\operatorname{Tr}[\rho\{A_0,B_0\}^0]|^2$.

(4) $|U_{\rho}|(A)|U_{\rho}|(B) \geq \frac{1}{4}|\operatorname{Tr}[\rho[A,B]^0]|^2$, *where*

$$|U_{\rho}|(A) = \sqrt{\big(\big|\operatorname{Var}_{\rho}^0\big|(A)\big)^2 - \big(\big|\operatorname{Var}_{\rho}^0\big|(A) - |I_{\rho}|(A)\big)^2},$$

$$|I_{\rho}|(A) = \frac{1}{2}\operatorname{Tr}\big[\big(i\big[\rho^{1/2},A^*\big]\big)\big(i\big[\rho^{1/2},A\big]\big)\big].$$

3 Quantum Fisher information

A function $f:(0,+\infty) \to \mathbb{R}$ is said operator monotone if, for any $n \in \mathbb{N}$, and $A,B \in M_{n,+}(\mathbb{C})$ such that $0 \leq A \leq B$, the inequalities $0 \leq f(A) \leq f(B)$ hold. An operator monotone function is said symmetric if $f(x) = xf(x^{-1})$ and normalized if $f(1) = 1$. The following definitions were given by Hansen and Gibilisco, *etc.*

Definition 2 $\mathcal{F}_{\text{op}}$ is the class of functions $f:(0,+\infty) \to (0,+\infty)$ satisfying

(1) $f(1) = 1$,

(2) $tf(t^{-1}) = f(t)$,

(3) f is operator monotone.

For $f \in \mathcal{F}_{\mathrm{op}}$ define $f(0) = \lim_{x\to 0} f(x)$. We introduce the sets of regular and non-regular functions

$$\mathcal{F}_{\mathrm{op}}^r = \{f \in \mathcal{F}_{\mathrm{op}} | f(0) \neq 0\}, \qquad \mathcal{F}_{\mathrm{op}}^n = \{f \in \mathcal{F}_{\mathrm{op}} | f(0) = 0\}$$

and notice that trivially $\mathcal{F}_{\mathrm{op}} = \mathcal{F}_{\mathrm{op}}^r \cup \mathcal{F}_{\mathrm{op}}^n$.

Definition 3 For $f \in \mathcal{F}_{\mathrm{op}}^r$ we define

$$\tilde{f}(x) = \frac{1}{2}\left[(x+1) - (x-1)^2 \frac{f(0)}{f(x)}\right], \quad x > 0.$$

Proposition 1 ([11]) *The correspondence $f \to \tilde{f}$ is a bijection between $\mathcal{F}_{\mathrm{op}}^r$ and $\mathcal{F}_{\mathrm{op}}^n$.*

Example 1

$$f_{RLD}(x) = \frac{2x}{x+1}, \qquad f_{BKN}(x) = \frac{x-1}{\log x},$$
$$f_{SLD}(x) = \frac{x+1}{2}, \qquad \tilde{f}_{SLD}(x) = \frac{2x}{x+1},$$
$$f_{WY}(x) = \left(\frac{\sqrt{x}+1}{2}\right)^2, \qquad \tilde{f}_{WY}(x) = \sqrt{x},$$
$$f_{WYD}(x) = \alpha(1-\alpha)\frac{(x-1)^2}{(x^\alpha - 1)(x^{1-\alpha} - 1)}, \quad \alpha \in (0,1),$$
$$\tilde{f}_{WYD}(x) = \frac{1}{2}\left\{x + 1 - (x^\alpha - 1)(x^{1-\alpha} - 1)\right\},$$
$$\frac{2x}{x+1} < \sqrt{x} < \frac{x-1}{\log x} < f_{WYD} < \left(\frac{\sqrt{x}+1}{2}\right)^2 < \frac{x+1}{2} \quad (x \neq 1).$$

In the Kubo-Ando theory of matrix means one associates a mean to each operator monotone function $f \in \mathcal{F}_{\mathrm{op}}$ by the formula

$$m_f(A,B) = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2},$$

where $A, B \in M_{n,+}(\mathbb{C})$.

By using the notion of matrix means one may define the class of monotone metrics (also called quantum Fisher information) by the following formula:

$$\langle A, B\rangle_{\rho,f} = \mathrm{Tr}\left[A \cdot m_f(L_\rho, R_\rho)^{-1}(B)\right],$$

where $L_\rho(A) = \rho A$, $R_\rho(A) = A\rho$.

4 Generalized quasi-metric adjusted skew information and correlation measure

Definition 4 Let $g, f \in \mathcal{F}_{\mathrm{op}}^r$ satisfy

$$g(x) \geq k\frac{(x-1)^2}{f(x)}$$

for some $k > 0$. We define

$$\Delta_g^f(x) = g(x) - k\frac{(x-1)^2}{f(x)} \in \mathcal{F}_{\mathrm{op}}. \tag{1}$$

By Lemma 5.2 in [12], $-k\frac{(x-1)^2}{f(x)}$ is operator concave. Then $\Delta_g^f(x)$ is also operator concave. Since $\Delta_g^f(x) > 0$ for $(0,\infty)$, $\Delta_g^f(x)$ is operator monotone.

Definition 5 For $A, B \in M_n(\mathbb{C})$ and $\rho \in M_{n,+,1}(\mathbb{C})$, we define the following quantities:

$$\begin{aligned}
&\left|\mathrm{Corr}_\rho^{(g,f)}\right|(A,B) = k\bigl\langle i[\rho,A], i[\rho,B]\bigr\rangle_{\rho,f},\\
&\left|I_\rho^{(g,f)}\right|(A) = \left|\mathrm{Corr}_\rho^{(g,f)}\right|(A,A),\\
&\left|C_\rho^f\right|(A,B) = \mathrm{Tr}\bigl[A^* m_f(L_\rho,R_\rho)B\bigr], \qquad \left|C_\rho^f\right|(A) = \left|C_\rho^f\right|(A,A),\\
&\left|U_\rho^{(g,f)}\right|(A) = \sqrt{\bigl(\left|C_\rho^g\right|(A) + \left|C_\rho^{\Delta_g^f}\right|(A)\bigr)\bigl(\left|C_\rho^g\right|(A) - \left|C_\rho^{\Delta_g^f}\right|(A)\bigr)}.
\end{aligned}$$

The quantity $|I_\rho^{(g,f)}|(A)$ and $|\mathrm{Corr}_\rho^{(g,f)}|(A,B)$ are called generalized quasi-metric adjusted skew information and generalized quasi-metric adjusted correlation measures, respectively.

Then we have the following proposition.

Proposition 2 *For $A, B \in M_n(\mathbb{C})$ and $\rho \in M_{n,+,1}(\mathbb{C})$, we have the following relations*:

(1) $|I_\rho^{(g,f)}|(A) = |I_\rho^{(g,f)}|(A_0) = |C_\rho^g|(A_0) - |C_\rho^{\Delta_g^f}|(A_0)$,

(2) $|J_\rho^{(g,f)}|(A) = |C_\rho^g|(A_0) + |C_\rho^{\Delta_g^f}|(A_0)$,

(3) $|U_\rho^{(g,f)}|(A) = \sqrt{|I_\rho^{(g,f)}|(A) \cdot |J_\rho^{(g,f)}|(A)}$,

(4) $|\mathrm{Corr}_\rho^{(g,f)}|(A,B) = |\mathrm{Corr}_\rho^{(g,f)}|(A_0,B_0)$, *where* $A_0 = A - \mathrm{Tr}[\rho A]I$ *and* $B_0 = B - \mathrm{Tr}[\rho B]I$.

Theorem 2 (Schrödinger type) *For $f \in \mathcal{F}_{\mathrm{op}}^r$, we have*

$$\left|I_\rho^{(g,f)}\right|(A) \cdot \left|I_\rho^{(g,f)}\right|(B) \geq \left|\left|\mathrm{Corr}_\rho^{(g,f)}\right|(A,B)\right|^2,$$

where $A, B \in M_n(\mathbb{C})$ and $\rho \in M_{n,+,1}(\mathbb{C})$.

Proof of Theorem 1 By Schwarz's inequality we have

$$\langle A,A\rangle_{\rho,f}\langle B,B\rangle_{\rho,f} \geq \left|\langle A,B\rangle_{\rho,f}\right|^2.$$

Then we have

$$\begin{aligned}
&\left|I_\rho^{(g,f)}\right|(A) \cdot \left|I_\rho^{(g,f)}\right|(B)\\
&\quad = \left|\mathrm{Corr}_\rho^{(g,f)}\right|(A,A) \cdot \left|\mathrm{Corr}_\rho^{(g,f)}\right|(B,B)\\
&\quad \geq \left|\left|\mathrm{Corr}_\rho^{(g,f)}\right|(A,B)\right|^2.
\end{aligned}$$

□

Theorem 3 (Heisenberg type) *For $f \in \mathcal{F}_{\mathrm{op}}^r$, if*

$$g(x) + \Delta_g^f(x) \geq \ell f(x) \tag{2}$$

for some $\ell > 0$, then we have

$$\left|U_\rho^{(g,f)}\right|(A) \cdot \left|U_\rho^{(g,f)}\right|(B) \geq k\ell \left|\mathrm{Tr}\big(\rho[A,B]\big)\right|^2, \tag{3}$$

where $A, B \in M_n(\mathbb{C})$ and $\rho \in M_{n,+,1}(\mathbb{C})$.

We need the following lemma in order to prove Theorem 3.

Lemma 1 *If* (1) *and* (2) *is satisfied, then*

$$m_g(x,y)^2 - m_{\Delta_g^f}(x,y)^2 \geq k\ell(x-y)^2.$$

Proof By (1) and (2),

$$m_{\Delta_g^f}(x,y) = m_g(x,y) - k\frac{(x-y)^2}{m_f(x,y)}, \tag{4}$$

$$m_g(x,y) + m_{\Delta_g^f}(x,y) \geq \ell m_f(x,y). \tag{5}$$

Then by (4) and (5), we have

$$\begin{aligned}
&m_g(x,y)^2 - m_{\Delta_g^f}(x,y)^2 \\
&\quad = \left\{m_g(x,y) - m_{\Delta_g^f}(x,y)\right\}\left\{m_g(x,y) + m_{\Delta_g^f}(x,y)\right\} \\
&\quad \geq \frac{k(x-y)^2}{m_f(x,y)} \ell m_f(x,y) \\
&\quad = k\ell(x-y)^2.
\end{aligned}$$

□

Proof of Theorem 2 Let $\{|\phi_1\rangle, |\phi_2\rangle, \ldots, |\phi_n\rangle\}$ be a basis of eigenvectors of ρ, corresponding to the eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. We put $a_{jk} = \langle\phi_j|A_0|\phi_k\rangle$, $b_{jk} = \langle\phi_j|B_0|\phi_k\rangle$. Then we have

$$\begin{aligned}
&\left|I_\rho^{(g,f)}\right|(A) \\
&\quad = \left|C_\rho^g\right|(A_0) - \left|C_\rho^{\Delta_g^f}\right|(A_0) \\
&\quad = \mathrm{Tr}\left[A_0^* m_f(L_\rho, R_\rho) A_0\right] - \mathrm{Tr}\left[A_0^* m_{\Delta_g^f}(L_\rho, R_\rho) A_0\right] \\
&\quad = \sum_{i,j}\left\{m_f(\lambda_i,\lambda_j) - m_{\Delta_g^f}(\lambda_i,\lambda_j)\right\}|a_{ij}|^2
\end{aligned}$$

and

$$\left|J_\rho^{(g,f)}\right|(A) = \sum_{i,j}\left\{m_f(\lambda_i,\lambda_j) + m_{\Delta_g^f}(\lambda_i,\lambda_j)\right\}|a_{ij}|^2.$$

We remark that

$$\mathrm{Tr}\big(\rho[A,B]\big) = \sum_i \sum_j (\lambda_i - \lambda_j) a_{ij} b_{ji}$$

and

$$\big|\mathrm{Tr}\big(\rho[A,B]\big)\big| \le \sum_i \sum_j |\lambda_i - \lambda_j||a_{ij}||b_{ji}|.$$

By Lemma 1 we have

$$\begin{aligned}
&k\ell\big|\mathrm{Tr}\big[\rho[A,B]\big]\big|^2 \\
&\quad \le \left\{\sum_i \sum_j \sqrt{kl}|\lambda_i - \lambda_j||a_{ij}||b_{ji}|\right\}^2 \\
&\quad \le \left\{\sum_i \sum_j \big(m_g(\lambda_i,\lambda_j)^2 - m_{\Delta_g^f}(\lambda_i,\lambda_j)^2\big)^{1/2}|a_{ij}||b_{ji}|\right\}^2 \\
&\quad \le \left\{\sum_i \sum_j \big(m_g(\lambda_i,\lambda_j) - m_{\Delta_g^f}(\lambda_i,\lambda_j)\big)|a_{ij}|^2\right\}\left\{\sum_i \sum_j \big(m_g(\lambda_i,\lambda_j) + m_{\Delta_g^f}(\lambda_i,\lambda_j)\big)|b_{ji}|^2\right\} \\
&\quad = I_\rho^{(g,f)}(A) J_\rho^{(g,f)}(B).
\end{aligned}$$

Similarly we have

$$k\ell\big|\mathrm{Tr}\big[\rho[A,B]\big]\big|^2 \le I_\rho^{(g,f)}(B) J_\rho^{(g,f)}(A).$$

Therefore

$$\big|U_\rho^{(g,f)}\big|(A) \cdot \big|U_\rho^{(g,f)}\big|(B) \ge k\ell\big|\mathrm{Tr}\big(\rho[A,B]\big)\big|^2.$$ □

Example 2 When

$$\begin{aligned}
&g(x) = \frac{x+1}{2}, \qquad f(x) = \alpha(1-\alpha)\frac{(x-1)^2}{(x^\alpha - 1)(x^{1-\alpha} - 1)} \quad (0 < \alpha < 1), \\
&k = \frac{f(0)}{2}, \qquad \ell = 2,
\end{aligned}$$

we can show positivity:

$$\Delta_g^f(x) = g(x) - k\frac{(x-1)^2}{f(x)} = \frac{1}{2}\big(x^\alpha + x^{1-\alpha}\big) \ge 0$$

and

$$\begin{aligned}
g(x) + \Delta_g^f(x) - \ell f(x) &= \frac{1}{2(x^\alpha - 1)(x^{1-\alpha} - 1)}\left\{\big(x^{2\alpha} - 1\big)\big(x^{2(1-\alpha)} - 1\big) - 4\alpha(1-\alpha)(x-1)^2\right\} \\
&\ge 0.
\end{aligned}$$

Then

$$\begin{aligned}\left|I_\rho^{(f,g)}\right|(A) = \left|I_\rho^{(f,g)}\right|(A_0) &= \frac{1}{2}\operatorname{Tr}\bigl[\rho A_0 A_0^*\bigr] + \frac{1}{2}\operatorname{Tr}\bigl[\rho A_0^* A_0\bigr] \\ &\quad - \frac{1}{2}\operatorname{Tr}\bigl[\rho^\alpha A_0 \rho^{1-\alpha} A_0^*\bigr] - \frac{1}{2}\operatorname{Tr}\bigl[\rho^{1-\alpha} A_0^* \rho^\alpha A_0\bigr].\end{aligned}$$

In particular, for $\alpha = 1/2$,

$$\left|I_\rho^{(f,g)}\right|(A) = \left|I_\rho^{(f,g)}\right|(A_0) = \frac{1}{2}\operatorname{Tr}\bigl[\rho A_0 A_0^*\bigr] + \frac{1}{2}\operatorname{Tr}\bigl[\rho A_0^* A_0\bigr] - \operatorname{Tr}\bigl[\rho^{1/2} A_0 \rho^{1/2} A_0^*\bigr].$$

In this case we can give some results by Dou-Du as corollaries.

Corollary 1 (Dou-Du (4)) *For $A, B \in M_n(\mathbb{C})$ and $\rho \in M_{n,+,1}(\mathbb{C})$,*

$$\begin{aligned}&|U_\rho|(A)\cdot|U_\rho|(B) \\ &\quad \geq \frac{1}{4}\bigl|\operatorname{Tr}\bigl[\rho[A,B]\bigr]\bigr|^2 \geq \frac{1}{4}\bigl|\operatorname{Im}\operatorname{Tr}\bigl[\rho[A,B]\bigr]\bigr|^2 \\ &\quad = \frac{1}{4}\left|\frac{1}{2}\operatorname{Tr}\bigl[\rho[A,B]\bigr] - \frac{1}{2}\overline{\operatorname{Tr}\bigl[\rho[A,B]\bigr]}\right|^2 \\ &\quad = \frac{1}{4}\left|\frac{1}{2}\operatorname{Tr}\bigl[\rho[A,B]\bigr] + \frac{1}{2}\operatorname{Tr}\bigl[\rho[A^*,B^*]\bigr]\right|^2 \\ &\quad = \frac{1}{4}\left|\operatorname{Tr}\left[\rho\frac{[A,B]+[A^*,B^*]}{2}\right]\right|^2 \\ &\quad = \frac{1}{4}\bigl|\operatorname{Tr}\bigl[\rho[A,B]^0\bigr]\bigr|^2.\end{aligned}$$

Corollary 2 (Dou-Du (1), (2)) *For $A, B \in M_n(\mathbb{C})$ and $\rho \in M_{n,+,1}(\mathbb{C})$,*

(1) $|V_\rho|(A)\cdot|V_\rho|(B) \geq |U_\rho|(A)\cdot|U_\rho|(B) \geq \frac{1}{4}|\operatorname{Tr}[\rho[A,B]]|^2$.
(2) $|V_\rho|(A)\cdot|V_\rho|(B) \geq \frac{1}{4}|\operatorname{Tr}[\rho\{A_0,B_0\}]|^2$.

Proof

(1) It is clear by Theorem 3.
(2) For $A, B \in M_n(\mathbb{C})$ and $f(x) = \frac{x+1}{2}$, we define an inner product on $M_n(\mathbb{C})$ by

$$\langle A, B\rangle = \operatorname{Tr}\bigl[A_0^* m_f(L_\rho, R_\rho) B_0\bigr].$$

By Schwarz's inequality we have

$$\bigl|\langle A,B\rangle\bigr|^2 \leq \langle A,A\rangle\langle B,B\rangle.$$

Then we have

$$\begin{aligned}\bigl|\langle A,B\rangle\bigr|^2 &= \left|\operatorname{Tr}\left[A_0^*\frac{L_\rho + R_\rho}{2}B_0\right]\right|^2 \\ &= \left|\frac{1}{2}\operatorname{Tr}\bigl[A_0^*\rho B_0\bigr] + \frac{1}{2}\operatorname{Tr}\bigl[A_0^* B_0 \rho\bigr]\right|^2\end{aligned}$$

$$= \left|\frac{1}{2}\operatorname{Tr}\left[\rho B_0 A_0^*\right] + \frac{1}{2}\operatorname{Tr}\left[\rho A_0^* B_0\right]\right|^2$$
$$= \frac{1}{4}\left|\operatorname{Tr}\left[\rho\left\{A_0^*, B_0\right\}\right]\right|^2$$

and

$$\begin{aligned}\langle A, A\rangle &= \operatorname{Tr}\left[A_0^* \frac{L_\rho + R_\rho}{2} A_0\right] \\ &= \frac{1}{2}\operatorname{Tr}\left[A_0^* \rho A_0\right] + \frac{1}{2}\operatorname{Tr}\left[A_0^* A_0 \rho\right] \\ &= \frac{1}{2}\operatorname{Tr}\left[\rho A_0 A_0^*\right] + \frac{1}{2}\operatorname{Tr}\left[\rho A_0^* A_0\right] \\ &= \frac{1}{2}\left|\operatorname{Var}_\rho^0\right|(A).\end{aligned}$$

Then

$$\left|\operatorname{Tr}\left[\rho\left\{A_0^*, B_0\right\}\right]\right|^2 \leq \left|\operatorname{Var}_\rho^0\right|(A) \cdot \left|\operatorname{Var}_\rho^0\right|(B).$$

By taking A^* in place of A

$$\left|\operatorname{Tr}\left[\rho\{A_0, B_0\}\right]\right|^2 \leq \left|\operatorname{Var}_\rho^0\right|\left(A^*\right) \cdot \left|\operatorname{Var}_\rho^0\right|(B).$$

Since $|\operatorname{Var}_\rho^0|(A) = |\operatorname{Var}_\rho^0|(A^*)$, we have the result. □

5 Remark

Dou-Du's result (3) For $A, B \in M_n(\mathbb{C})$ and $\rho \in M_{n,+,1}(\mathbb{C})$,

$$\left|\operatorname{Var}_\rho^0\right|(A) \cdot \left|\operatorname{Var}_\rho^0\right|(B) \geq \frac{1}{4}\left|\operatorname{Tr}\left[\rho[A,B]^0\right]\right|^2 + \frac{1}{4}\left|\operatorname{Tr}\left[\rho\{A_0, B_0\}^0\right]\right|^2. \tag{6}$$

Corollary 3 *For $A, B \in M_n(\mathbb{C})$ and $\rho \in M_{n,+,1}(\mathbb{C})$,*

$$\left|\operatorname{Var}_\rho^0\right|(A) \cdot \left|\operatorname{Var}_\rho^0\right|(B) \geq |U_\rho|(A) \cdot |U_\rho|(B) \geq \frac{1}{4}\left|\operatorname{Tr}\left[\rho[A,B]\right]\right|^2. \tag{7}$$

We cannot compare the RHS of (6) *with the RHS of* (7). *In fact, let*

$$\rho = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}, \qquad A = \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}.$$

Since the RHS of (6) = 0 *and the RHS of* (7) = $\frac{9}{16}$, *we have the RHS of* (6) < *the RHS of* (7). *On the other hand let*

$$\rho = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}, \qquad A = \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Since the RHS of (6) = 1 *and the RHS of* (7) = 0, *we have the RHS of* (6) > *the RHS of* (7).

6 Conclusion

The results (1), (4) of Dou-Du are given as corollaries of Theorem 3. Result (2) is proved by Schwarz's inequality. Also is shown, that (3) cannot be compared with our result.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
This work was carried out in collaboration between all authors. Example 2 and the comparison between (6) and (7) were given by KS. With the exception of them, the proofs of all results were given by KY. All authors have contributed to, checked, and approved the manuscript.

Acknowledgements
The first author (KY) was partially supported by JSPS KAKENHI Grant Number 26400119.

Received: 29 September 2015 Accepted: 13 November 2015 Published online: 03 December 2015

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