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Weight Distribution for Non-binary Cluster LDPC Code Ensemble*

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SUMMARY This paper derives the average symbol and bit weight distributions for the irregular non-binary cluster low-density parity-check (LDPC) code ensembles. Moreover, we give the exponential growth rates of the average weight distributions in the limit of large code length. We show the condition that the typical minimum distances linearly grow with the code length.

key words: *non-binary cluster LDPC code, weight distribution, exponential growth rate*

1. Introduction

Gallager invented low-density parity-check (LDPC) codes [1]. Due to the sparseness of the parity check matrices, LDPC codes are efficiently decoded by the belief propagation (BP) decoder. Optimized LDPC codes exhibit performance very close to the Shannon limit [2]. Davey and MacKay [3] have found that non-binary LDPC codes outperform binary ones.

The LDPC codes are defined by sparse parity check matrices or sparse Tanner graphs. For the non-binary LDPC codes, the Tanner graphs are represented by bipartite graphs with variable nodes, check nodes and labeled edges. The LDPC codes defined by Tanner graphs with the variable nodes of degree d_{v} and the check nodes of degree d_{c} are called $(d_{\mathrm{v}}, d_{\mathrm{c}})$-regular LDPC codes. It is empirically known that the best performance is achieved by $(2, d_{\mathrm{c}})$-regular non-binary LDPC codes for large order of Galois field [4].

Savin and Declercq proposed the non-binary cluster LDPC codes [5]. For the non-binary cluster LDPC code, each edge in the Tanner graphs is labeled by a *cluster* which is a full-rank $p \times r$ binary matrix, where $p \geq r$. In [5], Savin and Declercq showed that there exist expurgated $(2, d_{\mathrm{c}})$-regular non-binary cluster LDPC code ensembles whose minimum distances in terms of bit weight linearly grow with the code length.

Deriving the weight distribution is important to analyze the decoding performances for the linear codes. In particular, in the case for LDPC codes, the weight distribution gives a bound of decoding error probability under maximum likelihood decoding [6] and error floors under belief propagation decoding and maximum likelihood decoding [7], [8].

Studies on weight distribution for non-binary LDPC codes date back to [1]. Gallager derived the symbol-weight distribution of Gallager code ensemble defined over $\mathbb{Z}/q\mathbb{Z}$ [1]. Kasai et al. derived the average symbol and bit weight distributions and the exponential growth rates for the irregular non-binary LDPC code ensembles defined over Galois field $\mathbb{F}_q$, and showed that the normalized typical minimum distance does not monotonically grow with q [9]. Andriyanova et al. derived the bit weight distributions and the exponential growth rates for the regular non-binary LDPC code ensembles defined over Galois field and general linear group [10].

This paper assumes the *random* irregular non-binary cluster LDPC code ensembles. Firstly, we derive the average symbol and bit weight distributions for the irregular non-binary cluster LDPC code ensembles. Secondly, we show the exponential growth rates of average symbol and bit weight distributions in the limit of large code length. Finally, we show the condition that the typical minimum distances linearly grow with the code length.

The remainder of this paper is organized as follows: Sect. 2 defines the irregular non-binary cluster LDPC code ensembles. Section 3 derives the average weight distributions for the irregular non-binary LDPC code ensembles. Section 4 gives the exponential growth rates of the average weight distributions in the limit of large code length and shows some numerical examples for the exponential growth rates.

2. Preliminaries

In this section, we review non-binary cluster LDPC codes [5] and define the irregular non-binary cluster LDPC code ensembles. We introduce some notations used throughout this paper.

2.1 Non-binary Cluster LDPC Code

The LDPC codes are defined by sparse parity check matrices

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or sparse Tanner graphs. For the non-binary LDPC codes, the Tanner graphs are represented by bipartite graphs with variable nodes, check nodes and *labeled* edges.

For the non-binary cluster LDPC codes, each edge in the Tanner graphs is labeled by a *cluster* which is a full-rank $p \times r$ binary matrix, where $p \geq r$. Let $\mathbb{F}_2$ be the finite field of order 2. Note that the non-binary LDPC codes defined by Tanner graphs labeled by general linear group $\mathrm{GL}(p, \mathbb{F}_2)$ are special cases for the non-binary cluster LDPC codes with $p = r$.

We denote the cluster in the edge between the v-th variable node and the c-th check node, by $h_{c,v}$. For the cluster LDPC codes, r-bits are assigned to each variable node in the Tanner graphs. We refer to the r-bits assigned to the v-th variable node as *symbol* assigned to the v-th variable node, and denote it by $\boldsymbol{x}_v \in \mathbb{F}_2^r$.

For integers a, b, we denote the set of integers between a and b, as $[a; b]$. More precisely, we define

$$[a; b] := \begin{cases} \{n \in \mathbb{N} \mid a \leq n \leq b\}, & a \leq b, \\ \emptyset = \{\}, & a > b. \end{cases}$$

The non-binary cluster LDPC code defined by a Tanner graph G is given as follows:

$$C(\mathsf{G}) = \{(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \in (\mathbb{F}_2^r)^N \mid \textstyle\sum_{v \in \mathcal{N}_c(c)} h_{c,v} \boldsymbol{x}_i^T = \mathbf{0}^T \in \mathbb{F}_2^p \ \ \forall c \in [1; M]\},$$

where $\mathcal{N}_c(c)$ represents the set of indexes of the variable nodes adjacent to the c-th check node. Note that N is called symbol code length and the bit code length n is given by rN.

2.2 Irregular Non-binary Cluster LDPC Code Ensemble

Let $\mathcal{L}$ and $\mathcal{R}$ be the sets of degrees of the variable nodes and the check nodes, respectively. Irregular non-binary cluster LDPC codes are characterized with the number of variable nodes N, the size of cluster p, r and a pair of *degree distributions*, $\lambda(x) = \sum_{i \in \mathcal{L}} \lambda_i x^{i-1}$ and $\rho(x) = \sum_{i \in \mathcal{R}} \rho_i x^{i-1}$, where λ_i and ρ_i are the fractions of the edges connected to the variable nodes and the check nodes of degree i, respectively.

The total number of the edges in the Tanner graph is

$$E := N / \textstyle\int_0^1 \lambda(x) dx.$$

The number of check node M is given by

$$M = \Big(\textstyle\int_0^1 \rho(x) dx / \int_0^1 \lambda(x) dx\Big) N =: \kappa N.$$

Let L_i and R_j be the fraction of the variable nodes of degree i and the check nodes of degree j, respectively, i.e.,

$$L_i := \lambda_i / \Big(i \textstyle\int_0^1 \lambda(x) dx\Big), \quad R_j := \rho_j / \Big(j \int_0^1 \rho(x) dx\Big).$$

The design rate is given as follows:

$$1 - \kappa p / r.$$

Assume that we are given the number of variable nodes N, the size of the clusters p, r and the degree distribution pair (λ, ρ). An irregular non-binary cluster LDPC code ensemble $\mathcal{G}(N, p, r, \lambda, \rho)$ is defined as the following way. There exist $L_i N$ variable nodes of degree i and $R_j M$ check nodes of degree j. A node of degree i has i sockets for its connected edges. Consider a permutation π on the number of edges. Join the i-th socket on the variable node side to the $\pi(i)$-th socket on the check node side. The bipartite graphs are chosen with equal probability from all the permutations on the number of edges. Each cluster in the edges is chosen a full-rank $p \times r$ binary matrix with equal probability.

3. Weight Distribution for Non-binary Cluster LDPC Code

In this section, we derive the average symbol and bit weight distributions for the irregular non-binary cluster LDPC code ensemble $\mathcal{G}(N, p, r, \lambda, \rho)$.

We denote the r-bit representation of $\boldsymbol{x}_i \in \mathbb{F}_2^r$, by $(x_{i,1}, \ldots, x_{i,r})$. For a given codeword $\boldsymbol{x} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N)$, we denote the symbol and bit weight of $\boldsymbol{x}$, by $w(\boldsymbol{x})$ and $w_{\mathrm{b}}(\boldsymbol{x})$. More precisely, we define

$$w(\boldsymbol{x}) := |\{i \in [1; N] \mid \boldsymbol{x}_i \neq \mathbf{0}\}|,$$
$$w_{\mathrm{b}}(\boldsymbol{x}) := |\{(i, j) \in [1; N] \times [1; r] \mid x_{i,j} \neq 0\}|.$$

For a given Tanner graph G, let $A^{\mathsf{G}}(\ell)$ (resp. $A_{\mathrm{b}}^{\mathsf{G}}(\ell)$) be the number of codewords of symbol (resp. bit) weight ℓ in $C(\mathsf{G})$, i.e.,

$$A^{\mathsf{G}}(\ell) = |\{\boldsymbol{x} \in C(\mathsf{G}) \mid w(\boldsymbol{x}) = \ell\}|,$$
$$A_{\mathrm{b}}^{\mathsf{G}}(\ell) = |\{\boldsymbol{x} \in C(\mathsf{G}) \mid w_{\mathrm{b}}(\boldsymbol{x}) = \ell\}|.$$

For the irregular non-binary cluster LDPC code ensemble $\mathcal{G}(N, r, p, \lambda, \rho)$, we denote the average number of codewords of symbol and bit weight ℓ, by $A(\ell)$ and $A_{\mathrm{b}}(\ell)$, respectively. Since each Tanner graph in the ensemble $\mathcal{G} = \mathcal{G}(N, r, p, \lambda, \rho)$ is chosen with uniform probability, the following equations hold:

$$A(\ell) = \textstyle\sum_{\mathsf{G} \in \mathcal{G}} A^{\mathsf{G}}(\ell) / |\mathcal{G}|, \quad A_{\mathrm{b}}(\ell) = \sum_{\mathsf{G} \in \mathcal{G}} A_{\mathrm{b}}^{\mathsf{G}}(\ell) / |\mathcal{G}|.$$

Since the number of full-rank binary $p \times r$ matrix is $\prod_{i=0}^{r-1} (2^p - 2^i)$, the number of codes in the ensemble $\mathcal{G} = \mathcal{G}(N, r, p, \lambda, \rho)$ is derived as

$$|\mathcal{G}| = E! \{\textstyle\prod_{i=0}^{r-1} (2^p - 2^i)\}^E. \tag{1}$$

3.1 Symbol Codeword Weight Distribution

At first, we will derive the average symbol weight distributions for the irregular non-binary cluster LDPC code ensembles.

Theorem 1: The average number $A(\ell)$ of codewords of symbol weight ℓ for the irregular non-binary cluster LDPC code ensemble $\mathcal{G}(N, p, r, \lambda, \rho)$ is

$$A(\ell) = \sum_{k=0}^{E} \frac{(2^r-1)^\ell \mathrm{coef}((P(s,t)Q(u))^N, s^\ell t^k u^k)}{\binom{E}{k}(2^p-1)^k}, \quad (2)$$

$$P(s,t) := \textstyle\prod_{i\in\mathcal{L}}(1+st^i)^{L_i}, \quad Q(u) := \prod_{j\in\mathcal{R}} f_j(u)^{\kappa R_j},$$

$$f_j(u) := \frac{1}{2^p}[\{1+(2^p-1)u\}^j + (2^p-1)(1-u)^j], \quad (3)$$

where $\mathrm{coef}(g(s,t,u), s^i t^j u^k)$ is the coefficient of the term $s^i t^j u^k$ of a polynomial $g(s,t,u)$.

proof: We follow a similar way in [9, Theorem 1].

We refer to an edge as *active* if the edge connects to a variable node to which a non-zero symbol is assigned. We will derive the average number of codewords $A(\ell, k)$ with symbol weight ℓ and the number of active edges k.

Firstly, we count the edge constellations satisfying the constraints of the variable nodes. Consider a variable node v of degree i. Define the parameter $\tilde{\ell}$ as 1 if a non-zero symbol is assigned to the variable node v, and otherwise 0. For a given $\tilde{\ell} \in [0;1]$ and $\tilde{k} \in [0;i]$, let $a_i(\tilde{\ell},\tilde{k})$ be the number of constellations of $\tilde{k}$ active edges which stem from a variable node of degree i. The i edges connected to v are active if and only if a non-zero symbol is assigned to the variable node v. Hence, we have

$$a_i(\tilde{\ell},\tilde{k}) = \begin{cases} 1, & \tilde{\ell}=0,\ \tilde{k}=0, \\ 2^r-1, & \tilde{\ell}=1,\ \tilde{k}=i, \\ 0, & \text{otherwise.} \end{cases}$$

The generating function of $a_i(\tilde{\ell},\tilde{k})$ is written as follows:

$$\sum_{\tilde{\ell},\tilde{k}} a_i(\tilde{\ell},\tilde{k}) s^{\tilde{\ell}} t^{\tilde{k}} = 1 + (2^r-1)st^i.$$

Since there are $L_i N$ variable nodes of degree i, for a given ℓ and k, the number of edge constellations satisfying constraints of the N variable nodes in the Tanner graph is given by

$$\mathrm{coef}\Big(\textstyle\prod_{i\in\mathcal{L}}\{1+(2^r-1)st^i\}^{L_iN}, s^\ell t^k\Big).$$

This equation is simplified as follows:

$$(2^r-1)^\ell \mathrm{coef}\Big(\textstyle\prod_{i\in\mathcal{L}}(1+st^i)^{L_iN}, s^\ell t^k\Big). \quad (4)$$

Secondly, we count the edge constellations satisfying all the constraints of the check nodes. Consider a check node c of degree j. Let $m_j(\tilde{k})$ be the number of constellations of the $\tilde{k}$ active edges satisfying a check node of degree j. In other words,

$$m_j(\tilde{k}) = \Big|\Big\{(\boldsymbol{y}_1,\boldsymbol{y}_2,\ldots,\boldsymbol{y}_j) \in (\mathbb{F}_2^p)^j \mid \textstyle\sum_{i=1}^{j}\boldsymbol{y}_i = \mathbf{0}, |\{i \mid \boldsymbol{y}_i \neq \mathbf{0}\}| = \tilde{k}\Big\}\Big|.$$

As in [1, Eq. (5.3)], $m_j(\tilde{k})$ is given as follows:

$$m_j(\tilde{k}) = \binom{j}{\tilde{k}}\frac{1}{2^p}\Big\{(2^p-1)^{\tilde{k}} + (-1)^{\tilde{k}}(2^p-1)\Big\}$$

The generating function of $m_j(\tilde{k})$ is written as follows:

$$f_j(u) = \sum_{\tilde{k}} m_j(\tilde{k})u^{\tilde{k}} = \frac{1}{2^p}\Big[\{1+(2^p-1)u\}^j + (2^p-1)(1-u)^j\Big].$$

Since there are $\kappa R_j N$ check nodes of degree j, for a given number of active edge k, the number of the constellations satisfying all the constraints of the check nodes is given as:

$$\mathrm{coef}\Big(\textstyle\prod_{j\in\mathcal{R}} f_j(u)^{\kappa R_j N}, u^k\Big). \quad (5)$$

Thirdly, we count the edge permutation and the number of clusters which satisfy the edge constraints. For a given number of active edge k, the number of permutations of edges is given by $k!(E-k)!$ and the number of clusters which satisfy the edge constraints is equal to $\Big\{\prod_{i=1}^{r-1}(2^p-2^i)\Big\}^k \Big\{\prod_{i=0}^{r-1}(2^p-2^i)\Big\}^{E-k}$. Hence, for a given number of active edge k, the number of choices for the permutation of edges and clusters is

$$k!(E-k)!\Big(\textstyle\prod_{i=1}^{r-1}(2^p-2^i)\Big)^k \Big(\prod_{i=0}^{r-1}(2^p-2^i)\Big)^{E-k}. \quad (6)$$

By multiplying Eqs. (4), (5) and (6), and dividing by Eq. (1), we obtain the average number of codewords $A(\ell,k)$ with symbol weight ℓ and the number of active edges k as

$$A(\ell,k) = \frac{(2^r-1)^\ell \mathrm{coef}((P(s,t)Q(u))^N, s^\ell t^k u^k)}{\binom{E}{k}(2^p-1)^k}.$$

Since $A(\ell) = \sum_{k=0}^{E} A(\ell,k)$, we get Theorem 1. □

Theorem 1 gives the following corollary.

Corollary 1: For the irregular non-binary cluster LDPC code ensemble $\mathcal{G}(N,p,r,\lambda,\rho)$, the following equations hold:

$$A(0) = 1,$$

$$A(N) = \frac{(2^r-1)^N \prod_{j\in\mathcal{R}}\{(2^p-1)^j + (-1)^j(2^p-1)\}^{\kappa R_j N}}{(2^p-1)^E (2^p)^{\kappa N}}.$$

3.2 Bit Codeword Weight Distribution

In a similar way to the average symbol weight distribution, we are able to derive the average bit weight distribution for the irregular non-binary cluster LDPC code ensemble $\mathcal{G}(N,r,p,\lambda,\rho)$. At first, we consider a variable node of degree i. For a given bit weight $\tilde{\ell} \in [0;r]$, let $a_{\mathrm{b},i}(\tilde{\ell},\tilde{k})$ be the number of constellations of $\tilde{k}$ active edges which stem from a variable node of degree i. From the definition of active edges, we have

$$a_{\mathrm{b},i}(\tilde{\ell},\tilde{k}) = \begin{cases} 1, & \tilde{\ell}=0, \tilde{k}=0, \\ \binom{r}{\tilde{\ell}}, & \tilde{\ell}\in[1;r], \tilde{k}=i, \\ 0, & \text{otherwise.} \end{cases}$$

The generating function of $a_{\mathrm{b},i}(\tilde{\ell},\tilde{k})$ is given as:

$$\sum_{\tilde{\ell},\tilde{k}} a_{\mathrm{b},i}(\tilde{\ell},\tilde{k}) s^{\tilde{\ell}} t^{\tilde{k}} = 1 + \{(1+s)^r - 1\} t^i.$$

Since there are $L_i N$ variable nodes of degree i, the number of constellations of k active edges satisfying constraints of the N variable nodes with bit weight ℓ is

$$\mathrm{coef}\Big(\textstyle\prod_{i\in\mathcal{L}}[1 + \{(1+s)^r - 1\}t^i]^{L_i N}, s^\ell t^k\Big).$$

By using this equation, in a similar way to proof of the average symbol weight distributions, we obtain the average number $A_\mathrm{b}(\ell)$ of codewords of bit weight ℓ as follows:

Theorem 2: Let $n = rN$ be the bit code length. Define $f_j(u)$ as in Eq. (3). The average number $A_\mathrm{b}(\ell)$ of codewords of bit weight ℓ for the irregular non-binary cluster LDPC code ensemble $\mathcal{G}(N, p, r, \lambda, \rho)$ is

$$A_\mathrm{b}(\ell) = \sum_{k=0}^{E} \frac{\mathrm{coef}((P_\mathrm{b}(s,t)Q_\mathrm{b}(u))^n, s^\ell t^k u^k)}{\binom{E}{k}(2^p - 1)^k},$$
$$P_\mathrm{b}(s,t) := \textstyle\prod_{i\in\mathcal{L}}[1 + \{(1+s)^r - 1\}t^i]^{L_i/r},$$
$$Q_\mathrm{b}(u) := \textstyle\prod_{j\in\mathcal{R}} f_j(u)^{\kappa R_j/r}.$$

Theorem 2 gives the following corollary.

Corollary 2: For the irregular non-binary cluster LDPC code ensemble $\mathcal{G}(N, p, r, \lambda, \rho)$, the following equations hold:

$$A_\mathrm{b}(0) = 1,$$
$$A_\mathrm{b}(n) = \frac{\prod_{j\in\mathcal{R}}\{(2^p-1)^j + (-1)^j(2^p-1)\}^{\kappa R_j N}}{(2^p-1)^E (2^p)^{\kappa N}}.$$

4. Asymptotic Analysis

In this section, we investigate the asymptotic behavior of the average symbol and bit weight distributions for the non-binary cluster LDPC code ensembles in the limit of large code length.

4.1 Growth Rate

We define

$$\gamma(\omega) := \lim_{N\to\infty} \frac{1}{N} \log_{2^r} A(\omega N) = \lim_{N\to\infty} \frac{1}{rN} \log_2 A(\omega N),$$
$$\gamma_\mathrm{b}(\omega_\mathrm{b}) := \lim_{n\to\infty} \frac{1}{n} \log_2 A_\mathrm{b}(\omega_\mathrm{b} n),$$

and refer to them as the *exponential growth rates* or simply *growth rates* of the average number of codewords in terms of symbol and bit weight, respectively. To simplify the notation, we denote $\log_2(\cdot)$ as $\log(\cdot)$.

With the growth rate, we are able to roughly estimate the average number of codewords of symbol weight ωN (resp. bit weight $\omega_\mathrm{b} n$) by

$$A(\omega N) \sim (2^r)^{\gamma(\omega)N}, \quad (\text{resp. } A_\mathrm{b}(\omega_\mathrm{b} n) \sim 2^{\gamma_\mathrm{b}(\omega_\mathrm{b})n},)$$

where $a_N \sim b_N$ means that $\lim_{N\to\infty} N^{-1} \log a_N/b_N = 0$.

4.1.1 Growth Rate of Symbol Weight Distribution

Theorem 3: Define $\omega = \ell/N$ and $\epsilon := E/N$. The growth rate $\gamma(\omega)$ of the average number of codewords of normalized symbol weight ω for the irregular non-binary cluster LDPC code ensemble $\mathcal{G}(N, p, r, \lambda, \rho)$ with sufficiently large N is given by, for $0 < \omega < 1$,

$$\begin{aligned}\gamma(\omega) &= \sup_{0<\beta<\epsilon} \inf_{s>0,t>0,u>0} \frac{1}{r}\Big[\log P(s,t) + \log Q(u) - \epsilon h\Big(\frac{\beta}{\epsilon}\Big) \\ &\qquad - \beta\log(tu(2^p-1)) - \omega\log\Big(\frac{s}{2^r-1}\Big)\Big] \\ &=: \sup_{0<\beta<\epsilon} \inf_{s>0,t>0,u>0} \gamma(\omega,\beta,s,t,u) \\ &=: \sup_{0<\beta<\epsilon} \gamma(\omega,\beta), \end{aligned} \tag{7}$$

where $h(x) := -x\log x - (1-x)\log(1-x)$ for $0 < x < 1$. A point (s,t,u) which achieves the infimum of the function $\gamma(\omega,\beta,s,t,u)$ is given in a solution of the following equations:

$$\omega = \frac{s}{P}\frac{\partial P}{\partial s} = \sum_{i\in\mathcal{L}} L_i \frac{st^i}{1+st^i}, \tag{8}$$

$$\beta = \frac{t}{P}\frac{\partial P}{\partial t} = \sum_{i\in\mathcal{L}} L_i \frac{ist^i}{1+st^i}, \tag{9}$$

$$\beta = \frac{u}{Q}\frac{\partial Q}{\partial u} = \sum_{j\in\mathcal{R}} \kappa R_j \frac{u}{f_j(u)}\frac{\partial f_j}{\partial u}(u), \tag{10}$$

where

$$\frac{\partial f_j}{\partial u}(u) = j\frac{2^p-1}{2^p}[\{1 + (2^p-1)u\}^{j-1} - (1-u)^{j-1}].$$

The value β which gives the supremum of $\gamma(\omega,\beta)$ needs to satisfy the stationary condition

$$\beta = (2^p-1)tu(\epsilon - \beta). \tag{11}$$

The proof of Theorem 3 is in Appendix.

From Corollary 1 and the definition of growth rate, we derive the growth rate of average number of codewords with $\omega = 0, 1$ as follows:

Corollary 3: For the irregular non-binary cluster LDPC code ensemble $\mathcal{G}(N, p, r, \lambda, \rho)$ in the limit of large symbol code length N, the following equations hold:

$$\gamma(0) = 0,$$
$$\begin{aligned}\gamma(1) &= r^{-1}[\log(2^r-1) - \epsilon\log(2^p-1) - \kappa p \\ &\qquad + \textstyle\sum_{j\in\mathcal{R}} \kappa R_j \log\{(2^p-1)^j + (-1)^j(2^p-1)\}].\end{aligned}$$

Moreover, by letting p, r tend to infinity with a fixed ratio, we have

$$\gamma(1) \to 1 - \kappa p/r,$$

namely, $\gamma(1)$ tends to the design rate.

For a fixed normalized symbol weight ω, the intermediate variables s, t, u and β are derived from Eqs. (8), (9), (10) and (11). Hence, the intermediate variables s, t, u and β are represented as functions of ω. Thus, we denote those intermediate variables, by $s(\omega), t(\omega), u(\omega), \beta(\omega)$.

The derivation of $\gamma(\omega)$ in terms of ω is simply expressed as the following lemma.

Lemma 1: For $s > 0$ such that Eqs. (8), (9), (10) and (11) hold, we have

$$\frac{d\gamma}{d\omega}(\omega) = -\frac{1}{r}\log\frac{s(\omega)}{2^r - 1}.$$

proof: We follow a similar way in [11]. For a fixed ω, we denote the value achieving the supremum of $\gamma(\omega, \beta)$ by $\hat{\beta}$ and the point achieving the infimum of $\gamma(\omega, \hat{\beta}, s, t, u)$ by $(\hat{s}, \hat{t}, \hat{u})$. Then, $\gamma(\omega) = \gamma(\omega, \hat{\beta}, \hat{s}, \hat{t}, \hat{u})$ holds and $\hat{\beta}, \hat{s}, \hat{t}, \hat{u}$ satisfy Eqs. (8), (9), (10) and (11). From Eq. (7), we have

$$\begin{aligned}\frac{d\gamma(\omega)}{d\omega} &= \frac{d}{d\omega}\gamma(\omega, \hat{\beta}, \hat{s}, \hat{t}, \hat{u}) \\ &= \frac{1}{r\ln 2}\left[\frac{1}{P}\frac{dP}{d\omega} - \frac{\omega}{\hat{s}}\frac{d\hat{s}}{d\omega} - \frac{\hat{\beta}}{\hat{t}}\frac{d\hat{t}}{d\omega} + \frac{1}{Q}\frac{dQ}{d\omega} - \frac{\hat{\beta}}{\hat{u}}\frac{d\hat{u}}{d\omega}\right. \\ &\quad \left. + \frac{d\hat{\beta}}{d\omega}\ln\frac{\epsilon - \hat{\beta}}{(2^p - 1)\hat{\beta}\hat{t}\hat{u}} - \ln\frac{\hat{s}}{(2^r - 1)}\right] \end{aligned} \tag{12}$$

From (8) and (9), we have

$$\frac{1}{P}\frac{dP}{d\omega} = \frac{1}{P}\frac{\partial P}{\partial \hat{s}}\frac{d\hat{s}}{d\omega} + \frac{1}{P}\frac{\partial P}{\partial \hat{t}}\frac{d\hat{t}}{d\omega} = \frac{\omega}{\hat{s}}\frac{d\hat{s}}{d\omega} + \frac{\hat{\beta}}{\hat{t}}\frac{d\hat{t}}{d\omega}.$$

In other words, the sum of the first three terms of Eq. (12) is equal to 0. Similarly, from (10), we have

$$\frac{1}{Q}\frac{dQ}{d\omega} = \frac{1}{Q}\frac{\partial Q}{\partial \hat{u}}\frac{d\hat{u}}{d\omega} = \frac{\hat{\beta}}{\hat{u}}\frac{d\hat{u}}{d\omega},$$

i.e., the sum of forth and fifth terms of Eq. (12) is equal to 0. From (11), we see that the sixth term of Eq. (12) is equal to 0. This concludes the proof. □

4.1.2 Growth Rate of Bit Weight Distribution

In a similar way to symbol weight, we are able to derive the growth rate for the average number of codewords in terms of bit weight. Hence, we omit the proofs in this section.

Theorem 4: Define $\omega_b = \ell/n$ and $\epsilon_b := E/n$. The growth rate $\gamma_b(\omega_b)$ of the average number of codewords of normalized bit weight ω_b for the irregular non-binary cluster LDPC code ensemble $\mathcal{G}(N, p, r, \lambda, \rho)$ with sufficiently large N is given by, for $0 < \omega_b < 1$,

$$\begin{aligned}\gamma_b(\omega_b) &= \sup_{0<\beta_b<\epsilon_b}\inf_{s>0,t>0,u>0}\Big[\log P_b(s,t) + \log Q_b(u) \\ &\quad - \epsilon_b h\Big(\frac{\beta_b}{\epsilon_b}\Big) - \beta_b\log(tu(2^p - 1)) - \omega_b\log s\Big] \\ &=: \sup_{0<\beta_b<\epsilon_b}\inf_{s>0,t>0,u>0}\gamma_b(\omega_b, \beta_b, s, t, u) \\ &=: \sup_{0<\beta_b<\epsilon_b}\gamma_b(\omega_b, \beta_b).\end{aligned}$$

A point (s, t, u) which achieves the infimum of the function $\gamma_b(\omega_b, \beta_b, s, t, u)$ is given in a solution of the following equations:

$$\omega_b = \frac{s}{P_b}\frac{\partial P_b}{\partial s} = \sum_{i\in\mathcal{L}} L_i\frac{(1+s)^{r-1}st^i}{1 + \{(1+s)^r - 1\}t^i}, \tag{13}$$

$$\beta_b = \frac{t}{P_b}\frac{\partial P_b}{\partial t} = \sum_{i\in\mathcal{L}}\frac{L_i}{r}\frac{i\{(1+s)^r - 1\}t^i}{1 + \{(1+s)^r - 1\}t^i}, \tag{14}$$

$$\beta_b = \frac{u}{Q_b}\frac{\partial Q_b}{\partial u} = \sum_{j\in\mathcal{R}}\frac{\kappa R_j}{r}\frac{u}{f_j(u)}\frac{\partial f_j(u)}{\partial u} \tag{15}$$

The value β_b which gives the supremum of $\gamma_b(\omega_b, \beta_b)$ needs to satisfy the stationary condition

$$\beta_b = (2^p - 1)tu(\epsilon_b - \beta_b).$$

Corollary 4: For the irregular non-binary cluster LDPC code ensemble $\mathcal{G}(N, p, r, \lambda, \rho)$ in the limit of large bit code length n, the following equations hold:

$$\begin{aligned}\gamma_b(0) &= 0, \\ \gamma_b(1) &= -\epsilon_b\log(2^p - 1) - \kappa\frac{p}{r} \\ &\quad + \sum_{j\in\mathcal{R}}\frac{\kappa R_j}{r}\log\{(2^p - 1)^j + (-1)^j(2^p - 1)\}.\end{aligned}$$

Moreover, by letting p, r tend to infinity with fixed ratio, we have

$$\gamma_b(1) \to -\kappa p/r.$$

Lemma 2: For $s > 0$ such that Eqs. (13), (14) and (15) hold, we have

$$\frac{d\gamma_b}{d\omega_b}(\omega_b) = -\log s(\omega_b).$$

4.2 Analysis of Small Weight Codeword

In this section, we investigate the growth rate of the average number of codewords of symbol and bit weight with small ω. We denote the *right-hand limit* of f at x, by $\lim_{t\searrow x} f(t)$.

Theorem 5: For the irregular non-binary cluster LDPC code ensemble $\mathcal{G}(N, p, r, \lambda, \rho)$ with $\lambda_2 > 0$, the growth rate $\gamma(\omega)$ of the average number of codewords in terms of symbol weight, in the limit of large symbol code length for small ω, is given by

$$\gamma(\omega) = -\frac{\omega}{r}\log\Big[\frac{2^p - 1}{(2^r - 1)\lambda'(0)\rho'(1)}\Big] + o(\omega), \tag{16}$$

where $f(x) = o(g(x))$ means $\lim_{x\searrow 0}\left|\frac{f(x)}{g(x)}\right| = 0$ and where $\lambda'(0)\rho'(1) = \lambda_2\sum_{j\in\mathcal{R}}(j-1)\rho_j$.

proof: Note that for $\omega > 0$,

$$\gamma(\omega) = \gamma(0) + \omega \frac{d^+\gamma}{d\omega}(0) + o(\omega), \tag{17}$$

where

$$\frac{d^+\gamma}{d\omega}(0) := \lim_{\omega\searrow 0} \frac{\gamma(\omega) - \gamma(0)}{\omega} = \lim_{\omega\searrow 0} \frac{d\gamma}{d\omega}(\omega).$$

From Corollary 3, we have $\gamma(0) = 0$. Hence, we will calculate $\lim_{\omega\searrow 0} \frac{d\gamma}{d\omega}(\omega)$. From Lemma 1, we have

$$\lim_{\omega\searrow 0} \frac{d\gamma}{d\omega}(\omega) = -\frac{1}{r} \lim_{\omega\searrow 0} \log \frac{s(\omega)}{2^r - 1}. \tag{18}$$

Recall that $s(\omega)$ satisfies Eqs. (8), (9), (10) and (11). From Eq. (8), for $\omega \searrow 0$, it holds that $st^i \searrow 0$ for $i \in \mathcal{L}$. By using this and Eq. (9), we have $\beta \searrow 0$. Notice that

$$f_j(u) = 1 + \binom{j}{2}(2^p - 1)u^2 + o(u^2). \tag{19}$$

By combining Eqs. (10) and (19), and $\beta \searrow 0$, we get

$$\beta = \epsilon\rho'(1)(2^p - 1)u^2 + o(u^2).$$

Substitution of this equation into Eq. (11) yields

$$t = \rho'(1)u + o(u). \tag{20}$$

The combination of this equation and $u \searrow 0$ gives $t \searrow 0$. Since $t \searrow 0$ and $\lambda_2 > 0$, from Eq. (9), we get

$$\beta = \epsilon\lambda_2 st^2 + o(t^2).$$

Substituting this equation into Eq. (11), we have

$$u = \frac{1}{2^p - 1}\lambda_2 st + o(t). \tag{21}$$

Combining Eqs. (20) and (21), we have for $\omega \searrow 0$

$$s(\omega) = (2^p - 1)\frac{1}{\lambda'(0)\rho'(1)}.$$

Thus, from Eq. (18), we obtain

$$\lim_{\omega\searrow 0} \frac{d\gamma}{d\omega}(\omega) = \frac{1}{r} \log\left[\frac{2^r - 1}{2^p - 1}\lambda'(0)\rho'(1)\right].$$

From this equation and Eq. (17), we obtain Theorem 5. □

Similarly, the growth rate of the average number of codewords in terms of bit weight with small weight ω_{b} is given in the following theorem.

Theorem 6: For the irregular non-binary cluster LDPC code ensemble $\mathcal{G}(N, p, r, \lambda, \rho)$ with $\lambda_2 > 0$, the growth rate $\gamma_{\mathrm{b}}(\omega_{\mathrm{b}})$ of the average number of codewords in terms of bit weight, in the limit of large bit code length for small ω_{b}, is given by

$$\gamma_{\mathrm{b}}(\omega_{\mathrm{b}}) = -\omega_{\mathrm{b}} \log\left[\left(\frac{2^p - 1}{\lambda'(0)\rho'(1)} + 1\right)^{1/r} - 1\right] + o(\omega_{\mathrm{b}}). \tag{22}$$

We define

$$\delta^* := \inf\{\omega > 0 \mid \gamma(\omega) \geq 0\},$$
$$\delta^*_{\mathrm{b}} := \inf\{\omega_{\mathrm{b}} > 0 \mid \gamma_{\mathrm{b}}(\omega_{\mathrm{b}}) \geq 0\},$$

and refer to them as the *normalized typical minimum distance* in terms of symbol and bit weight, respectively. Recall that the average number of codewords of symbol weight ωN (resp. bit weight $\omega_{\mathrm{b}} n$) is approximated by $A(\omega N) \sim 2^{r\gamma(\omega)N}$ (resp. $A_{\mathrm{b}}(\omega_{\mathrm{b}} n) \sim 2^{\gamma_{\mathrm{b}}(\omega_{\mathrm{b}})n}$). Since $\gamma(\omega) < 0$ (resp. $\gamma_{\mathrm{b}}(\omega_{\mathrm{b}}) < 0$) for $\omega \in (0, \delta^*)$ (resp. for $\omega_{\mathrm{b}} \in (0, \delta^*_{\mathrm{b}})$), there are exponentially few codewords of symbol weight ωN (resp. bit weight $\omega_{\mathrm{b}} n$) for $\omega \in (0, \delta^*)$ (resp. for $\omega_{\mathrm{b}} \in (0, \delta^*_{\mathrm{b}})$).

Theorem 5 and 6 gives the following corollary.

Corollary 5: For the irregular non-binary cluster LDPC code ensemble $\mathcal{G}(N, p, r, \lambda, \rho)$ with sufficiently large N, the normalized typical minimum distances δ^* and δ^*_{b} in terms of symbol and bit weight, respectively, are strictly positive if

$$\lambda'(0)\rho'(1) < \frac{2^p - 1}{2^r - 1}. \tag{23}$$

Moreover, $\delta^* = 0$ and $\delta^*_{\mathrm{b}} = 0$ if

$$\lambda'(0)\rho'(1) > \frac{2^p - 1}{2^r - 1}.$$

proof: At first, we derive a sufficient condition for $\delta^*_{\mathrm{b}} > 0$. The normalized typical minimum distance δ^*_{b} is strictly positive if $\mathrm{coef}(\gamma_{\mathrm{b}}(\omega_{\mathrm{b}}), \omega_{\mathrm{b}}) < 0$ for small ω_{b}. From Eq. (22), $\mathrm{coef}(\gamma_{\mathrm{b}}(\omega_{\mathrm{b}}), \omega_{\mathrm{b}}) < 0$ for small ω_{b} if and only if

$$\left(\frac{2^p - 1}{\lambda'(0)\rho'(1)} + 1\right)^{1/r} - 1 > 1 \iff \frac{2^p - 1}{\lambda'(0)\rho'(1)} + 1 > 2^r.$$

This leads Eq. (23).

Secondly, we derive a necessary condition for $\delta^*_{\mathrm{b}} > 0$. If $\lambda'(0)\rho'(1) > (2^p - 1)/(2^r - 1)$, then $\gamma_{\mathrm{b}}(\omega_{\mathrm{b}}) > 0$ for small ω from Theorem 5. Hence, if $\lambda'(0)\rho'(1) > (2^p - 1)/(2^r - 1)$, then $\delta^*_{\mathrm{b}} = 0$.

Similarly, we are able to derive a necessary condition and a sufficient condition for $\delta^* > 0$ by using Theorem 5. □

Remark 1: For the non-binary LDPC code ensembles defined over finite field $\mathbb{F}_{2^p}$, the normalized typical minimum distances are 0 if $\lambda'(0)\rho'(1) > 1$ [9]. For the non-binary LDPC code ensembles defined by the parity check matrices over general linear group $\mathrm{GL}(p, \mathbb{F}_2)$, a sufficient condition that the normalized typical minimum distances are 0 is also $\lambda'(0)\rho'(1) > 1$ from Corollary 5 with $p = r$. Hence, we see that the typical minimum distances are 0 if we employ $(2,d_{\mathrm{c}})$-regular non-binary LDPC code ensembles defined by Galois fields and general linear groups.

On the other hand, in the case for the non-binary cluster LDPC code ensembles, a sufficient condition that the normalized typical minimum distances are strictly positive depends on not only $\lambda'(0)\rho'(1)$ but also the size of cluster p, r as in Corollary 5. Therefore, for arbitrary degree distribution pair (λ, ρ) (even for $(2, d_{\mathrm{c}})$-regular LDPC code), we are able to satisfy Eq. (23) with fixed ratio p, r.

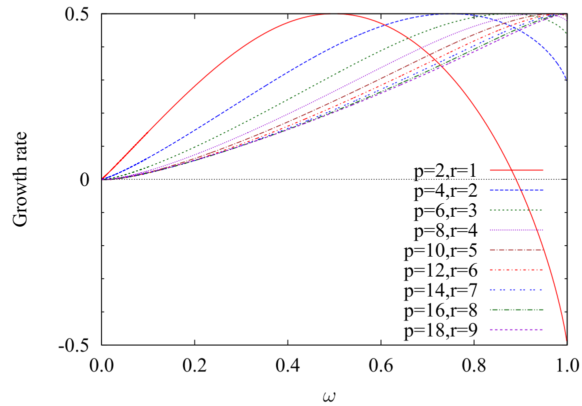

Fig. 1 Growth rates to the average symbol weight distributions for the (2, 8)-regular non-binary cluster LDPC code ensembles with the cluster size $(p, r) = (2, 1), (4, 2), \ldots, (18, 9)$.

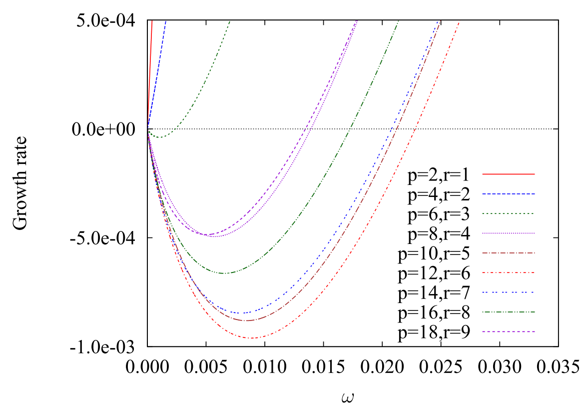

Fig. 2 Growth rates to the average symbol weight distributions for the (2, 8)-regular non-binary cluster LDPC code ensembles with the cluster size $(p, r) = (2, 1), (4, 2), \ldots, (18, 9)$.

Remark 2: For c-th check node, Savin and Declercq [5] defined the c-th *component code* by the null space of the following matrix:

$$H_c := \begin{pmatrix} h_{c,v_1} & h_{c,v_2} & \ldots & h_{c,v_k} \end{pmatrix},$$

where $v_1, v_2, \ldots, v_k$ are the elements in $\mathcal{N}_c(c)$. The *local minimum distance* is defined by $\min_{c \in [1;M]} \Delta_c$, where Δ_c is the minimum distance of the c-th component code. The expurgated ensemble $\mathcal{E}(N, p, r, d_c, \Delta)$ consists of the subset of the codes $\mathcal{G}(N, p, r, x, x^{d_c - 1})$ whose local minimum distance are Δ. In the above setting, Savin and Declercq [5] showed that there exist $\mathcal{E}(N, p, r, d_c, \Delta)$ whose minimum distance in terms of bit weight grows linearly with the code length.

On the other hand, Corollary 5 shows conditions that the typical minimum distances in terms of symbol and bit weight linearly grow with the code length for the *random* irregular non-binary cluster LDPC code ensembles.

4.3 Numerical Examples

In this section, we show some numerical examples of the growth rates for the cluster non-binary LDPC code ensembles. As mentioned in Remark 1, we are able to obtain the $(2,d_c)$-regular non-binary cluster LDPC code ensemble with strictly positive normalized typical minimum distance. In this section, to confirm the above, we employ the (2, 8)-regular non-binary cluster LDPC code ensembles. To keep the design rate at half, we fix the ratio of the cluster size as $p/r = 2$.

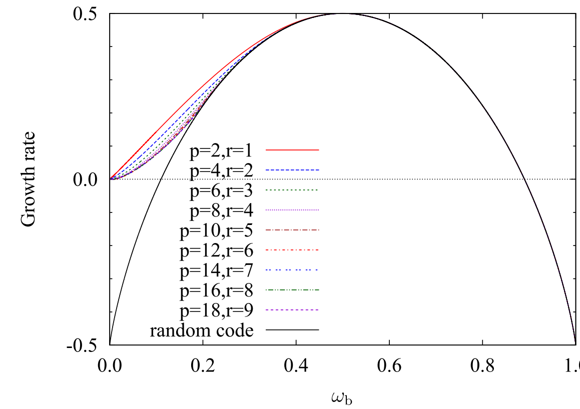

Fig. 3 Growth rates to the average bit weight distributions for the (2, 8)-regular non-binary cluster LDPC code ensembles with the cluster size $(p, r) = (2, 1), (4, 2), \ldots, (18, 9)$. The black solid curve (random code) gives the growth rate for the binary random code ensemble of rate 0.5.

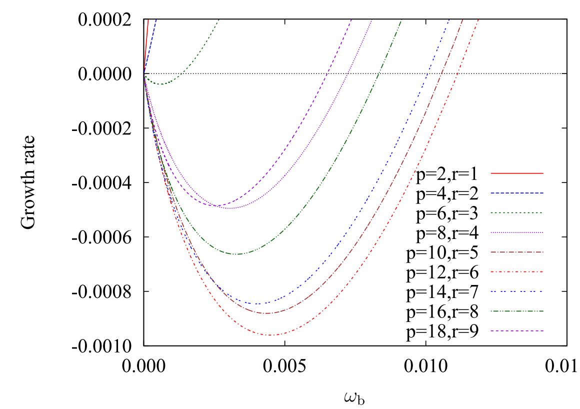

Fig. 4 Growth rates to the average bit weight distributions for the (2, 8)-regular non-binary cluster LDPC code ensembles with the cluster size $(p, r) = (2, 1), (4, 2), \ldots, (18, 9)$.

Figures 1 and 2 give the growth rates to the average symbol weight distributions for the cluster size $(p, r) = (2, 1), (4, 2), \ldots, (18, 9)$. As shown in Corollary 3, $\gamma(1)$ tends to the design rate 0.5. From Fig. 2, we see that the slopes of the growth rates at $\omega = 0$ are negative and the normalized typical minimum distance δ^* is strictly positive for $(p, r) = (6, 3), (8, 4), \ldots, (18, 9)$. This confirms Corollary 5.

Figures 3 and 4 give the growth rates to the average bit weight distributions for the cluster size $(p, r) = (2, 1), (4, 2), \ldots, (18, 9)$. The black solid curve in Fig. 3 shows the growth rate of the binary random code ensemble of rate 0.5. As shown in Corollary 4, $\gamma_b(1)$ tends to -0.5. Moreover, we see that the curves in $\omega_b > 1/2$ converge to

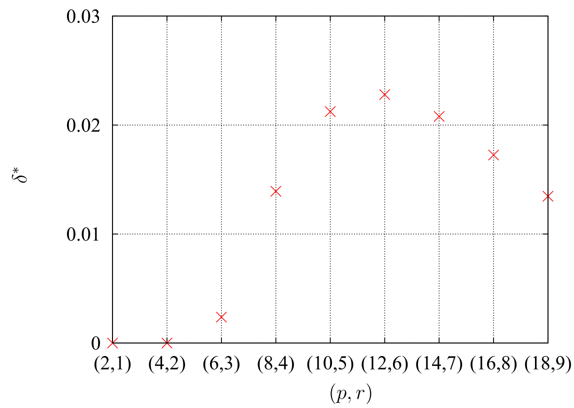

Fig. 5 The normalized typical minimum distance δ^* of the symbol weight distribution for the (2,8)-regular non-binary cluster LDPC code ensemble with the cluster size $(p, r) = (2, 1), (4, 2), \dots, (18, 9)$.

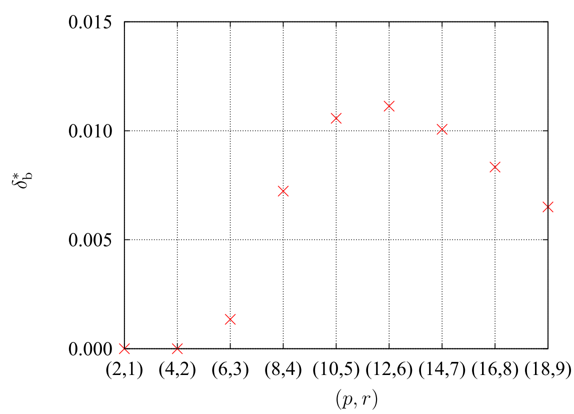

Fig. 6 The normalized typical minimum distance δ^*_b of the bit weight distribution for the (2,8)-regular non-binary cluster LDPC code ensemble with the cluster size $(p, r) = (2, 1), (4, 2), \dots, (18, 9)$.

the growth rate of the binary random code ensemble. From Fig. 4, we see that the slopes of the growth rates at $\omega_b = 0$ are negative and the normalized typical minimum distance δ^*_b is strictly positive for $(p, r) = (6, 3), (8, 4), \dots, (18, 9)$. This confirms Corollary 5.

Figures 5 and 6 give the normalized typical minimum distance δ^* and δ^*_b of the symbol and bit weight distribution, respectively, for the cluster size $(p, r) = (2, 1), (4, 2), \dots, (18, 9)$. From Figs. 5 and 6, we see that the normalized typical minimum distances δ^* and δ^*_b do not monotonically increase with the size of cluster (p, r). In this case, the normalized typical minimum distances δ^*, δ^*_b have the local maximum at $(p, r) = (12, 6)$.

5. Conclusion

This paper has derived the average symbol and bit weight distributions for the irregular non-binary cluster LDPC code ensembles. Moreover, we have given the exponential growth rates of the average symbol and bit weight distributions in the limit of large code length. Furthermore, we have shown a condition that the typical minimum distances linearly grow with the code length.

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Appendix: Proof of Theorem 3

To prove Theorem 3, we employ the following lemma.

Lemma 3: [12, Theorem 2] Let $p(x_1, x_2)$ be a multivariate polynomial with non-negative coefficients. Let $\alpha_1 > 0$ and $\alpha_2 > 0$ be some rational numbers and let n_i be the series of all indexes j such that $\mathrm{coef}(p(x_1, x_2)^j, x_1^{\alpha_1 j} x_2^{\alpha_2 j}) \neq 0$. Then

$$n_i^{-1} \log \mathrm{coef}(p(x_1, x_2)^{n_i}, (x_1^{\alpha_1} x_2^{\alpha_2})^{n_i}) \leq \inf_{x_1, x_2 > 0} \{\log p(x_1, x_2) - \alpha_1 \log x_1 - \alpha_2 \log x_2\}, \quad \text{(A· 1)}$$

and

$$\lim_{i \to \infty} n_i^{-1} \log \mathrm{coef}(p(x_1, x_2)^{n_i}, (x_1^{\alpha_1} x_2^{\alpha_2})^{n_i}) = \inf_{x_1, x_2 > 0} \{\log p(x_1, x_2) - \alpha_1 \log x_1 - \alpha_2 \log x_2\}. \quad \text{(A· 2)}$$

A point (x_1, x_2) achieves the minimum of the function $p(x_1, x_2)/(x_1^{\alpha_1} x_2^{\alpha_2})$, if and only if it satisfies the following equations for $k = 1, 2$:

$$x_k \frac{\partial p}{\partial x_k}(x_1, x_2) = \alpha_k p(x_1, x_2).$$

proof of Theorem 3: Since the number of terms in Eq. (2) is equal to $E + 1$, we get

$$\sup_{k\in[0;E]} A(\ell, k) \leq A(\ell) \leq (E + 1) \sup_{k\in[0;E]} A(\ell, k).$$

Hence, we have

$$\lim_{N\to\infty} \frac{1}{rN} \log A(\ell) = \lim_{N\to\infty} \frac{1}{rN} \sup_{k\in[0;E]} \log A(\ell, k). \qquad (\text{A}\cdot 3)$$

From this equation, we have for $0 < \omega < 1$

$$\gamma(\omega) = \lim_{N\to\infty} \sup_{0<\beta<\epsilon} \frac{1}{rN} \log A(\omega N, \beta N). \qquad (\text{A}\cdot 4)$$

At first, we derive an upper bound of $\gamma(\omega)$. Since $\log \binom{n}{k} \geq nh(k/n) - \log(n + 1)$, we get

$$-\frac{1}{rN} \log \binom{\epsilon N}{\beta N} \leq \frac{1}{rN} \log(\epsilon N + 1) - \frac{\epsilon}{r} h(\beta/\epsilon).$$

By combining this inequality and Eq. (A·1), we obtain

$$\begin{aligned}
&\sup_{0<\beta<\epsilon} \frac{1}{rN} \log A(\omega N, \beta N) \\
&\leq \sup_{0<\beta<\epsilon} r^{-1}\Big[\omega \log(2^r - 1) - \epsilon h(\beta/\epsilon) - \beta \log(2^p - 1) \\
&\qquad + \inf_{s,t>0} \{\log P(s, t) - \omega \log s - \beta \log t\} \\
&\qquad + \inf_{u>0} \{\log Q(u) - \beta \log u\}\Big] + (rN)^{-1} \log(\epsilon N + 1) \\
&= \sup_{0<\beta<\epsilon} \gamma(\omega, \beta) + (rN)^{-1} \log(\epsilon N + 1).
\end{aligned}$$

This upper bound yields

$$\begin{aligned}
\gamma(\omega) &= \lim_{N\to\infty} \sup_{0<\beta<\epsilon} \frac{1}{rN} \log A(\omega N, \beta N) \\
&\leq \sup_{0<\beta<\epsilon} \gamma(\omega, \beta). \qquad (\text{A}\cdot 5)
\end{aligned}$$

Secondly, we derive a lower bound of $\gamma(\omega)$. Since $\lim_{i\to\infty} \sup_{x\in X} f_i(x) \geq \sup_{x\in X} \lim_{i\to\infty} f_i(x)$ for any sequence of functions $\{f_i(x)\}$ converging on X, Eq. (A·4) gives

$$\gamma(\omega) \geq \sup_{0<\beta<\epsilon} \lim_{N\to\infty} \frac{1}{rN} \log A(\omega N, \beta N).$$

From Eq. (A·2), the right-hand side of this inequality is equal to $\sup_{0<\beta<\epsilon} \gamma(\omega, \beta)$. Thus, we obtain

$$\gamma(\omega) \geq \sup_{0<\beta<\epsilon} \gamma(\omega, \beta). \qquad (\text{A}\cdot 6)$$

By combining Eqs. (A·5) and (A·6), we leads Eq.(7).

Lemma 3 derives a point (s, t, u) which achieves the infimum of the function $\gamma(\omega, \beta, s, t, u)$. Since value β which gives the supremum of $\gamma(\omega, \beta)$ needs to satisfy the stationary condition $\frac{\partial \gamma}{\partial \beta}(\omega, \beta) = 0$, we get Eq. (11). □

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