Analysis of Error Floors of Non-binary LDPC Codes over MBIOS Channel*

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SUMMARY   In this paper, we investigate the error floors of non-binary low-density parity-check (LDPC) codes transmitted over the memoryless binary-input output-symmetric (MBIOS) channels. We provide a necessary and sufficient condition for successful decoding of zigzag cycle codes over the MBIOS channel by the belief propagation decoder. We consider an expurgated ensemble of non-binary LDPC codes by using the above necessary and sufficient condition, and hence exhibit lower error floors. Finally, we show lower bounds of the error floors for the expurgated LDPC code ensembles over the MBIOS channels.
key words: non-binary LDPC code, error floor, MBIOS channel

1. Introduction

Gallager invented low-density parity-check (LDPC) codes [1]. Due to the sparseness of the parity check matrices, LDPC codes are efficiently decoded by the belief propagation (BP) decoder. Optimized LDPC codes can exhibit performance very close to the Shannon limit [2].

Davey and MacKay [3] have found that non-binary LDPC codes can outperform binary ones. In this paper, we consider the non-binary LDPC codes defined over the Galois field $\mathbb{F}_q$ with $q = 2^m$.

A non-binary LDPC code C over $\mathbb{F}_q$ is defined by the null space of a sparse $M \times N$ parity-check matrix $H = (h_{i,j})$ over $\mathbb{F}_q$:

$$C = \left\{ \boldsymbol{x} \in \mathbb{F}_q^N \mid H\boldsymbol{x} = \boldsymbol{0} \in \mathbb{F}_q^M \right\}.$$

The Tanner graph for a non-binary LDPC code is represented by a bipartite graph with variable nodes, check nodes and labeled edges. The v-th variable node and the c-th check node are connected with an edge labeled $h_{c,v} \in \mathbb{F}_q \setminus \{0\}$ if $h_{c,v} \neq 0$. The LDPC codes defined by Tanner graphs with the variable nodes of degree j and the check nodes of degree k are called (j, k)-regular LDPC codes. It is empirically known that $(2, k)$-regular non-binary LDPC codes exhibit good decoding performance among other LDPC codes for $q \geq 64$ [4]. However, this is not the case for $q < 64$. In this paper, we consider the irregular non-binary LDPC codes which contain variable nodes of degree two for the generality of code ensemble.

A zigzag cycle is a cycle such that the degrees of all the variable nodes in the cycle are two. In order to reduce the error floors of codes under maximum likelihood decoding, Poulliat et al. proposed *cycle cancellation* [5]. The cycle cancellation is a method to design the edge labels in zigzag cycles so that the corresponding submatrices are nonsingular. We see that from the simulation results [5] the resulting codes have lower error floors under BP decoding. However, it is found in our analyses that some zigzag cycles, even if their submatrices are nonsingular, degrade decoding performance. In this paper, we analyze a condition for successful decoding of zigzag cycles under BP decoding over the memoryless binary-input output-symmetric (MBIOS) channel. Based on this condition, we propose a design method of selecting labels so as to eliminate small zigzag cycles which degrade decoding performance.

In [6], we analyze the decoding erasure rate in the error floors of non-binary LDPC codes over the binary erasure channel (BEC) under BP decoding. In this paper, we analyze the error floors of non-binary LDPC codes over the MBIOS channel. First, we expurgate non-binary LDPC code ensembles. Next, we show lower bounds for the symbol error rates in the error floors of the expurgated LDPC code ensembles over the MBIOS channel. More precisely, those lower bounds are derived from the decoding errors caused by the zigzag cycles. Furthermore, simulation results show that the lower bounds on symbol error rates are tight for the expurgated ensembles constructed by our proposed method over the MBIOS channels.

This paper is organized as follows. In Sect. 2, we briefly review the BP decoder for non-binary LDPC codes. In Sect. 3, we clarify a necessary and sufficient condition for successful decoding of zigzag cycle codes over the MBIOS channels by the BP decoder and propose a design method to lower the error floor. In Sect. 4, we give lower bounds for the symbol error rates in the error floors for code ensembles constructed by our proposed method and show that the proposed method gives better performance than the cycle cancellation [5] and the method which uses both the cycle cancellation and the stopping set mitigation [5].

2. Preliminaries

In this section, we recall the BP decoder for the non-binary LDPC codes [3]. We introduce some notations used

throughout this paper.

2.1 Channel Model

Let α be a primitive element of $\mathbb{F}_{2^m}$. Once a primitive element of α is fixed, each element in $\mathbb{F}_{x^m}$ is given an m-bit representation [7, p. 110]. We denote the m-bit representation for $\gamma \in \mathbb{F}_{2^m}$, by $(\gamma_1, \gamma_2, \ldots, \gamma_m)$. Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_N)$ denote the codeword over $\mathbb{F}_{2^m}$. Since each symbol of $\mathbb{F}_{2^m}$ is given an m-bit representation, a codeword is represented as a binary codeword of length Nm, $\boldsymbol{x} = (x_{1,1}, x_{1,2}, \ldots, x_{N,m})$.

We denote the received word by $\boldsymbol{y} = (y_{1,1}, y_{1,2}, \ldots, y_{N,m})$. A channel is called *memoryless* binary-input channel if

$$p(\boldsymbol{y} \mid \boldsymbol{x}) = \prod_{i=1}^{N} \prod_{j=1}^{m} p(y_{i,j} \mid x_{i,j}).$$

It is convenient to transform the binary input alphabet $\{0, 1\}$ into $\{+1, -1\}$ by a binary phase shift keying (BPSK). With some abuse of notation, we make no distinction between $\{0, 1\}$ and $\{+1, -1\}$. A memoryless binary-input channel is called *output-symmetric* if

$$p(y \mid x) = p(-y \mid -x).$$

We assume the transmission over the MBIOS channel. The MBIOS channels are characterized by its L-density a [8]. Examples of the MBIOS channels include the BEC, the binary symmetric channel (BSC) and the additive white Gaussian noise (AWGN) channel.

2.2 BP Decoder for Non-binary LDPC Codes

BP decoding proceeds by sending messages along the edges in the Tanner graph. The messages arising in the BP decoder for LDPC codes over $\mathbb{F}_{2^m}$ are vectors of length 2^m. Let $\Psi_{v,c}^{(\ell)}$ be the message from the v-th variable node to the c-th check node at the ℓ-th iteration. Let $\Phi_{c,v}^{(\ell)}$ be the message from the c-th check node to the v-th variable node at the ℓ-th iteration.

2.2.1 Initialization

Set $\ell = 0$. Let N and M be the number of variable nodes and check nodes in a Tanner graph, respectively. For $v = 1, 2, \ldots, N$, let $C_v = (C_v(0), C_v(1), \ldots, C_v(\alpha^{2^m-2}))$ denote the initial message of the v-th variable node. For $\gamma \in \mathbb{F}_{2^m}$, the element of the initial message $C_v(\gamma)$ is given from the channel outputs as follows:

$$C_v(\gamma) = \prod_{i=1}^{m} \Pr(Y_{v,i} = y_{v,i} \mid X_{v,i} = \gamma_i).$$

Let $\mathcal{N}_{\mathsf{c}}(c)$ be the set of the positions of the variable nodes connecting to the c-th check node. Set for all $c = 1, 2, \ldots, M$ and $v \in \mathcal{N}_{\mathsf{c}}(c)$,

$$\Phi_{c,v}^{(0)} = (2^{-m}, 2^{-m}, \ldots, 2^{-m}).$$

2.2.2 Iteration

(1) Variable node action

Let $\mathcal{N}_{\mathsf{v}}(v)$ be the set of the positions of the check nodes connected to the v-th variable node. The message $\Psi_{v,c}^{(\ell)}$ is given by the component-wise multiplication of the initial message C_v and the incoming messages $\Phi_{c',v}^{(\ell)}$ from check nodes whose positions c' are in $\mathcal{N}_{\mathsf{v}}(v)$, i.e., for $x \in \mathbb{F}_{2^m}$

$$\Psi_{v,c}^{(\ell)}(x) = \frac{1}{\xi} C_v(x) \prod_{c' \in \mathcal{N}_{\mathsf{v}}(v) \setminus \{c\}} \Phi_{c',v}^{(\ell)}(x),$$

where ξ is the normalization factor such that $1 = \sum_{x \in \mathbb{F}_{2^m}} \Psi_{v,c}^{(\ell)}(x)$.

(2) Check node action

The convolution of two vectors Ψ_1 and Ψ_2 is given by

$$[\Psi_1 \oplus \Psi_2](x) = \sum_{y,z \in \mathbb{F}_{2^m} : x = y + z} \Psi_1(y)\Psi_2(z),$$

where $\sum_{y,z \in \mathbb{F}_{2^m} : x = y + z} \Psi_1(y)\Psi_2(z)$ is the sum of $\Psi_1(y)\Psi_2(z)$ over all $y, z \in \mathbb{F}_{2^m}$ such that $x = y + z$. To simplify the notation, we define $\bigoplus_{i \in \{1,2,\ldots,k\}} \Psi_i := \Psi_1 \oplus \Psi_2 \oplus \cdots \oplus \Psi_k$.

Let $h_{c,v}$ be the label of the edge adjacent to the c-th check node and the v-th variable node. The message $\Phi_{c,v}^{(\ell+1)}$ is given as, for $x \in \mathbb{F}_{2^m}$

$$\check{\Psi}_{v,c}^{(\ell)}(x) = \Psi_{v,c}^{(\ell)}\left(h_{c,v}^{-1}x\right),$$
$$\check{\Phi}_{c,v}^{(\ell+1)} = \bigoplus_{v' \in \mathcal{N}_{\mathsf{c}}(c) \setminus \{v\}} \check{\Psi}_{v',c}^{(\ell)},$$
$$\Phi_{c,v}^{(\ell+1)}(x) = \check{\Phi}_{c,v}^{(\ell+1)}(h_{c,v}x).$$

2.2.3 Decision

Define

$$\arg\max_{x \in \mathbb{F}_{2^m}} \Psi := \{x \in \mathbb{F}_{2^m} \mid \forall y \in \mathbb{F}_{2^m} : \Psi(x) \geq \Psi(y)\},$$

and for $x \in \mathbb{F}_{2^m}$

$$D_v^{(\ell)}(x) := \frac{1}{\xi} C_v(x) \prod_{c \in \mathcal{N}_{\mathsf{v}}(v)} \Phi_{c,v}^{(\ell)}(x),$$

where ξ is the normalization factor such that $1 = \sum_{x \in \mathbb{F}_{2^m}} D_v^{(\ell)}(x)$. For $v = 1, 2, \ldots, N$, let $\hat{x}_v^{(\ell)}$ be the decoding output of the v-th variable node. Define

$$\mathcal{D}_v^{(\ell)} := \arg\max_{x \in \mathbb{F}_{2^m}} D_v^{(\ell)}(x).$$

If $|\mathcal{D}_v^{(\ell)}| = 1$, the decoding output $\hat{x}_v^{(\ell)}$ is the element of $\mathcal{D}_v^{(\ell)}$. If $|\mathcal{D}_v^{(\ell)}| > 1$, the decoder chooses $\hat{x}_v^{(\ell)} \in \mathcal{D}_v^{(\ell)}$ with probability $1/|\mathcal{D}_v^{(\ell)}|$.

2.3 All-Zero Assumption and Defining Failure

For the MBIOS channels, we assume that all-zero codeword

is sent without loss of generality to analyze the decoding error rate [9].

The v-th symbol is *eventually correct* [10] if there exists L_v such that for all $\ell > L_v$, $\hat{x}_v^{(\ell)} = 0$. The symbol error rate is defined by the fraction of the symbols which are not eventually correct.

3. Condition for Successful Decoding over MBIOS Channel

A zigzag cycle is a cycle such that the degrees of all the variable nodes in the cycles are two. A zigzag cycle of weight s consists of s variable nodes of degree two. The zigzag cycle code is defined by a Tanner graph which forms a single zigzag cycle. Figure 1 shows a zigzag cycle code of symbol code length s. In this section, we give a condition for successful decoding of the zigzag cycle codes over the MBIOS channels under BP decoding.

We consider the zigzag cycle code of symbol code length s with labels $h_{1,1}, h_{1,2}, \ldots, h_{s,1}$ as shown in Fig. 1. We define $\gamma_i := h_{i,i}^{-1} h_{i,i+1}$ for $i = 1, 2, \ldots, s$ where $h_{s,s+1} := h_{s,1}$. Define $\beta := \prod_{i=1}^{s} \gamma_i$. We refer to the parameter β as *cycle parameter* [6]. The following theorem shows a necessary and sufficient condition for successful decoding of the zigzag cycle codes over the MBIOS channels by the BP decoder.

Theorem 1: Let σ be the order of β, i.e., let σ be the smallest positive integer such that $\beta^{\sigma} = 1$. We consider a zigzag cycle code of symbol code length s defined over $\mathbb{F}_{2^m}$ with the cycle parameter β over the MBIOS channel. In the limit of large ℓ, all the symbols in the zigzag cycle code are eventually correct under BP decoding if and only if for all $x \in A_\beta := \{\alpha^j \mid j = 0, 1, \ldots, \frac{2^m - 1}{\sigma} - 1\}$

$$\prod_{k=1}^{s} (C_k(0))^{\sigma} > \prod_{t=0}^{\sigma-1} \prod_{k=1}^{s} C_k \left(\beta^t x \prod_{j=1}^{k-1} \gamma_j \right).$$

Moreover, in the limit of large ℓ, no symbols in the zigzag cycle code are eventually correct under BP decoding if and only if there exists $x \in A_\beta := \{\alpha^j \mid j = 0, 1, \ldots, \frac{2^m - 1}{\sigma} - 1\}$ such that

$$\prod_{k=1}^{s} (C_k(0))^{\sigma} \le \prod_{t=0}^{\sigma-1} \prod_{k=1}^{s} C_k \left(\beta^t x \prod_{j=1}^{k-1} \gamma_j \right).$$

proof: First, we write the messages $D_v^{(\ell)}$ by the initial messages C_v for the zigzag cycle code of symbol code length s with the cycle parameter β. Let $\tilde{\Psi}_{v,c}^{(\ell)}$ be the *unnormalized* message from the v-th variable node to the c-th check node at the ℓ-th iteration. For all $x \in \mathbb{F}_{2^m}$ and $i = 1, 2, \ldots, s$, the
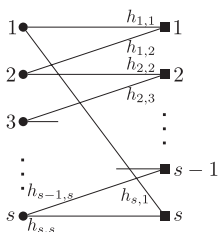

Fig. 1 A zigzag cycle code of symbol code length s.

unnormalized message for the zigzag cycle code of symbol code length s is written as follows:

$$\tilde{\Psi}_{i,i-1}^{(0)}(x) := C_i(x), \quad \tilde{\Psi}_{i,i-1}^{(\ell+1)}(x) := C_i(x)\tilde{\Psi}_{i+1,i}^{(\ell)}(\gamma_i^{-1} x),$$

$$\tilde{\Psi}_{i,i}^{(0)}(x) := C_i(x), \quad \tilde{\Psi}_{i,i}^{(\ell+1)}(x) := C_i(x)\tilde{\Psi}_{i-1,i-1}^{(\ell)}(\gamma_{i-1} x),$$

$$\tilde{D}_i^{(\ell+1)}(x) := C_i(x)\tilde{\Psi}_{i-1,i-1}^{(\ell)}(\gamma_{i-1} x)\tilde{\Psi}_{i+1,i}^{(\ell)}(\gamma_i^{-1} x),$$

where $\tilde{\Psi}_{0,0}^{(\ell)} = \tilde{\Psi}_{s,s}^{(\ell)}$, $\tilde{\Psi}_{1,0}^{(\ell)} = \tilde{\Psi}_{s+1,s}^{(\ell)} = \tilde{\Psi}_{1,s}^{(\ell)}$, $\tilde{\Psi}_{s+1,s+1}^{(\ell)} = \tilde{\Psi}_{1,1}^{(\ell)}$ and $\gamma_0 = \gamma_s$. Then, for zigzag cycle code, the messages $\Psi_{i,j}^{(\ell)}$ and $D_i^{(\ell)}$ are written as follows:

$$\Psi_{i,j}^{(\ell)}(x) = \frac{\tilde{\Psi}_{i,j}^{(\ell)}(x)}{\sum_{x' \in \mathbb{F}_{2^m}} \tilde{\Psi}_{i,j}^{(\ell)}(x')}, \quad D_i^{(\ell)}(x) = \frac{\tilde{D}_i^{(\ell)}(x)}{\sum_{x' \in \mathbb{F}_{2^m}} \tilde{D}_i^{(\ell)}(x')}.$$

From the definition, we have

$$\tilde{D}_i^{(\ell)}(x) = C_i(x) \prod_{k=1}^{\ell} \left\{ C_{i-k} \left(x \prod_{j=1}^{k} \gamma_{i-j} \right) C_{i+k} \left(x \prod_{j=0}^{k-1} \gamma_{i+j}^{-1} \right) \right\}, \tag{1}$$

where $C_{i+ns}(x) = C_i(x)$ and $\gamma_{i+ns} = \gamma_i$ for $n = 0, \pm 1, \ldots$. Define $\chi_i = \prod_{j=1}^{i-1} \gamma_j$ and

$$B(x) := \prod_{t=0}^{\sigma-1} \prod_{k=1}^{s} C_k \left(\beta^t x \prod_{j=k}^{s} \gamma_j \right).$$

From Eq. (1), we have for $i = 1, 2, \ldots, n$

$$\tilde{D}_i^{(\ell+s\sigma)}(x) = \{B(\chi_i x)\}^2 \tilde{D}_i^{(\ell)}(x).$$

By using this equation, we have

$$D_i^{(\ell_1 s\sigma + \ell_2)}(0)$$
$$= \frac{\tilde{D}_i^{(\ell_2)}(0)}{\tilde{D}_i^{(\ell_2)}(0) + \sum_{x \in A_\beta} \left\{ \frac{B(\chi_i x)}{B(0)} \right\}^{2\ell_1} \sum_{t=0}^{\sigma-1} \tilde{D}_i^{(\ell_2)}(x\beta^t)}.$$

If $B(0) > B(x)$ for all $x \in A_\beta = \{\alpha^j \mid j = 0, 1, \ldots, \frac{2^m - 1}{\sigma} - 1\}$, for all $i = 1, 2, \ldots, s$, we have $\lim_{\ell \to \infty} D_i^{(\ell)}(0) = 1$, i.e., the decoding is successful. If there exists $x \in A_\beta$ such that $B(0) < B(x)$, for all $i = 1, 2, \ldots, s$ we have $\lim_{\ell \to \infty} D_i^{(\ell)}(0) = 0$, i.e., no symbols are eventually correct.

Finally, we claim that no symbols are eventually correct, if there exists $x \in A_\beta$ such that $B(0) = B(x)$. Note that for all $t \ge 1$ and $i \in \{1, 2, \ldots, s\}$,

$$\tilde{D}_i^{(s\sigma t)}(\chi_i^{-1} x) = \{B(x)\}^{2t} C_i(\chi_i^{-1} x)$$

$$\tilde{D}_i^{(s\sigma t-1)}(\chi_i^{-1} x) = \{B(x)\}^{2t} \{C_i(\chi_i^{-1} x)\}^{-1}.$$

Hence for $t \ge 1$ and $i \in \{1, 2, \ldots, s\}$

$$\tilde{D}_i^{(s\sigma t)}(\chi_i^{-1} x)\tilde{D}_i^{(s\sigma t-1)}(\chi_i^{-1} x) = \{B(x)\}^{4t}$$
$$= \{B(0)\}^{4t}$$
$$= \tilde{D}_i^{(s\sigma t)}(0)\tilde{D}_i^{(s\sigma t-1)}(0). \tag{2}$$

The i-th symbol is eventually correct if there exist L such that $\tilde{D}_i^{(\ell)}(0) > \tilde{D}_i^{(\ell)}(x)$ for $\ell > L$ and $x \in \mathbb{F}_{2^m} \setminus \{0\}$. However, from Eq. (2), for all $i \in \{1, 2, \dots\}$, if $\tilde{D}_i^{(s\sigma t-1)}(0) > \tilde{D}_i^{(s\sigma t-1)}(\chi_i^{-1} x)$, then $\tilde{D}_i^{(s\sigma t)}(0) < \tilde{D}_i^{(s\sigma t)}(\chi_i^{-1} x)$. Thus, no symbols are eventually correct. This completes the proof. □

By Using Theorem 1, we have the following corollary.

Corollary 1: Let σ be the order of β. For a given channel output, if the zigzag cycle with cycle parameter β such that $\sigma \neq 2^m - 1$ is successfully decoded, then the zigzag cycle with cycle parameter β such that $\sigma = 2^m - 1$ is also successfully decoded.

proof: We consider zigzag cycle of symbol code length s. Since the channel output is given, we are able to fix the initial message C_i for $i = 1, 2, \dots, s$. From Theorem 1, if the zigzag cycle with cycle parameter β such that $\sigma \neq 2^m - 1$ is successfully decoded, then for all $x \in A_\beta = \{\alpha^j \mid j = 0, 1, \dots, \frac{2^m-1}{\sigma} - 1\}$

$$\prod_{k=1}^{s}(C_k(0))^\sigma > \prod_{t=0}^{\sigma-1}\prod_{k=1}^{s} C_k\left(\beta^t x \prod_{j=1}^{k-1} \gamma_j\right).$$

From the product of the above equation over all $x \in A_\beta$, we have

$$\prod_{x \in A_\beta}\prod_{k=1}^{s}(C_k(0))^\sigma$$
$$> \prod_{x \in A_\beta}\prod_{t=0}^{\sigma-1}\prod_{k=1}^{s} C_k\left(\beta^t x \prod_{j=1}^{k-1} \gamma_j\right)$$
$$\iff \prod_{k=1}^{s}(C_k(0))^{2^m-1} > \prod_{x \in \mathbb{F}_{2^m}\setminus\{0\}}\prod_{k=1}^{s} C_k(x).$$

From this condition, the zigzag cycle with cycle parameter β such that $\sigma = 2^m - 1$ is also successfully decoded. □

Discussion 1: Corollary 1 shows that the zigzag cycles with cycle parameter β such that the order of β is $2^m - 1$ have the best decoding performance. We claim that the order of β is $2^m - 1$ if and only if $\beta \notin \mathcal{H}_m$, where

$$\mathcal{H}_m := \bigcup_{0 < r < 2^m - 1 : r \mid 2^m - 1} \left\{\alpha^{i\frac{2^m-1}{r}} \mid i = 0, \dots, r-1\right\}. \quad (3)$$

Firstly, we show that the order of β is $2^m - 1$ if $\beta \notin \mathcal{H}_m$. For $r < 2^m - 1$, we define

$$\mathcal{H}_m^{(r)} := \left\{\alpha^{i\frac{2^m-1}{r}} \mid i = 0, \dots, r-1\right\}.$$

If $\beta \notin \mathcal{H}_m^{(r)}$, there exist integers $i \in \{0, 1, \dots, r-1\}$ and $j \in \{1, \dots, (2^m-1)/r - 1\}$ such that $\beta = \alpha^{i(2^m-1)/r+j}$. Hence, we have

$$\beta^r = \alpha^{\{i(2^m-1)/r+j\}r} = \alpha^{jr}.$$

Since $jr < 2^m - 1$, we get $\beta^r = \alpha^{jr} \neq 1$. Thus, we have the order of β is not r if $\beta \notin \mathcal{H}_m^{(r)}$. Since the order of β is less than or equal to $2^m - 1$ for $\beta \in \mathbb{F}_{2^m} \setminus \{0\}$, the order of β is $2^m - 1$ if $\beta \notin \mathcal{H}_m$. Secondly, we show that $\beta \notin \mathcal{H}_m$ if the order of β is $2^m - 1$. Obviously, the order of $\beta \in \mathcal{H}_m^{(r)}$ is less than or equal to r. Hence, the order of $\beta \in \mathcal{H}_m$ is less than $2^m - 1$. From the contraposition, $\beta \notin \mathcal{H}_m$ if the order of β is $2^m - 1$. Therefore, we see that the order of β is $2^m - 1$ if and

Table 1 The elements in $\mathcal{H}_m$ for $m = 2, 3, 4, 5, 6$.

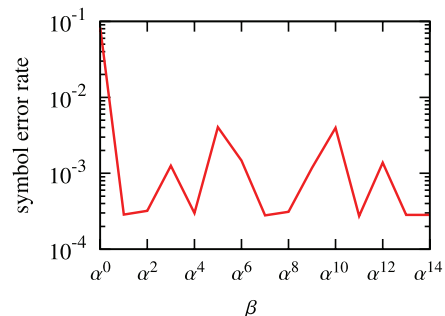| Field	The elements of $\mathcal{H}_m$
$\mathbb{F}_{2^2}$	1
$\mathbb{F}_{2^3}$	1
$\mathbb{F}_{2^4}$	$1, \alpha^3, \alpha^5, \alpha^6, \alpha^9, \alpha^{10}, \alpha^{12}$
$\mathbb{F}_{2^5}$	1
$\mathbb{F}_{2^6}$	$1, \alpha^3, \alpha^6, \alpha^7, \alpha^9, \alpha^{12}, \alpha^{14}, \alpha^{15}, \alpha^{18}, \alpha^{21}, \alpha^{24}, \alpha^{25}, \alpha^{27}, \alpha^{28}, \alpha^{30}, \alpha^{33}, \alpha^{35}, \alpha^{36}, \alpha^{39}, \alpha^{42}, \alpha^{45}, \alpha^{48}, \alpha^{49}, \alpha^{51}, \alpha^{54}, \alpha^{56}, \alpha^{57}, \alpha^{60}$

Fig. 2 The symbol error rate for the zigzag cycle code defined over $\mathbb{F}_{2^4}$ of symbol code length 3 over the AWGN channel with channel variance $\sigma^2 = 1$. The horizontal line corresponds to the cycle parameter.

only if $\beta \notin \mathcal{H}_m$.

Thus, the zigzag cycles with the cycle parameter $\beta \notin \mathcal{H}_m$ have the best decoding performance. Note that $\{\alpha^{i(2^m-1)/r} \mid i = 0, \dots, r-1\}$ represents a proper subgroup of $\mathbb{F}_{2^m}$. Table 1 shows the elements in $\mathcal{H}_m$ for $m = 2, 3, 4, 5, 6$. Figure 2 shows the symbol error rate for the zigzag cycle code define over $\mathbb{F}_{2^4}$ of symbol code length 3 over the AWGN channel with channel variance $\sigma^2 = 1$. From Fig. 2, we see that the zigzag cycle codes with the cycle parameter $\beta \notin \mathcal{H}_4$ have the best decoding performance.

By using the log-likelihood ratio, Theorem 1 is simplified for the zigzag cycle codes with the cycle parameter $\beta \notin \mathcal{H}_m$ over the MBIOS channel.

Corollary 2: We consider the zigzag cycle codes of symbol code length s with the cycle parameter $\beta \notin \mathcal{H}_m$ over the MBIOS channel. Let $Z_{v,i}(Y_{v,i})$ be the log-likelihood ratio corresponding to the i-th channel output in the v-th variable node, i.e.,

$$Z_{v,i}(Y_{v,i}) = \log \frac{\Pr(Y_{v,i} \mid X_{v,i} = 1)}{\Pr(Y_{v,i} \mid X_{v,i} = -1)}.$$

In the limit of large ℓ, all the symbols in the zigzag cycle code are eventually correct if and only if

$$\sum_{v=1}^{s}\sum_{i=1}^{m} Z_{v,i}(Y_{v,i}) > 0.$$

Moreover, in the limit of large ℓ, no symbols in the zigzag cycle code are eventually correct if and only if

$$\sum_{v=1}^{s}\sum_{i=1}^{m} Z_{v,i}(Y_{v,i}) \leq 0.$$

proof: To simplify the notation, we define for $k \in$

$\{1, 2, \ldots, s\}$ and $i \in \{1, 2, \ldots, m\}$

$$p_{k,i} := \Pr(Y_{k,i} \mid X_{k,i} = 1),$$
$$\bar{p}_{k,i} := \Pr(Y_{k,i} \mid X_{k,i} = -1).$$

Note that for $k \in \{1, 2, \ldots, m\}$

$$C_k(0) = \prod_{i=1}^{m} p_{k,i},$$
$$\prod_{t=0}^{2^m-2} C_k \left(\beta^t \prod_{s=1}^{k-1} \gamma_s \right) = \prod_{x \in \mathbb{F}_{2^m} \setminus \{0\}} C_k(x)$$
$$= \prod_{i=1}^{m} p_{k,i}^{2^{m-1}-1} \bar{p}_{k,i}^{2^{m-1}}.$$

From Theorem 1, all the symbols in the zigzag cycle are eventually correct if and only if

$$\prod_{k=1}^{s} (C_k(0))^{2^m-1} > \prod_{t=0}^{2^m-2} \prod_{k=1}^{s} C_k \left(\beta^t x \prod_{j=1}^{k-1} \gamma_j \right)$$
$$\iff \prod_{k=1}^{s} \prod_{i=1}^{m} p_{k,i}^{2^m-1} > \prod_{k=1}^{s} \prod_{i=1}^{m} p_{k,i}^{2^{m-1}-1} \bar{p}_{k,i}^{2^{m-1}}$$
$$\iff \sum_{k=1}^{s} \sum_{i=1}^{m} \log \frac{p_{k,i}}{\bar{p}_{k,i}} > 0.$$

Hence, we see that all the symbols in the zigzag cycle code are eventually correct if and only if $\sum_{k=1}^{s} \sum_{i=1}^{m} Z_{k,i}(Y_{k,i}) > 0$. Similarly, no symbols in the zigzag cycle code are eventually correct if and only if $\sum_{k=1}^{s} \sum_{i=1}^{m} Z_{k,i}(Y_{k,i}) \leq 0$. □

Let $P_{zz}(s, m, \mathsf{a})$ be the symbol error rate for the zigzag cycle code defined over $\mathbb{F}_{2^m}$ of symbol code length s with cycle parameter $\beta \notin \mathcal{H}_m$ over the MBIOS channel characterized by its L-density [8] a under BP decoding. Let $\mathsf{a}_1, \mathsf{a}_2, \ldots, \mathsf{a}_k$ denote independent and identically distributed random variables with density function a. Define $Z^{(k)} := \sum_{i=1}^{k} \mathsf{a}_i$. From Corollary 2, we have the symbol error rate of zigzag cycle code is given by

$$P_{zz}(s, m, \mathsf{a}) = \Pr(Z^{(sm)} \leq 0). \tag{4}$$

Figure 3 shows the symbol error rate for the zigzag cycle code defined over $\mathbb{F}_{2^4}$ of symbol code length 3 with the cycle parameter $\beta \notin \mathcal{H}_4$ over the AWGN channel. The circles in Fig. 3 show the simulation results. The solid curve shows the theoretical symbol error rate. For the AWGN c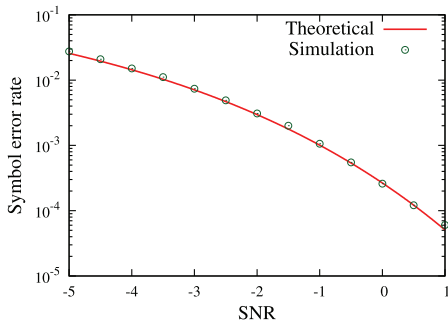hannel with channel variance σ, the theoretical symbol error rate of the zigzag cycle codes defined over $\mathbb{F}_{2^4}$ of symbol code length s with cycle parameter $\beta \notin \mathcal{H}_m$ is given by $Q(\frac{\sqrt{sm}}{\sigma})$, where $Q(y) := \frac{1}{\sqrt{2\pi}} \int_{y}^{\infty} \exp[-\frac{x^2}{2}] \mathrm{d}x$. From Fig. 3,

Fig. 3 Symbol error rate of zigzag cycle codes defined over $\mathbb{F}_{2^4}$ of symbol code length 3 with cycle parameter $\beta \notin \mathcal{H}_4$. The solid curve shows the theoretical symbol error rate. The circles show the simulation result.

we see that the theoretical result gives the symbol error rate of zigzag cycle code with the cycle parameter $\beta \notin \mathcal{H}_m$.

4. Analysis of Error Floors

In the previous section, we give a condition for the decoding error to the zigzag cycle code. By using this result, in this section, we give lower bounds of the symbol error rates in the error floors of non-binary LDPC code ensembles over the MBIOS channel under BP decoding.

4.1 Code Ensemble

A stopping set $\mathcal{S}$ is a set of variable nodes such that all the neighbors of $\mathcal{S}$ are connected to $\mathcal{S}$ at least twice. Since the stopping sets depend only on the structure of a Tanner graph, we are able to extend the definition of the stopping set for the non-binary LDPC codes. Obviously, the zigzag cycles form stopping sets.

It is empirically known that $(2, k)$-regular non-binary LDPC codes exhibit good decoding performance among other LDPC codes for $q \geq 64$ [4]. However, this is not the case for $q < 64$. In this paper, we consider the irregular non-binary LDPC codes which contain variable nodes of degree two for the generality of code ensemble. Note that the $(2,k)$-regular non-binary LDPC code ensembles are included in the irregular non-binary LDPC code ensembles which contain variable nodes of degree two.

From Discussion 1, we see that the cycle parameter β is an important parameter to improve the decoding error rate in the error floor. The following definition gives expurgated ensembles parameterized by the cycle parameter β.

Definition 1: Let $LDPC(N, m, \lambda, \rho)$ denote LDPC code ensemble of symbol code length N over $\mathbb{F}_{2^m}$ defined by Tanner graphs with a degree distribution pair (λ, ρ) [8] and labels of edges chosen elements from $\mathbb{F}_{2^m} \setminus \{0\}$ with equal probability. Let $s_g \in \mathbb{N} \setminus \{1\}$ be an expurgation parameter. The expurgated ensemble $ELDPC(N, m, \lambda, \rho, s_g)$ consists of the subset of codes in $LDPC(N, m, \lambda, \rho)$ which contain no stopping sets of weight in $\{1, \ldots, s_g - 1\}$. Note that the expurgated ensemble $ELDPC(N, m, \lambda, \rho, 1)$ is equivalent to $LDPC(N, m, \lambda, \rho)$. Let $s_c \in \mathbb{N}$ be an expurgation parameter for labeling in the Tanner graph, where $s_g < s_c$. Define expurgated ensemble $\mathcal{E}(N, m, \lambda, \rho, s_g, s_c, \mathcal{H})$ as the subset of codes in $ELDPC(N, m, \lambda, \rho, s_g)$ which contain no zigzag cycles of weight in $\{s_g, \ldots, s_c - 1\}$ with the cycle parameter $\beta \in \mathcal{H}$.

Recall that α is a primitive element of $\mathbb{F}_{2^m}$. Define $\mathcal{H}_m$ as in Eq. (3). Note that the expurgated ensemble constructed by our proposed method and the cycle cancellation is represented as $\mathcal{E}(N, m, \lambda, \rho, s_g, s_c, \mathcal{H}_m)$ and $\mathcal{E}(N, m, \lambda, \rho, s_g, s_c, \{1\})$, respectively.

4.2 Analysis of Error Floors

In this section, we analyze the symbol error rates in the error

floors for the expurgated ensembles defined in Definition 1. The following theorem gives a lower bound on the symbol error rate under BP decoding for the expurgated ensemble $\mathcal{E}(N, m, \lambda, \rho, s_g, s_c, \mathcal{H}_m)$.

Theorem 2: Let $P_s(\mathcal{E}, a)$ be the symbol error rate of the expurgated ensemble $\mathcal{E}(N, m, \lambda, \rho, s_g, s_c, \mathcal{H}_m)$ over the MBIOS channel characterized by its L-density a under BP decoding. Let $a_1, a_2, \ldots, a_k$ denote independent and identically distributed random variables with density function a. Define $Z^{(k)} := \sum_{i=1}^{k} a_i$ and $\mu := \lambda'(0)\rho'(1)$. Let $\mathfrak{B}(a)$ be the Battacharyya functional, i.e., $\mathfrak{B}(a) := \int a(x)e^{-x/2}dx$. The symbol error rate for sufficiently large N and $\mathfrak{B}(a) < \mu^{-1/m}$ is bounded by

$$P_s(\mathcal{E}, a) \geq \frac{1}{2N} \sum_{s=s_g}^{\infty} \mu^s \Pr(Z^{(sm)} \leq 0) + o\left(\frac{1}{N}\right). \quad (5)$$

proof: Let $\tilde{P}(\mathcal{E}, a)$ be the symbol rate rate caused by the zigzag cycles under BP decoding over the MBIOS channel with characterized by its L-density a for $\mathcal{E}(N, m, \lambda, \rho, s_g, s_c, \mathcal{H})$. Hence, we have

$$P_s(\mathcal{E}, a) \geq \tilde{P}(\mathcal{E}, a).$$

We will consider $\tilde{P}(\mathcal{E}, a)$. Let $\tilde{P}_1(\mathcal{E}, a, s)$ be the symbol error rate caused by the zigzag cycles of weight s under BP decoding over the MBIOS channel with characterized by its L-density a for $\mathcal{E}(N, m, \lambda, \rho, s_g, s_c, \mathcal{H})$. For the expurgated ensemble $\mathcal{E}(N, m, \lambda, \rho, s_g, s_c, \mathcal{H})$, the weights of zigzag cycles are at least s_g. By [8, C. 37], if we fix a finite W and let N tend to infinity, the zigzag cycles of weight at most W become asymptotically non-overlapping with high probability [8, p. 155]. Hence, for fixed W and sufficiently large N we have

$$\tilde{P}(\mathcal{E}, a) \geq \sum_{s=s_g}^{W} \tilde{P}_1(\mathcal{E}, a, s).$$

From Corollary 2, no symbols in zigzag cycle codes of weight s with cycle parameter $\beta \notin \mathcal{H}_m$ are eventually correct if $Z^{(sm)} \leq 0$. The symbol error rate for the zigzag cycle codes with $\beta \notin \mathcal{H}_m$ is smaller than that for the zigzag cycle codes with $\beta \in \mathcal{H}_m$. Hence, no symbols in the zigzag cycle of weight s are eventually correct, with probability at least $\Pr(Z^{(sm)} \leq 0)$. By [8, C. 37] for fixed W, the expected number of zigzag cycles of weight $s \leq W$ is given by $\mu^s/2s$, for sufficiently large N. Each zigzag cycle of weight s causes a symbol error probability s/N. Hence, for sufficiently large N, we have for $s \in \{s_g, \ldots, s_c - 1\}$

$$\tilde{P}_1(\mathcal{E}, a, s) = \frac{1}{2N} \mu^s \Pr(Z^{(sm)} \leq 0) + o\left(\frac{1}{N}\right), \quad (6)$$

and for $s \in \{s_c, \ldots, W\}$

$$\tilde{P}_1(\mathcal{E}, a, s) \geq \frac{1}{2N} \mu^s \Pr(Z^{(sm)} \leq 0) + o\left(\frac{1}{N}\right), \quad (7)$$

Thus, we have

$$P_s(\mathcal{E}, a) \geq \tilde{P}(\mathcal{E}, a) \geq \frac{1}{2N} \sum_{s=s_g}^{W} \mu^s \Pr(Z^{(sm)} \leq 0) + o\left(\frac{1}{N}\right).$$

Note that $\Pr(Z^{(sm)} \leq 0) \leq \{\mathfrak{B}(a)\}^{sm}$. Hence, we have

$$\frac{1}{2N} \sum_{s=s_g}^{W} \mu^s \Pr(Z^{(sm)} \leq 0) \leq \frac{1}{2N} \sum_{s=s_g}^{W} \left(\mu\{\mathfrak{B}(a)\}^m\right)^s.$$

Thus, if $\mathfrak{B}(a) < \mu^{-\frac{1}{m}}$ and W tends to infinity, the left hand side of this inequality converges. $\square$

Corollary 3: Define

$$\epsilon_m^* := \begin{cases} \frac{1}{2} & \text{for } \mu \leq 1, \\ \frac{1-\sqrt{1-\mu^{-2/m}}}{2} & \text{for } \mu > 1. \end{cases}$$

For the BSC with crossover probability ϵ and $\epsilon < \epsilon_m^*$, the symbol error rate is lower bounded by

$$P_s(\mathcal{E}, a)$$
$$\geq \frac{1}{2N} \sum_{s=s_c}^{\infty} \mu^s \sum_{i \leq ms/2} \binom{ms}{i} \epsilon^{ms-i}(1 - \epsilon)^i + o\left(\frac{1}{N}\right). \quad (8)$$

Corollary 4: Define

$$\sigma_m^* := \begin{cases} \infty & \text{for } \mu \leq 1, \\ \sqrt{\frac{m}{2\ln\mu}} & \text{for } \mu > 1. \end{cases}$$

For the AWGN channel with channel variance σ^2 and $\sigma < \sigma_m^*$, the symbol error rate is lower bounded by

$$P_s(\mathcal{E}, a) \geq \frac{1}{2N} \sum_{s=s_g}^{\infty} \mu^s Q\left(\frac{\sqrt{sm}}{\sigma}\right) + o\left(\frac{1}{N}\right), \quad (9)$$

where $Q(y) = \frac{1}{\sqrt{2\pi}} \int_y^{\infty} \exp[-\frac{x^2}{2}]dx$.

4.3 Simulation Results

In this section, we compare the symbol error rate in the error floor for the expurgated ensemble constructed by our proposed method with (i) that constructed by the cycle cancellation [5] in Sects. 4.3.1 and 4.3.2, and (ii) that constructed by the combination of the cycle cancellation and the stopping set mitigation [5] in Sect. 4.3.3.

4.3.1 AWGN Channel Case

First, we show the cases for regular non-binary LDPC code ensembles. From Table 1, we have $\mathcal{H}_4 = \{1, \alpha^3, \alpha^5, \alpha^6, \alpha^9, \alpha^{10}, \alpha^{12}\}$. Figures 4 and 5 compare the symbol error rates for the expurgated ensembles constructed by our proposed method $\mathcal{E}(315, 4, x, x^2, 1, 8, \mathcal{H}_4)$ and $\mathcal{E}(1200, 4, x, x^2, 2, 11, \mathcal{H}_4)$ with the expurgated ensembles constructed by the cycle cancellation [5] $\mathcal{E}(315, 4, x, x^2, 1, 8, \{1\})$ and $\mathcal{E}(1200, 4, x,$
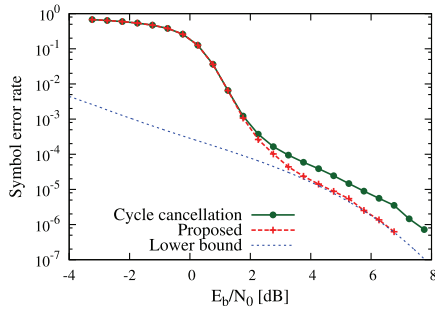
Fig. 4 Comparison of the symbol error rate for the expurgated ensemble $\mathcal{E}(315, 4, x, x^2, 1, 8, \mathcal{H}_4)$ (proposed) with the expurgated ensemble $\mathcal{E}(315, 4, x, x^2, 1, 8, \{1\})$ (cycle cancellation). The lower bound is given by Eq. (9).
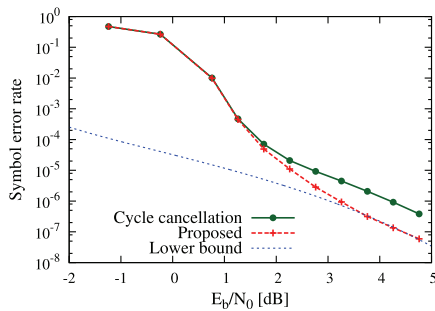

Fig. 5 Comparison of the symbol error rate for the expurgated ensemble $\mathcal{E}(1200, 4, x, x^2, 2, 11, \mathcal{H}_4)$ (proposed) with the expurgated ensemble $\mathcal{E}(1200, 4, x, x^2, 2, 11, \{1\})$ (cycle cancellation). The lower bound is given by Eq. (9).
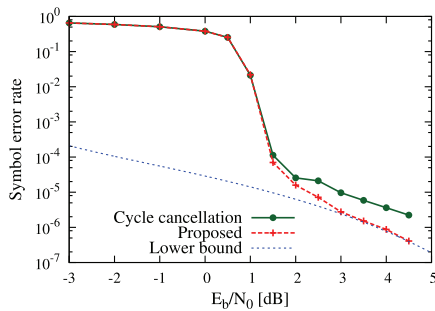

Fig. 6 Comparison of the symbol error rate for the expurgated ensemble $\mathcal{E}(1000, 4, \lambda, \rho, 1, 8, \mathcal{H}_4)$ (proposed) with the expurgated ensemble $\mathcal{E}(1000, 4, \lambda, \rho, 1, 8, \{1\})$ (cycle cancellation), where $\lambda = 0.5x + 0.5x^2$ and $\rho = 0.5x^3 + 0.5x^5$. The lower bound is given by Eq. (9).

$x^2, 2, 11, \{1\})$, respectively. The lower bounds on symbol error rate are given by Eq. (9). Figure 5 is the case for $s_g > 1$. We see that our proposed codes exhibit better decoding performance than codes designed by the cycle cancellation. We see that Theorem 2 gives tight lower bounds for the symbol error rates to the expurgated ensembles constructed by our proposed method in the error floor.

Next, we show the case for an irregular non-binary LDPC code ensemble. As an example, we employ the degree distribution pair $\lambda = 0.5x + 0.5x^2$ and $\rho = 0.5x^3 + 0.5x^5$. Figure 6 compares the symbol error rate for the
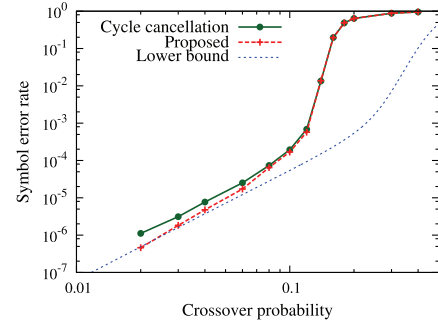

Fig. 7 Comparison of the symbol error rate for the expurgated ensemble $\mathcal{E}(315, 6, x, x^2, 1, 8, \mathcal{H}_6)$ (proposed) with the expurgated ensemble $\mathcal{E}(315, 6, x, x^2, 1, 8, \{1\})$ (cycle cancellation). The lower bound is given by Eq. (8).
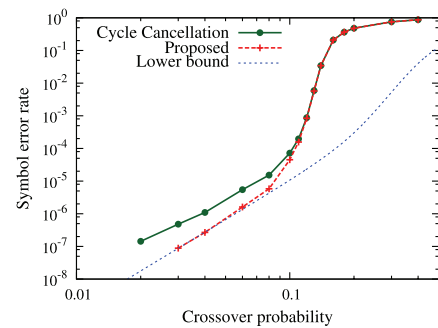

Fig. 8 Comparison of the symbol error rate for the expurgated ensemble $\mathcal{E}(1200, 4, x, x^2, 2, 11, \mathcal{H}_4)$ (proposed) with the expurgated ensemble $\mathcal{E}(1200, 4, x, x^2, 2, 11, \{1\})$ (cycle cancellation). The lower bound is given by Eq. (8).

expurgated ensemble constructed by our proposed method $\mathcal{E}(1000, 4, \lambda, \rho, 1, 8, \mathcal{H}_4)$ with the expurgated ensemble constructed by the cycle cancellation. $\mathcal{E}(1000, 4, \lambda, \rho, 1, 8, \{1\})$. The lower bounds on the symbol error rates are given by Eq. (9). We see that our proposed codes exhibit better decoding performance than codes designed by the cycle cancellation. We see that Theorem 2 gives a tight lower bounds for the symbol error rates to the expurgated ensembles constructed by our proposed method in the error floor.

4.3.2 BSC Case

Figures 7 and 8 compare the symbol error rates for the expurgated ensembles constructed by our proposed method $\mathcal{E}(315, 6, x, x^2, 1, 8, \mathcal{H}_6)$ and $\mathcal{E}(1200, 4, x, x^2, 2, 11, \mathcal{H}_4)$ with the expurgated ensembles constructed by the cycle cancellation $\mathcal{E}(315, 6, x, x^2, 1, 8, \{1\})$ and $\mathcal{E}(1200, 4, x, x^2, 2, 11, \{1\})$, respectively. The lower bounds for the symbol error rates are given by Eq. (8). Figure 8 is the case for $s_g > 1$. From Figs. 7 and 8, we see that our proposed codes exhibit better decoding performance than codes designed by the cycle cancellation.

We see that Theorem 2 gives tight lower bounds for the symbol error rates to the expurgated ensembles constructed by our proposed method in the error floor.
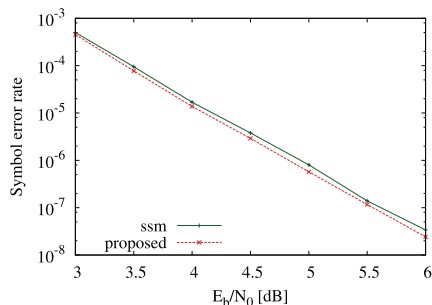
Fig. 9 Comparison of the symbol error rate for the codes designed by the proposed method and the codes designed by the method which uses both the cycle cancellation and the stopping set mitigation. The base code ensemble is $ELDPC(60, 4, x, x^3, 3)$. The solid curve (proposed) shows the symbol error rate for the codes designed by our proposed method. The dotted curve (ssm) shows the symbol error rate for the codes designed by the method which uses both the cycle cancellation and the stopping set mitigation.

4.3.3 Comparison with Stopping Set Mitigation

In [5], Poulliat et al. also proposed the *stopping set mitigation*. To lower the error floor further, Poulliat et al. proposed to use both the cycle cancellation and the stopping set mitigation. We refer to the Hamming weight of the binary represented non-binary codeword as binary weight. The stopping set mitigation is a method to design the labels on the edges, which are connecting to the nodes in the smallest stopping set, so that the binary minimum distance in the stopping sets takes the maximum value.

Figure 9 compares the symbol error rate for the codes designed by the proposed method and the codes designed by the method which uses both the cycle cancellation and the stopping set mitigation [5]. In order to make the stopping set mitigation work effectively, we employ as the base codes the codes whose Tanner graphs include many small stopping sets. For example, this condition is met by the code ensemble $ELDPC(60, 4, x, x^3, 3)$. By applying our proposed method and the method which uses both the cycle cancellation and stopping set mitigation, we get resulting codes which are the subsets of $ELDPC(60, 4, x, x^3, 3)$. We see Fig. 9 that the symbol error rate for our proposed method is lower than that for the method using both the cycle cancellation and the stopping set mitigation.

5. Conclusion

In this paper, we provide a necessary and sufficient condition for successful decoding of zigzag cycle codes over the MBIOS channel by the BP decoder. Based on this condition, we propose a design method of selecting labels so as to eliminate small zigzag cycles which degrade decoding performance for non-binary LDPC codes. Finally, we show lower bounds of the error floors for the expurgated LDPC code ensembles over the MBIOS channel.

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