Mathematics and Sudoku II

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(Received September 26, 2014)

We discuss on the worldwide famous Sudoku puzzule by using mathematical approach. In this paper we discuss on some basic techniques in Sudoku.
This paper is the second paper in our series, so we use the same notations and terminologies in [1] without any descriptions.

4. Intersectable systems.
Let $S=\{s_1,s_2,...,s_n\}, T=\{t_1,t_2,...,t_n\}\subset \boldsymbol{BLK}$ for n, $1\leqq n\leqq 9$. The pair (S,T) is intersectable $n-$system provided that it satisfies the following conditions:
(i) $s_i\cap s_j=\phi,\ t_i\cap t_j=\phi\quad 1\leqq i\leqq n, 1\leqq j\leqq n,\ i\neq j$
(ii) $s_i\cap t_j\neq\phi\ ,\quad 1\leqq i\leqq n, 1\leqq j\leqq n$
(iii) $s=s_1\cup s_2\cup ...\cup s_n\ ,\ t=t_1\cup t_2\cup ...\cup t_n.$

We discuss on intersectable systems in this paper. First, we need the following.
Proposition 13. Let $s\subset b\in \boldsymbol{BLK}$. For each $\boldsymbol{K}=(K_\alpha)_{\alpha\in J_1\times J_2}\in \boldsymbol{SMTX}(f,f_0)$, we have that
(a) $J_3-K_s\subset f(b-s)$.
Proof. Since $\boldsymbol{K}$ is a sudoku matrix associated with (f,f_0), we have
(1) $f(\alpha)\in K_\alpha$ for each $\alpha\in s$,
By (1) we have
(2) $f(s)=\cup\{f(\alpha):\alpha\in s\}\subset\cup\{K_\alpha:\alpha\in s\}=K_s$.
Then by (2) we have
(3) $J_3-f(s)\supset J_3-K_s$.
On the other hand, by (SDM), $f\mid b:b\to J_3$ is bijective. Thus we have that
(4) $J_3=f(b)=f(s)\cup f(b-s),\ f(s)\cap f(b-s)=\phi$.
By (4) we have that
(5) $J_3-f(s)=f(b-s)$.
By (3) and (5), we have that
(6) $f(b-s)\supset J_3-K_s$.
Hence, we show Proposition 13.

Under the conditions (i)−(iii) we show the following:
Proposition 14. Let $\boldsymbol{K}=(K_\alpha)_{\alpha\in J_1\times J_2}\in \boldsymbol{SMTX}(f,f_0)$ and (S,T) be an intersectable n−system. Then we have that
(b) $J_3-K_{s-t}\subset J_3-f(t-s)$, and
(c) $J_3-K_{t-s}\subset J_3-f(s-t)$.
Proof. We show the condition (b). We assume that the condition (b) does not hold. Thus there exists a $k\in J_3$ such that
(1) $k\in J_3-K_{s-t}$ and $k\notin J_3-f(t-s)$.

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We have that

(2) $\cap\{J_3-K_{s_j-s_j\cap t}: j=1,2,...,n\} = \cap\{J_3-K_{s_j-t}: j=1,2,...,n\}$

$= J_3-\cup\{K_{s_j-t}: j=1,2,...n\} = J_3-K_{\cup\{s_j-t: j=1,2,...,n\}} = J_3-K_{\cup\{s_j: j=1,2,...,n\}-t}$

$= J_3-K_{s-t}$.

By (1) and (2), we have that

(3) $k\in\cap\{J_3-K_{s_j-s_j\cap t}: j=1,2,...,n\}$ and $k\in f(t-s)$.

Now, take each i. Thus we have that

(4) $s_i\cap t = s_i\cap(t_1\cup t_2\cup...\cup t_n) = (s_i\cap t_1)\cup(s_i\cap t_2)\cup...\cup(s_i\cap t_n)$.

Since $s_i\in \boldsymbol{BLK}$, by (4) and Proposition 13 we have that

(5) $J_3-M_{s_i-s_i\cap t}\subset f(s_i\cap t)=f(s_i\cap t_1)\cup f(s_i\cap t_2)\cup...\cup f(s_i\cap t_n)$ and

(6) $f\mid s_i: s_i\to J_3$ is bijective.

By the condition (i), for each $j\neq j'$, we have $(s_i\cap t_j)\cap(s_i\cap t_{j'})=s_i\cap t_j\cap t_{j'}=\phi$.
Thus by (6) we have that

(7) $f(s_i\cap t_j)\cap f(s_i\cap t_{j'})=\phi$ for $j\neq j'$.

By (3) we have that $k\in J_3-M_{s_i-s_i\cap t}$. Therefore by (5),(7) there is the unique j_i such that

(8) $1\leqq j_i\leqq n$ and $k\in f(s_i\cap t_{j_i})$.

Moreover by (6) and (8) there is the unique α_{i,j_i} such that

(9) $\alpha_{i,j_i}\in s_i\cap t_{j_i}$ and $f(\alpha_{i,j_i})=k$.

Now, we put $j(k)=\{j_1,j_2,....,j_n\}$ and show the following:

Claim 1. $j(k)=\{j_1,j_2,....,j_n\}=\{1,2,...,n\}$.

Proof of Claim 1. By (8) we have that $j(k)\subset\{1,2,...,n\}$. We assume that Claim 1 does not hold. Since $j(k)=\{j_1,j_2,....,j_n\}\neq\{1,2,...,n\}$ by the assumption, there are i_1, i_2 such that

(10) $i_1\neq i_2$ $1\leqq i_1\leqq n, 1\leqq i_2\leqq n$, $j_{i_1}=j_{i_2}$.

Here, we put $j^*=j_{i_1}=j_{i_2}$. By (9) we have that

(11) $\alpha_{i_1,j^*}\in s_{i_1}\cap t_{j^*}$, $f(\alpha_{i_1,j^*})=k$ and

(12) $\alpha_{i_2,j^*}\in s_{i_2}\cap t_{j^*}$, $f(\alpha_{i_2,j^*})=k$.

We show that

(13) $\alpha_{i_1,j^*}\neq\alpha_{i_2,j^*}$.

We assume that (13) does not hold, that is, $\alpha_{i_1,j^*}=\alpha_{i_2,j^*}$. By (11) and (12) we have that

(14) $\alpha_{i_1,j^*}=\alpha_{i_2,j^*}\in(s_{i_1}\cap t_{j^*})\cap(s_{i_2}\cap t_{j^*})\subset s_{i_1}\cap s_{i_2}$.

By (i) and (10), we have that $s_{i_1}\cap s_{i_2}=\phi$. This contradicts to (14). Hence we have (13).

Also by (11) and (12) we have that

(15) $\alpha_{i_1,j^*}\in t_{j^*}$, $\alpha_{i_2,j^*}\in t_{j^*}$, $f(\alpha_{i_1,j^*})=k$, $f(\alpha_{i_2,j^*})=k$.

Since $t_{j^*}\in BLK$, by (SDM) we have that

(16) $f \mid t_{j^*}: t_{j^*} \to J_3$ is bijective.

However, (13) and (15) contradict to (16). Hence we complete the proof of Claim 1.

Now we back to the main proof. By (3) we have that

(17) $k \in f(t-s) = f(t_1 \cup t_2 \cup \ldots \cup t_n - s) = f((t_1 - s) \cup (t_2 - s) \cup \ldots \cup (t_n - s))$
$= f(t_1 - s) \cup f(t_2 - s) \cup \ldots \cup f(t_n - s)$.

Thus, by (17) there exists a j' such that

(18) $1 \leqq j' \leqq n$ and $k \in f(t_{j'} - s)$.

By Claim 1 there exists an i' such that

(19) $1 \leqq i' \leqq n$ and $j_{i'} = j'$.

By (9) and (19) we have that

(20) $\alpha_{i',j'} \in s_{i'} \cap t_{j'}$ and $f(\alpha_{i',j'}) = k$.

By (18) we have that

(21) $k \in f(t_{j'} - s) = f(t_{j'} - s_1 \cup s_2 \cup \ldots \cup s_n) = f((t_{j'} - s_1) \cap (t_{j'} - s_2) \cap \ldots \cap (t_{j'} - s_n))$
$\subset f(t_{j'} - s_1) \cap f(t_{j'} - s_2) \cap \ldots \cap f(t_{j'} - s_n)$.

By (21) we have that

(22) $k \in f(t_{j'} - s_{i'})$.

By (22) there exists an α^* such that

(23) $\alpha^* \in t_{j'} - s_{i'}$ and $f(\alpha^*) = k$.

Since $\alpha^* \in t_{j'} - s_{i'} = t_{j'} - s_{i'} \cap t_{j'}$, by (20) and (23) we have that

(24) $\alpha_{i',j'}, \alpha^* \in t_{j'}, \alpha_{i',j'} \neq \alpha^*,\ f(\alpha_{i',j'}) = f(\alpha^*) = k$.

On the other hand, since $t_{j'} \in \boldsymbol{BLK}$, by (SDM)

(25) $f \mid t_{j'}: t_{j'} \to J_3$ is bijective.

However (24) contradicts to (25). Thus we have the condition (b).

Since our conditions (i)−(iii) are symmetrical, similarly we have (c). Hence we complete the proof of Proposition 14.

Proposition 15. Let $\boldsymbol{K} = (K_\alpha)_{\alpha \in J_1 \times J_2} \in \boldsymbol{SMTX}(f, f_0)$ and (S,T) be an intersectable n−system. Then $\boldsymbol{K}' = (K'_\alpha)_{\alpha \in J_1 \times J_2} \in \boldsymbol{SMTX}(f, f_0)$ and $\boldsymbol{K}' \leqq \boldsymbol{K}$, where $\boldsymbol{K}'$ is defined as follows:

$$\text{(iv)} \quad K'_\alpha = \begin{cases} K_\alpha & \textit{for } \alpha \in (J_1 \times J_2 - s \cup t) \cup (s \cap t) \\ K_\alpha \cap K_{t-s} = K_\alpha - (J_3 - K_{t-s}) & \textit{for } \alpha \in s - s \cap t \\ K_\alpha \cap K_{s-t} = K_\alpha - (J_3 - K_{s-t}) & \textit{for } \alpha \in t - s \cap t \end{cases}$$

In notation, we put $\boldsymbol{K}' = T(S,T)(\boldsymbol{K})$.

Proof. Since $\boldsymbol{K} \in \boldsymbol{SMTX}(f, f_0)$, by $(SMTX)$ we have that

(1) $f(\alpha) \in K_\alpha$ for each $\alpha \in J_1 \times J_2$.

By (iv) and (1) we have that

(2) $f(\alpha) \in K_\alpha = K'_\alpha$ for $\alpha \in (J_1 \times J_2 - s \cup t) \cup (s \cap t)$.

Take any $\alpha \in t - s \cap t = t - s$ and thus $f(\alpha) \in f(t-s)$, that is, we have

(3) $f(\alpha) \notin J_3 - f(t-s)$.

Since $J_3 - K_{s-t} \subset J_3 - f(t-s)$ by Proposition 14, by (3) we have that

(4) $f(\alpha) \notin J_3 - K_{s-t}$.

By (1),(4) we have that

(5) $f(\alpha)\in K_\alpha-(J_3-K_{s-t})$.
Since $K_\alpha-(J_3-K_{s-t})=K_\alpha\cap(J_3-(J_3-K_{s-t}))=K_\alpha\cap K_{s-t}=K'_\alpha$, by (5) we have that
(6) $f(\alpha)\in K'_\alpha$ for $\alpha\in t-s\cap t=t-s$.
Take any $\alpha\in s-s\cap t=s-t$ and thus $f(\alpha)\in f(s-t)$, that is, we have
(7) $f(\alpha)\notin J_3-f(s-t)$.
Since $J_3-K_{t-s}\subset J_3-f(s-t)$ by Proposition 14, by (7) we have that
(8) $f(\alpha)\notin J_3-K_{t-s}$.
By (1),(8) we have that
(9) $f(\alpha)\in K_\alpha-(J_3-K_{t-s})$.
Since $K_\alpha-(J_3-K_{t-s})=K_\alpha\cap(J_3-(J_3-K_{t-s}))=K_\alpha\cap K_{t-s}=K'_\alpha$, by (9) we have that
(10) $f(\alpha)\in K'_\alpha$ for $\alpha\in s-s\cap t=s-t$.
By (2),(6) and (10) we have that $\boldsymbol{K}'$ satisfies the condition $(SMTX)$ and thus $\boldsymbol{K}'\in \boldsymbol{SMTX}(f,f_0)$. By the definition (iv), $\boldsymbol{K}'\leqq\boldsymbol{K}$. Hence we complete the proof of Proposition 15.

Proposition 16. Let $\boldsymbol{K}\in\boldsymbol{SMTX}(f_0)$ and (S,T) be an intersectable n−system. Then $\boldsymbol{K}'=T(S,T)(\boldsymbol{K})\in\boldsymbol{SMTX}(f_0)$.
Proof. Since $\boldsymbol{K}\in\boldsymbol{STMX}(f_0)=\cap\{\boldsymbol{STMX}(f,f_0):f\in\boldsymbol{SOL}(f_0)\}$, for each $f\in\boldsymbol{SOL}(f_0)$, $\boldsymbol{K}\in\boldsymbol{STMX}(f,f_0)$. Thus, by Proposition 15, $\boldsymbol{K}'\in\boldsymbol{STMX}(f,f_0)$ for each $f\in\boldsymbol{SOL}(f_0)$, that is, $\boldsymbol{K}'\in\boldsymbol{STMX}(f_0)$. Hence we show Proposition 16.

Remark 17. In the latter paper we classify pairs of intersectable n−systems. Many cases give useful grid analysis techniques in sudoku. For example, X−wing, Swordfish are appeared as the special type of intersectable 2−system, intersectable 3−system, respectively. However, our representations of intersectable 2−system, intersectable 3−system and representations of X−wing, Swordfish are completely different, because we have the concept of sudoku matrices.

5. Transformations of sudoku matrices.
For each n, $1\leqq n\leqq 9$, we put $\boldsymbol{SFS}(n)=\{(s,b):b\in\boldsymbol{BLK},\ s\subset b\ and\ |s|=n\}$ and $\boldsymbol{SFS}=\cup_{n=1}^{9}\boldsymbol{SFS}(n)$.
Also we put $\boldsymbol{IS}(n)=\{(S,T):(S,T)\ is\ a\ pair\ of\ intersectable\ n-system\ of\ \boldsymbol{BLK}\}$ and $\boldsymbol{IS}=\cup_{n=1}^{9}\boldsymbol{IS}(n)$. We put $\boldsymbol{BTOOL}=\boldsymbol{SFS}\cup\boldsymbol{IS}$. Note that $|\boldsymbol{BTOOL}|<1.95\times10^{16}$.

For each $\omega\in\boldsymbol{BTOOL}$ we define a map $T_\omega:\boldsymbol{STMX}(f,f_0)\to\boldsymbol{STMX}(f,f_0)$ and a map $T_\omega:\boldsymbol{STMX}(f_0)\to\boldsymbol{STMX}(f_0)$ as follows:
Let $\omega=(s,b)\in\boldsymbol{SFS}(n)\subset\boldsymbol{BTOOL}$. For each $\boldsymbol{K}=(K_\alpha)_{\alpha\in J_1\times J_2}\in\boldsymbol{SMTX}(f,f_0)$,

$$T_\omega(\boldsymbol{K})=\begin{cases} nNSF((s,b),\boldsymbol{K}) & if\quad |K_s|=|s|=n \\ \boldsymbol{K} & if\quad |K_s|\neq|s|=n \end{cases}.$$

Let $\omega=(S,T)\in\boldsymbol{IS}(n)\subset\boldsymbol{IS}$. For each $\boldsymbol{K}=(K_\alpha)_{\alpha\in J_1\times J_2}\in\boldsymbol{SMTX}(f,f_0)$,

$$T_\omega(\boldsymbol{K})=T(S,T)(\boldsymbol{K}).$$

Now, we put $\boldsymbol{TRF}=\{T_\omega:\omega\in \boldsymbol{BTOOL}\}$. Since $\boldsymbol{BTOOL}$ is a finite set, $\boldsymbol{TRF}$ is also finite.

Proposition 18. For each $\omega\in \boldsymbol{BTOOL}$, we have the followings:

(a) $T_\omega(\boldsymbol{K})\in \boldsymbol{SMTX}(f,f_0)$ for each $\boldsymbol{K}\in \boldsymbol{SMTX}(f,f_0)$.

(b) $T_\omega(\boldsymbol{K})\leqq \boldsymbol{K}$ for each $\boldsymbol{K}\in \boldsymbol{SMTX}(f,f_0)$.

(c) $T_\omega(\boldsymbol{L})\leqq T_\omega(\boldsymbol{K})$ for each $\boldsymbol{K},\boldsymbol{L}\in \boldsymbol{SMTX}(f,f_0)$ with $\boldsymbol{L}\leqq \boldsymbol{K}$.

(d) $T_\omega(\boldsymbol{K})\in \boldsymbol{SMTX}(f_0)$ for each $\boldsymbol{K}\in \boldsymbol{SMTX}(f_0)$.

(e) $T_\omega(\boldsymbol{K})\leqq \boldsymbol{K}$ for each $\boldsymbol{K}\in \boldsymbol{SMTX}(f_0)$.

(f) $T_\omega(\boldsymbol{L})\leqq T_\omega(\boldsymbol{K})$ for each $\boldsymbol{K},\boldsymbol{L}\in \boldsymbol{SMTX}(f_0)$ with $\boldsymbol{L}\leqq \boldsymbol{K}$.

Proof. The facts (a) and (b) follow from Proposition 5 and Proposition 15.

We show the fact (c). Take any $\boldsymbol{K}=\{K_\alpha\}_{\alpha\in J_1\times J_2}$, $\boldsymbol{L}=\{L_\alpha\}_{\alpha\in J_1\times J_2}\in \boldsymbol{SMTX}(f,f_0)$ with $\boldsymbol{L}\leqq \boldsymbol{K}$. We must consider the following cases:

Case 1. $\omega=(s,b)\in \boldsymbol{SFS}(n)$ for some n and $|K_s|=|s|=n$.

Case 2. $\omega=(s,b)\in \boldsymbol{SFS}(n)$ for some n and $|K_s|\neq|s|=n$.

Case 3. $\omega=(S,T)\in \boldsymbol{IS}(n)$ for some n.

First, we consider Case 1. In this case, s is a naked $n-$self filled set of $\boldsymbol{K}$. Since $\boldsymbol{L}\leqq \boldsymbol{K}$, by Proposition 4, s is also a naked $n-$self$-$filled set of $\boldsymbol{L}$ and $K_s=L_s$. That is,

(1) $|L_s|=|s|=n$, and

(2) $K_s=L_s$.

By (1),

(3) $T_\omega(\boldsymbol{K})=nNSF((s,b),\boldsymbol{K})$ and $T_\omega(\boldsymbol{L})=nNSF((s,b),\boldsymbol{L})$.

By (3) and the definition we have

(4) $$T_\omega(\boldsymbol{K})_\alpha=\begin{cases}K_\alpha & for\ \alpha\in s\\ K_\alpha-K_s & for\ \alpha\in b-s\\ K_\alpha & for\ \alpha\in J_1\times J_2-b\end{cases}$$ and

(5) $$T_\omega(\boldsymbol{L})_\alpha=\begin{cases}L_\alpha & for\ \alpha\in s\\ L_\alpha-L_s & for\ \alpha\in b-s\\ L_\alpha & for\ \alpha\in J_1\times J_2-b\end{cases}$$.

Since $\boldsymbol{L}\leqq \boldsymbol{K}$,

(6) $L_\alpha\subset K_\alpha$ for each $\alpha\in J_1\times J_2$.

Take any $\alpha\in b-s$. By (2),(4),(5) and (6)

(7) $T_\omega(\boldsymbol{L})_\alpha=L_\alpha-L_s\subset K_\alpha-L_s=K_\alpha-K_s=T_\omega(\boldsymbol{K})_\alpha$.

Take any $\alpha\in s\cup(J_1\times J_2-b)$. By (4),(5) and (6)

(8) $T_\omega(\boldsymbol{L})_\alpha=L_\alpha\subset K_\alpha=T_\omega(\boldsymbol{K})_\alpha$.

Hence, by (7),(8)

(9) $T_\omega(\boldsymbol{L})\leqq T_\omega(\boldsymbol{K})$.

Secondly, we consider Case 2. In this case, by definition

(10) $T_\omega(\boldsymbol{K})=\boldsymbol{K}$.

Thus , by (10) and the fact (b),

(11) $T_\omega(\boldsymbol{K})=\boldsymbol{K}\geqq \boldsymbol{L}\geqq T_\omega(\boldsymbol{L})$.

Thirdly, we consider Case 3. In this case, $T_\omega(\boldsymbol{K})=T(S,T)(\boldsymbol{K})$ and

$T_\omega(\boldsymbol{L})=T(S,T)(\boldsymbol{L})$, that is,

(12) $T_\omega(\boldsymbol{K})_\alpha=\begin{cases} K_\alpha & for\ \alpha\in(J_1\times J_2-s\cup t)\cup(s\cap t) \\ K_\alpha\cap K_{t-s} & for\ \alpha\in s-s\cap t \\ K_\alpha\cap K_{s-t} & for\ \alpha\in t-s\cap t \end{cases}$

(13) $T_\omega(\boldsymbol{L})_\alpha=\begin{cases} L_\alpha & for\ \alpha\in(J_1\times J_2-s\cup t)\cup(s\cap t) \\ L_\alpha\cap L_{t-s} & for\ \alpha\in s-s\cap t \\ L_\alpha\cap L_{s-t} & for\ \alpha\in t-s\cap t \end{cases}$.

Take any $\alpha\in(J_1\times J_2-s\cup t)\cup(s\cap t)$. By (6),(11) and (12)

(14) $T_\omega(\boldsymbol{L})_\alpha=L_\alpha\subset K_\alpha=T_\omega(\boldsymbol{L})_\alpha$.

Take any $\alpha\in s-s\cap t$. By (6) we have that

(15) $L_{t-s}\subset K_{t-s}$.

By (6),(12),(13) and (15) we have that

(16) $T_\omega(\boldsymbol{L})_\alpha=L_\alpha\cap L_{t-s}\subset K_\alpha\cap K_{t-s}=T_\omega(\boldsymbol{K})_\alpha$.

Take any $\alpha\in t-s\cap t$. By (6) we have that

(17) $L_{s-t}\subset K_{s-t}$.

By (6),(12),(13) and (17) we have that

(18) $T_\omega(\boldsymbol{L})_\alpha=L_\alpha\cap L_{s-t}\subset K_\alpha\cap K_{s-t}=T_\omega(\boldsymbol{K})_\alpha$.

By (14),(16) and (18) we have that

(19) $T_\omega(\boldsymbol{L})=T(S,T)(\boldsymbol{L})\leqq T(S,T)(\boldsymbol{K})=T_\omega(\boldsymbol{K})$.

Thus, in any case we show that $T_\omega(\boldsymbol{L})\leqq T_\omega(\boldsymbol{K})$ and hence we have (c).

We show (d) by Proposition 6 and Proposition 16. Similarly we show (e) and (f) by (b) and (c). We complete the proof of Proposition 18.

References

[1] T.Kitamoto and T.Watanabe, Mathematics and Sudoku I, Bulletin of the Faculty of Education, Yamaguchi University, pp. 193–201, vol. 64, PT.2, 2015.