Mathematics and Sudoku I

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We discuss on the worldwide famous Sudoku puzzule by using mathematical approach. In this paper we discuss on some basic techniques in Sudoku.

1. Mathematical definition of Sudoku.

We use the following notations: For a set K, $|K|$ denotes the cardinal number of K. In this paper we use almost all finite sets. So we can say, $|K|$ means the number of elements in K. When K is an infinite set, we denote it by $|K|=\infty$.

We use the set $J=\{1,2,3,4,5,6,7,8,9\}$ and $J_1=J_2=J_3=J$. For each $i_0\in J_1$ and $j_0\in J_2$, the sets $col(i_0)=\{(i_0,j):j\in J_2\}\subset J_1\times J_2$ and $row(j_0)=\{(i,j_0):i\in J_1\}\subset J_1\times J_2$ are called by the i_0−th column and the j_0−th row of $J_1\times J_2$, respectively. And we define nine 3×3 blocks in $J_1\times J_2$ as follows: We use the sets $A=\{1,2,3\}$, $B=\{4,5,6\}$, $C=\{7,8,9\}\subset J$ and $blk(1)=A\times A$, $blk(2)=A\times B$, $blk(3)=A\times C$, $blk(4)=B\times A$, $blk(5)=B\times B$, $blk(6)=B\times C$, $blk(7)=C\times A$, $blk(8)=C\times B$ and $blk(9)=C\times C$, respectively. Sometimes we say, columns and rows are also column blocks and row blocks, respectively. We define the following sets: $rOW=\{row(i):i\in J_1\}$, $cOL=\{col(j):j\in J_2\}$, $bLK=\{blk(k):k\in J\}$ and $\boldsymbol{BLK}=rOW\cup cOL\cup bLK$.

We define SUDOKU as maps. A map $f:J_1\times J_2\to J_3$ is a sudoku map provided that it satifies the following condition:

(SDM) $f\mid b:b\to J_3$ is bijective for each $b\in\boldsymbol{BLK}$.

Here, $f\mid b$ is the restriction map of f to the subset $b\subset J_1\times J_2$.

Let L_0 be a subset of $J_1\times J_2$, and $f_0:L_0\to J_3$ be a map. We say, f_0 is a sudoku problem map. A map $f:J_1\times J_2\to J_3$ is a sudoku solution map of f_0 provided that it satisfies the following condition:

(SOL) f is a sudoku map with $f\mid L_0=f_0$.

We define the set $\boldsymbol{SOL}(f_0)=\{f: f \textit{ is a sudoku solution map of } f_0\}$. In general, f_0 has many sudoku solution maps.

We define the set $\boldsymbol{POW}(J_3)=\{K:K\subset J_3\}$, that is, it consists of all subsets of J_3. Let $\boldsymbol{K}=(K_\alpha)_{\alpha\in J_1\times J_2}$ be a 9×9 matrix with $K_\alpha\in\boldsymbol{POW}(J_3)$ for each $\alpha\in J_1\times J_2$. Let $\boldsymbol{K}'=(K'_\alpha)_{\alpha\in J_1\times J_2}$ be another 9×9 matrix with $K'_\alpha\in\boldsymbol{POW}(J_3)$ for each $\alpha\in J_1\times J_2$. We say, $\boldsymbol{K}'$ is smaller than $\boldsymbol{K}$, in notation $\boldsymbol{K}'\leqq\boldsymbol{K}$, provided that $K'_\alpha\subset K_\alpha$ for

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each $\alpha \in J_1 \times J_2$. Also we define matrices $\boldsymbol{K} \cup \boldsymbol{K}' = (S_\alpha)_{\alpha \in J_1 \times J_2}$ and $\boldsymbol{K} \cap \boldsymbol{K}' = (T_\alpha)_{\alpha \in J_1 \times J_2}$ such that $S_\alpha = K_\alpha \cup K'_\alpha$ and $T_\alpha = K_\alpha \cap K'_\alpha$ for each $\alpha \in J_1 \times J_2$. By the same way we can define union and intersection of many 9×9 matrices and we can use the ususal rules in the set theory for 9×9 matrices . In this paper, matrices are these 9×9 matrices. Let $\boldsymbol{K} = (K_\alpha)_{\alpha \in J_1 \times J_2}$ be a singleton matrix provided that $|K_\alpha| = 1$ for each $\alpha \in J_1 \times J_2$.

Let $f \in \boldsymbol{SOL}(f_0)$. We say, a 9×9 matrix $\boldsymbol{K} = (K_\alpha)_{\alpha \in J_1 \times J_2}$ is a sudoku matrix associated with (f, f_0) provided that it satisfies the following condition:

(*SMTX*) $f(\alpha) \in K_\alpha$ for each $\alpha \in J_1 \times J_2$.

We denote $\boldsymbol{STMX}(f, f_0) = \{\boldsymbol{K} : \boldsymbol{K}$ ***is a sudoku matrix associated with*** $(f, f_0)\}$ and $\boldsymbol{STMX}(f_0) = \cap \{\boldsymbol{STMX}(f, f_0) : f \in \boldsymbol{SOL}(f_0)\}$. Each matrix $\boldsymbol{K} \in \boldsymbol{STMX}(f_0)$ is called as a sudoku matrix associated with f_0.

We can easily show the followings by definitions:

Proposition 1. Let $\boldsymbol{K}$ and $\boldsymbol{K}'$ be 9×9 matrices.

(a) If $\boldsymbol{K}, \boldsymbol{K}' \in \boldsymbol{STMX}(f, f_0)$, then $\boldsymbol{K} \cap \boldsymbol{K}' \in \boldsymbol{STMX}(f, f_0)$.

(b) If $\boldsymbol{K} \leqq \boldsymbol{K}'$ and $\boldsymbol{K} \in \boldsymbol{STMX}(f, f_0)$, then $\boldsymbol{K}' \in \boldsymbol{STMX}(f, f_0)$.

(c) If $\boldsymbol{K}, \boldsymbol{K}' \in \boldsymbol{STMX}(f_0)$, then $\boldsymbol{K} \cap \boldsymbol{K}' \in \boldsymbol{STMX}(f_0)$.

(d) If $\boldsymbol{K} \leqq \boldsymbol{K}'$ and $\boldsymbol{K} \in \boldsymbol{STMX}(f_0)$, then $\boldsymbol{K}' \in \boldsymbol{STMX}(f_0)$.

For a sudoku problem map $f_0 : L_0 \to J_3$, we define the sudoku matrix $\boldsymbol{Ker}(f_0) = (Ker(f_0)_\alpha)_{\alpha \in J_1 \times J_2}$ such that $Ker(f)_\alpha = \{f_0(\alpha)\}$ for each $\alpha \in L_0$ and $Ker(f_0)_\alpha = J_3$ for each $\alpha \in J_1 \times J_2 - L_0$. Similarly, for a sudoku solution map f of f_0, we define the sudoku matrix $\boldsymbol{Ker}(f) = \{Ker(f)_\alpha\}_{\alpha \in J_1 \times J_2}$ such that $Ker(f)_\alpha = \{f(\alpha)\}$ for each $\alpha \in J_1 \times J_2$.

We can easily show the following by definitions.

Proposition 2. Let $f \in \boldsymbol{SOL}(f_0)$ and $\boldsymbol{K} \in \boldsymbol{STMX}(f, f_0)$.

(a) $\boldsymbol{Ker}(f_0) \in \boldsymbol{STMX}(f_0)$.

(b) $\boldsymbol{Ker}(f), \boldsymbol{Ker}(f_0) \in \boldsymbol{STMX}(f, f_0)$ and $\boldsymbol{Ker}(f) \leqq \boldsymbol{Ker}(f_0)$.

(c) $\boldsymbol{K} = \boldsymbol{Ker}(f)$ if and only if $\boldsymbol{K}$ is a singleton matrix.

Now, we make a mathematical sudoku game as follows: Let $f_0 : L_0 \to J_3$ be a sudoku problem map. We assume that f_0 has a solution sudoku map $f : J_1 \times J_2 \to J_3$. Our approach is to make a decreasing finite sequence

(*) $\boldsymbol{K}_0 \to \boldsymbol{K}_1 \to \boldsymbol{K}_2 \to \ldots\ldots \to \boldsymbol{K}_m$

of sudoku matrices in $\boldsymbol{STMX}(f_0)$ or in $\boldsymbol{STMX}(f, f_0)$ such that

(**) $\boldsymbol{K}_0 = \boldsymbol{Ker}(f_0)$ and $\boldsymbol{K}_m$ is a singleton sudoku matrix.

In this case, since $\boldsymbol{K}_m$ is a singleton sudoku matrix, by Proposition 2 we have $\boldsymbol{K}_m = \boldsymbol{Ker}(f)$, that is, we take a solution map f. We say, this sequence is a solution sequence by sudoku matrices of f_0, or of (f,f_0).

2. Naked self−filled sets.

We need some theorems, which give guarantees to make decreasing sudoku matrices. Let $\boldsymbol{K}=(K_\alpha)_{\alpha\in J_1\times J_2}$ be a matrix. For each set $W\subset J_1\times J_2$, we define $K_W=\cup\{K_\alpha:\alpha\in W\}$. Let $0\leqq n\leqq 9$ and $s=\{\alpha_1,\alpha_2,\ldots,\alpha_n\}\subset J_1\times J_2$. We say that s is a naked n−self−filled set of $\boldsymbol{K}$ provided that it satisfies the following condition:

$(nNSF)$ $|K_s|=|K_{\alpha_1}\cup K_{\alpha_2}\cup\ldots\cup K_{\alpha_n}|=|s|=n$.

Proposition 3. Let $\boldsymbol{K}=(K_\alpha)_{\alpha\in J_1\times J_2}\in\boldsymbol{STMX}(f,f_0)$ and $b\in\boldsymbol{BLK}$. If s is a naked n−self−filled set of $\boldsymbol{K}$ and $s\subset b$, then we have the followings:

(a) $f(s)=K_s$ and $f(b-s)=J_3-K_s$,

(b) $f\mid s:s\to K_s$ and $f\mid b-s:b-s\to J_3-K_s$ are bijective.

Proof. By the condition (SDM) we have

(1) $f\mid b:b\to J_3$ is bijective.

Since $s\subset b$, we have that

(2) $b=s\cup(b-s)$ and $s\cap(b-s)=\phi$.

By (1) and (2), we have that

(3) $J_3=f(b)=f(s)\cup f(b-s)$ and $f(s)\cap f(s-b)=\phi$.

By (3) we have

(4) $f(b-s)=J_3-f(s)$.

Let $s=\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$. By the condition (SMTX), we have

(5) $f(\alpha_i)\in K_{\alpha_i}$ for each i, $1\leqq i\leqq n$.

By (5) we have

(6) $f(s)=\{f(\alpha_1),f(\alpha_2),\ldots,f(\alpha_n)\}\subset K_{\alpha_1}\cup K_{\alpha_2}\cup\ldots\cup K_{\alpha_n}=K_s$.

By (6)

(7) $|f(s)|\leqq|K_s|$.

By (1), we have

(8) $|f(s)|=|s|=n$.

Since s is a naked n−self−filled set of $\boldsymbol{K}$, by the condition $(nNSF)$

(9) $|K_s|=|s|=n$.

By (6),(7),(8),(9) we have

(10) $f(s)=K_s$.

By (4) and (10) we have

(11) $f(b-s)=J_3-K_s$.

By (10),(11) we have (a). Also by (1) and (a), we have (b). This completes the proof of Proposition 3.

Proposition 4. Let $\boldsymbol{K}=(K_\alpha)_{\alpha\in J_1\times J_2}$, $\boldsymbol{K}'=(K'_\alpha)_{\alpha\in J_1\times J_2}\in\boldsymbol{STMX}(f,f_0)$ and $s\subset b\in\boldsymbol{BLK}$. If s is a naked n−self−filled set of $\boldsymbol{K}$ and $\boldsymbol{K}'\leqq\boldsymbol{K}$, then s is also a naked n−self−filled set of $\boldsymbol{K}'$ and $K'_s=K_s$.

Proof. Let $s=\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$. Since $\boldsymbol{K}'\leqq\boldsymbol{K}$ and these satisfy the condition $(SMTX)$, we have

(1) $f(\alpha_i)\in K'_{\alpha_i}\subset K_{\alpha_i}$ for each i, $1\leqq i\leqq n$.

By (1), we have

$f(s)=\{f(\alpha_1),f(\alpha_2),...,f(\alpha_n)\}\subset K'_s=K'_{\alpha_1}\cup K'_{\alpha_2}\cup...\cup K'_{\alpha_n}\subset K_s=K_{\alpha_1}\cup K_{\alpha_2}\cup...\cup K_{\alpha_n}$,

that is,

(2) $f(s)\subset K'_s\subset K_s$.

Since s is a naked n−self−filled set of $\boldsymbol{K}$, by Proposition 3 we have that

(3) $f(s)=K_s$.

By (2) and (3), we have

(4) $f(s)=K'_s=K_s$.

Since s is a naked n−self−filled set of $\boldsymbol{K}$, we have that

(5) $|K_s|=|s|=n$,

By (4),(5) we have

(6) $|K'_s|=|K_s|=|s|=n$.

(6) means that s is also a naked n−self−filled set of $\boldsymbol{K}'$. We have proved Proposition 4.

Proposition 5. Let $\boldsymbol{K}=(K_\alpha)_{\alpha\in J_1\times J_2}\in\boldsymbol{STMX}(f,f_0)$ and $b\in\boldsymbol{BLK}$. Let s be a naked n−self−filled set of $\boldsymbol{K}$ and $s\subset b$. Then we have $\boldsymbol{K}'=(K'_\alpha)_{\alpha\in J_1\times J_2}$ $\in\boldsymbol{STMX}(f,f_0)$ and $\boldsymbol{K}'\leqq\boldsymbol{K}$, where $\boldsymbol{K}'$ is defined by

$$(*)\quad K'_\alpha=\begin{cases}K_\alpha & for\ \alpha\in s,\\ K_\alpha-K_s & for\ \alpha\in b-s,\\ K_\alpha & for\ \alpha\in J_1\times J_2-b.\end{cases}$$

We use the notation as follows: $\boldsymbol{K}'=nNSF((s,b),\boldsymbol{K})$.

Proof. We show the condition $(SMTX)$ for $\boldsymbol{K}'$, that is,

(1) $f(\alpha)\in K'_\alpha$ for each $\alpha\in J_1\times J_2$.

Since $\boldsymbol{K}$ satisfies $(SMTX)$, then

(2) $f(\alpha)\in K_\alpha$ for each $\alpha\in J_1\times J_2$.

By the above definition $(*)$ and (1), we have

(3) $K'_\alpha=K_\alpha\ni f(\alpha)$ for each $\alpha\in s\cup(J_1\times J_2-b)$.

By Proposition 3, we have that

(4) $f(b-s)=J_3-K_s$.

Take any $\alpha\in b-s$. By (2),(4), we have

(5) $f(\alpha)\in K_\alpha\cap f(b-s)=K_\alpha\cap(J_3-K_s)=K_\alpha-K_s=K'_\alpha$ for $\alpha\in b-s$.

By (3) and (5), we have (1). Hence $\boldsymbol{K}'\in\boldsymbol{STMX}(f,f_0)$.

Easily by the above definition $(*)$, we can show $\boldsymbol{K}'\leqq\boldsymbol{K}$. Therefore, we complete the proof of Proposition 5.

Proposition 6. Let $\boldsymbol{K}\in\boldsymbol{STMX}(f_0)$ and $b\in\boldsymbol{BLK}$. Let s be a naked n−self−filled set of $\boldsymbol{K}$ and $s\subset b$. Then $\boldsymbol{K}'=nNSF((s,b),\boldsymbol{K})\in\boldsymbol{STMX}(f_0)$.

Proof. Since $\boldsymbol{K}\in\boldsymbol{STMX}(f_0)=\cap\{\boldsymbol{STMX}(f,f_0):f\in\boldsymbol{SOL}(f_0)\}$, for each $f\in\boldsymbol{SOL}(f_0)$, $\boldsymbol{K}\in\boldsymbol{STMX}(f,f_0)$. Thus, by Proposition 5, $\boldsymbol{K}'\in\boldsymbol{STMX}(f,f_0)$ for each $f\in\boldsymbol{SOL}(f_0)$, that is, $\boldsymbol{K}'\in\boldsymbol{STMX}(f_0)$. Hence we show Proposition 6.

Proposition 7. Let $\boldsymbol{K}=(K_\alpha)_{\alpha\in J_1\times J_2}\in\boldsymbol{STMX}(f,f_0)$, $b\in\boldsymbol{BLK}$ and $b\supset s\supset t$. If s is a naked $n-$self$-$filled set of $\boldsymbol{K}$ and t is a naked $m-$self$-$filled set of $\boldsymbol{K}$, then $(s-t)$ is a naked $(n-m)-$self$-$filled set of $\boldsymbol{K}'=mNSF((t,b),\boldsymbol{K})$.

Proof. Let $t=\{\alpha_1,\alpha_2,...,\alpha_m\}\subset s=\{\alpha_1,\alpha_2,...,\alpha_m,\alpha_{m+1},...,\alpha_n\}$. Since s and t are naked self$-$filled sets of $\boldsymbol{K}$, we have that

(1) $|K_t|=m$, $K_t=K_{\alpha_1}\cup K_{\alpha_2}\cup...\cup K_{\alpha_m}$,

(2) $|K_s|=n$, $K_s=K_{\alpha_1}\cup K_{\alpha_2}\cup...\cup K_{\alpha_m}\cup K_{\alpha_{m+1}}\cup...\cup K_{\alpha_n}=K_t\cup K_{s-t}$.

By (SDM) we have

(3) $f\mid b\to J_3$ is bijective.

By Proposition 3 we have

(4) $f(s)=\boldsymbol{K}_s$ and $f(t)=\boldsymbol{K}_t$.

By (3) we have

(5) $f(t)\cup f(s-t)=f(s)$ and

(6) $f(t)\cap f(s-t)=\phi$.

By (5) and (6), we have

(7) $f(s)-f(t)=f(s-t)$.

By (4) and (7) we have

(8) $\boldsymbol{K}_s-\boldsymbol{K}_t=f(s-t)$.

Since $\boldsymbol{K}'=mNSF((t,b),\boldsymbol{K})$, we have

(9) $$K'_\alpha=\begin{cases}K_\alpha & for\ \alpha\in t,\\ K_\alpha-K_t & for\ \alpha\in b-t,\\ K_\alpha & for\ \alpha\in J_1\times J_2-b.\end{cases}$$

By (9) we have

$K'_{s-t}=K'_{\alpha_{m+1}}\cup K'_{\alpha_{m+2}}\cup...\cup K'_{\alpha_n}=(K_{\alpha_{m+1}}-K_t)\cup(K_{\alpha_{m+2}}-K_t)\cup...\cup(K_{\alpha_n}-K_t)$
$=K_{\alpha_{m+1}}\cup K_{\alpha_{m+2}}\cup...\cup K_{\alpha_n}-K_t=K_{s-t}-K_t=K_{s-t}\cup K_t-K_t=K_s-K_t$, that is,

(10) $K'_{s-t}=K_s-K_t$.

By (3),(8) and (10) we have

(11) $|K'_{s-t}|=|K_s-K_t|=|f(s-t)|=|s-t|=n-m$.

By (11) we show that $s-t$ satisfies $(n-m)NSF$ for $\boldsymbol{K}'$, and hence $s-t$ is a naked $(n-m)-$self filled set of $\boldsymbol{K}'$. We complete the proof.

Proposition 8. Let $b\in \boldsymbol{BLK}$ and $\boldsymbol{K}\in \boldsymbol{STMX}(f,f_0)$.

(a) b is a naked $9-$self$-$filled set of $\boldsymbol{K}$.

(b) ϕ is a naked $0-$self$-$filled set of $\boldsymbol{K}$.

Proof. Let us show (a). Let $b=\{\alpha_1,\alpha_2,...,\alpha_9\}$. By (SDM), we have

(1) $f\mid b: b\to J_3$ is bijective.

By $(SMTX)$, we have

(2) $f(\alpha_i)\in K_{\alpha_i}\subset J_3$.

By (2), we have

(3) $f(b)=\{f(\alpha_1),f(\alpha_2),...,f(\alpha_9)\}\subset K_{\alpha_1}\cup K_{\alpha_2}\cup...\cup K_{\alpha_9}=K_b\subset J_3$.

By (1),

(4) $f(b)=J_3$.

By (3),(4)

(5) $K_b = J_3$.
By(4),(5) we have
(6) $|K_b| = |J_3| = 9 = |b|$.
This means that b is a naked 9−self−filled set. Thus we have (a).
Next (b) follows as the fact $K_\phi = \phi$. Hence, we complete the proof.

5. Hidden self−filled sets.
Let $\boldsymbol{K} = (K_\alpha)_{\alpha \in J_1 \times J_2} \in \boldsymbol{STMX}(f, f_0)$. Let $t \subset s \subset J_1 \times J_2$ and s be a naked n−self−filled set of $\boldsymbol{K}$. We say, t is a hidden m−self−filled set in s of $\boldsymbol{K}$ provided that it satisfies the following condition:
($mHSF$) $|\boldsymbol{K}_s - \boldsymbol{K}_{s-t}| = |t| = m$.

Proposition 9. Let $\boldsymbol{K} = (K_\alpha)_{\alpha \in J_1 \times J_2} \in \boldsymbol{STMX}(f, f_0)$. Let $t \subset s \subset J_1 \times J_2$ and s be a naked n−self−filled set of $\boldsymbol{K}$. Here $n = |s| \geqq |t| = m$. Then the following conditions are equivalent.
(a) The set t is a hidden m−self−filled set in s of $\boldsymbol{K}$.
(b) The set $(s - t)$ is a naked $(n - m)$−self filled set of $\boldsymbol{K}$.
Proof. Since s is a naked n−self−filled set of $\boldsymbol{K}$, by ($nNSF$) we have
(1) $|K_s| = |s| = n$.
Since $t \subset s$, we have $s - t \subset s$ and
(2) $K_s \supset K_{s-t}$.
By (2) we have
(3) $K_s = (K_s - K_{s-t}) \cup K_{s-t}$ and $(K_s - K_{s-t}) \cap K_{s-t} = \phi$.
By (3) we have
(4) $|K_s| = |K_s - K_{s-t}| + |K_{s-t}|$.
By (1) and (4) we have
(5) $n = |K_s - K_{s-t}| + |K_{s-t}|$.

First, we show that (a) implies (b), in notation, (a)→(b).
By the assumption (a), then the set t is a hidden m−self−filled set in s of $\boldsymbol{K}$.
By ($mHSF$), we have
(6) $|K_s - K_{s-t}| = |t| = m$.
By (5) and (6) we have
(7) $|K_{s-t}| = n - m = |s - t|$.
(7) means that $(s - t)$ is a naked $(n - m)$−self−filled set of $\boldsymbol{K}$. Hence, we have (b).

Secondly, we show (b)→(a).
By the assumption (b), then $(s - t)$ is a naked $(n - m)$−self−filled set of $\boldsymbol{K}$. By ($(n - m)NSF$), we have
(8) $|K_{s-t}| = n - m = |s - t|$.
By (5) and (8), we have
(9) $|K_s - K_{s-t}| = m = |t|$.
(9) means that t is a hidden m−self−filled set in s of $\boldsymbol{K}$. Hence, we have (a).
Therefore, we complete the proof.

Proposition 10. Let $K=(K_\alpha)_{\alpha\in J_1\times J_2}\in STMX(f,f_0)$ and $b\in BLK$. Let s be a naked n-self-filled set of K and $s\subset b$. If $t\subset s$ and t is a hidden m-self-filled set in s of K, then t is a naked m-self-filled set of $K'=(n-m)NSF((s-t,b),K)$.

Proof. By Proposition 9, $s-t$ is a naked $(n-m)$-self-filled set of K. Then by Proposition 7, $s-(s-t)=t$ is a naked $n-(n-m)=m$-self-filled set of K'. Hence we complete the proof.

Proposition 11. Let $K=(K_\alpha)_{\alpha\in J_1\times J_2}\in STMX(f,f_0)$ and $b\in BLK$. Let s be a naked n-self-filled set of K and $t\subset s\subset b$. Then the followings are equivalent:

(a) t is a hidden m-self-filled set in s of K.

(b) There exists a set $L\subset K_s$ such that

(i) $t=t_L$, $t_L=\{\beta\in s: L\cap K_\beta\neq\phi\}$

(ii) $|t_L|=|L|=m$.

Proof. First, we show that (a)→(b).
Let $L=K_s-K_{s-t}\subset K_s$. We need the following Claim 1 and Claim 2:

Claim 1. $t\supset\{\beta\in s: L\cap K_\beta\neq\phi\}=t_L$.

We assume that Claim 1 does not hold. Then there exists an $\alpha\in t_L$ such that $\alpha\notin t$. Then we have

(1) $\alpha\in s$

(2) $L\cap K_\alpha\neq\phi$ and

(3) $\alpha\notin t$.

By (1) and (3),

(4) $\alpha\in s-t$.

By (4),

(5) $K_\alpha\subset K_{s-t}\subset K_s$.

Then by (1),(4) and (5) we have

$K_\alpha\cap L=K_\alpha\cap(K_s-K_{s-t})=K_\alpha\cap K_s-K_\alpha\cap K_{s-t}=K_\alpha-K_\alpha=\phi$,

that is,

(6) $K_\alpha\cap L=\phi$.

(6) and (2) give a contradiction. Hence Claim 1 holds.

Claim 2. $t\subset\{\beta\in s: L\cap K_\beta\neq\phi\}=t_L$.

We assume that Claim 2 does not hold. Then there exists an α such that

(7) $\alpha\in t$

(8) $\alpha\notin\{\beta\in s: L\cap K_\beta\neq\phi\}=t_L$.

Since $t\subset s$ and (7), then

(9) $\alpha\in$ s.

Hence, by (9) and (8)

(10) $L\cap K_\alpha=\phi$.

By (9) we have

(11) $K_\alpha\subset K_s$.

By (10), (11) we have

$\phi=L\cap K_\alpha=(K_s-K_{s-t})\cap K_\alpha=K_s\cap K_\alpha-K_{s-t}\cap K_\alpha=K_\alpha-K_{s-t}\cap K_\alpha$.

Hence we have

(12) $K_\alpha\subset K_{s-t}\cap K_\alpha\subset K_{s-t}$.

Since t is a hidden $m-$self$-$filled set in s of $\boldsymbol{K}$, by Proposition 9,
(13) $(s-t)$ is a naked $(n-m)-$self$-$filled set of $\boldsymbol{K}$.
By (13) and Proposition 3 we have
(14) $f(s-t)=K_{s-t}$ and
(15) $f(b-(s-t))=J_3-K_{s-t}$.
Since $t\subset s\subset b$, we have $t\subset b-(s-t)$. Hence, by (15) we have
(16) $f(t)\subset J_3-K_{s-t}$.
By (7) and (16) we have
(17) $f(\alpha)\in f(t)\subset J_3-K_{s-t}$.
However, since $\boldsymbol{K}\in \boldsymbol{STMX}(f,f_0)$, by (SMTX)
(18) $f(\alpha)\in K_\alpha$.
By (12) and (18)
(19) $f(\alpha)\in K_{s-t}$.
Both (17) and (19) give a contradiction. Therefore, we have Claim 2.

By Claim 1 and Claim 2 we have $t=t_L$, that is, the condition (i).
Since t is a hidden $m-$self$-$filled set in s of $\boldsymbol{K}$ by (a), by $(mHSF)$ it satisfies
(20) $|L|=|K_s-K_{s-t}|=m=|t|=|t_L|$.
Thus (20) means the condition (ii). Hence we have (b).

Next, we show that (b)$\rightarrow$(a).
By the definition of t_L, we have $t_L\subset s$. Then $m=|t_L|\leqq|s|=n$. Thus we may put
(21) $t_L=\{\alpha_1,\alpha_2,...,\alpha_m\}\subset\{\alpha_1,\alpha_2,...,\alpha_m,\alpha_{m+1},...,\alpha_n\}=s$.
By the definition of t_L,
(22) $L\cap K_{\alpha_i}=\phi,\ \ m+1\leqq i\leqq n$.
By (22) we have $K_{s-t_L}\cap L=\left(\cup_{i=m+1}^{n}K_{\alpha_i}\right)\cap L=\cup_{i=m+1}^{n}\left(K_{\alpha_i}\cap L\right)=\phi$, that is,
(23) $K_{s-t_L}\cap L=\phi$.
Since $L\subset K_s$, by (23) we have
(24) $L\subset K_s-K_{s-t_L}$.
Since $s\subset b$ and s ia a naked $n-$self$-$filled set of $\boldsymbol{K}$, by Proposition 3 we have
(25) $f(s)=K_s$ and
(26) $f\mid s:s\rightarrow f(s)$ is bijective.
By (25), we have
(27) $\{f(\alpha_i):i=1,2,...,n\}=f(s)=K_s$.
Since $\boldsymbol{K}\in \boldsymbol{STMX}(f,f_0)$, we have
(28) $f(\alpha_i)\in K_{\alpha_i}$, $1\leqq i\leqq n$.
By (28) we have
(29) $\{f(\alpha_i):i=m+1,...,n\}\subset\cup_{i=m+1}^{n}K_{\alpha_i}=K_{s-t_L}$.
By (26), (27) and (29) we have
(30) $K_s-K_{s-t_L}\subset\{f(a_i):i=1,2,...,n\}-\{f(\alpha_i):i=m+1,...,n\}=\{f(\alpha_i):i=1,2,..,m\}$.
By (24) and (30) we have
(31) $L\subset K_s-K_{s-t_L}\subset\{f(\alpha_i):i=1,2,..,m\}$.
By (26) and (31) we have

(32) $|L| \leqq |K_s - K_{s-t_L}| \leqq |\{f(\alpha_i): i=1,\cdot 2,...,m\}|$.
By (26),
(33) $m=|t_L|=|\{\alpha_1,\alpha_2,...,\alpha_m\}|=|\{f(\alpha_1),f(\alpha_2),...,f(\alpha_m)\}|$.
By the assumption $|L|=|t_L|=m$, (32) and (33) imply that
(34) $|K_s - K_{s-t_L}|=m=|t_L|$ and $L=K_s-K_{s-t_L}$.
(33) and (34) mean that $t=t_L$ is a hidden m−self−filled set in s of $\boldsymbol{K}$. Hence we have (a).
Hence we complete the proof of Proposition 11.

Remark 12. Many sudoku puzzler use many terminologies without any mathematical definitions. Among of them are locked candidates, n−nations allianceis sets, hidden m−nations alliance sets and so on, for example see Mepham[2].

The attempt to give the logical definition of locked candidates, naked n−nations allianceis and of hidden m−nations allianceis by Crook[1] and Masuo[2], respectively. However, they can not show the relations among them. Because their definitions contains some ambiguities. So we need the notion of sudoku matrices and give mathematical definitions for naked n−self−filled sets and hidden m−self−filled sets by using sudoku matrices.

Our approach is new, but many our concepts depend on many previous authors ones. For example, the concept of n−self−filled sets comes from Crook [1] and the concept of hidden m−self−filled sets comes from Sasao[2].

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