Non-Hermitian Extensions of Schrödinger Type Uncertainty Relations

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***Abstract*—In quantum mechanics it is well known that observables are represented by hermitian matrices (or operators). Uncertainty relations are represented as some kinds of trace inequalities satisfied by two observables and one density matrix (or operator). Now we try to release the hermitian restriction on observables. This is only a mathematical interest. In this case we give several non-hermitian extensions of the Schrödinger type uncertainty relation for generalized skew information under some conditions.**

I. Introduction

Wigner-Yanase skew information

$$\begin{aligned} I_\rho(H) &= \frac{1}{2}Tr\left[\left(i\left[\rho^{1/2}, H\right]\right)^2\right] \\ &= Tr[\rho H^2] - Tr[\rho^{1/2}H\rho^{1/2}H] \end{aligned}$$

was defined in [15]. This quantity can be considered as a kind of the degree for non-commutativity between a quantum state ρ and an observable H. Here we denote the commutator by $[X, Y] = XY - YX$. This quantity was generalized by Dyson

$$\begin{aligned} I_{\rho,\alpha}(H) &= \frac{1}{2}Tr[(i[\rho^\alpha, H])(i[\rho^{1-\alpha}, H])] \\ &= Tr[\rho H^2] - Tr[\rho^\alpha H\rho^{1-\alpha}H], \alpha \in [0, 1] \end{aligned}$$

which is known as the Wigner-Yanase-Dyson skew information. Recently it is shown that these skew informations are connected to special choices of quantum Fisher information in [7]. The family of all quantum Fisher informations is parametrized by a certain class of operator monotone functions $\mathcal{F}_{op}$ which were justified in [13]. The Wigner-Yanase skew information and Wigner-Yanase-Dyson skew information are given by the following operator monotone functions

$$f_{WY}(x) = \left(\frac{\sqrt{x}+1}{2}\right)^2,$$

$$f_{WYD}(x) = \alpha(1-\alpha)\frac{(x-1)^2}{(x^\alpha - 1)(x^{1-\alpha}-1)},\ \alpha \in (0, 1),$$

respectively. In particular the operator monotonicity of the function f_{WYD} was proved in [14]. See also [5]. On the other hand the uncertainty relation related to Wigner-Yanase skew information was given by Luo [12] and the uncertainty relation related to Wigner-Yanase-Dyson skew information was given by Yanagi [16], respectively. Recently Dou and Du proposed the release the restriction on operators which are observables. And they defined the corresponding Wigner-Yanase-Dyson skew information and studied some properties of them in [1], [2]. Also they obtained non-hermitian extensions of Heisenberg or Schrödinger uncertainty relations which is a generalization of Luo's theorem. In this paper we give several kinds of non-hermitian extensions of Schrödinger type uncertainty relations which correspond to the results given in [3], [4], [18] in the case of hermitian observables.

II. Non-Hermitian Extension I

Let $M_n(\mathbb{C})$(resp. $M_{n,sa}(\mathbb{C})$) be the set of all $n \times n$ complex matrices (resp. all $n \times n$ self-adjoint matrices), endowed with the Hilbert-Schmidt scalar product $\langle A, B\rangle = Tr(A^*B)$. Let $M_{n,+}(\mathbb{C})$ be the set of strictly positive elements of $M_n(\mathbb{C})$ and $M_{n,+,1}(\mathbb{C})$ be the set of stricly positive density matrices, that is $M_{n,+,1}(\mathbb{C}) = \{\rho \in M_n(\mathbb{C}) | Tr\rho = 1, \rho > 0\}$. If it is not otherwise specified, from now on we shall treat the case of faithful states, that is $\rho > 0$.

Definition 2.1: For $X \in M_n(\mathbb{C})$ and $\rho \in M_{n,+,1}(\mathbb{C})$, non-hermitian Wigner-Yanase skew information is defined by

$$\begin{aligned} &|I_\rho|(X) \\ = &\frac{1}{2}Tr[(i[\rho^{1/2}, X^*])(i[\rho^{1/2}, X])] \\ = &\frac{1}{2}Tr[\rho X^*X] + \frac{1}{2}Tr[\rho XX^*] - Tr[\rho^{1/2}X^*\rho^{1/2}X]. \end{aligned}$$

We remark that $|I_\rho|(X) = |I_\rho|(X_0)$, where $X_0 = X - Tr[\rho X]I$. We also define

$$\begin{aligned} &|J_\rho|(X) \\ = &\frac{1}{2}Tr[\{\rho^{1/2}, X_0^*\}\{\rho^{1/2}, X_0\}] \\ = &\frac{1}{2}Tr[\rho X_0^*X_0] + \frac{1}{2}Tr[\rho X_0X_0^*] + Tr[\rho^{1/2}X_0^*\rho^{1/2}X_0], \end{aligned}$$

where $\{X, Y\} = XY + YX$ is anti-commutator. If we define

$$|U_\rho|(X) = \sqrt{|I_\rho|(X) \cdot |J_\rho|(X)},$$

then we have

$$0 \le |I_\rho|(X) \le |U_\rho|(X) \le |V_\rho|(X) \le |J_\rho|(X),$$

where

$$|V_\rho|(X) = \frac{1}{2}Tr[\rho X_0^*X_0] + \frac{1}{2}Tr[\rho X_0X_0^*]$$

is non-hermitian variance. And also we define for $X, Y \in M_n(\mathbb{C})$

$$\begin{aligned} & |Corr_\rho|(X,Y) \\ = \ & Tr[\rho X_0^* Y_0] - Tr[\rho^{1/2} X_0^* \rho^{1/2} Y_0], \end{aligned}$$

and

$$\begin{aligned} & |Corr_\rho^0|(X,Y) \\ = \ & \frac{1}{2}|Corr_\rho|(X,Y) + \frac{1}{2}|Corr_\rho|(X^*,Y^*). \end{aligned}$$

Now we have the following theorem which is non-hermitian extension of Theorem 1 given by Furuichi [3].

Theorem 2.1: For $X, Y \in M_n(\mathbb{C})$ and $\rho \in M_{n,+,1}(\mathbb{C})$

$$|U_\rho|(X) \cdot |U_\rho|(Y) \geq ||Corr_\rho^0|(X,Y)|^2.$$

Proof. Let $\rho = \sum_i \lambda_i |\phi_i\rangle\langle\phi_i|, a_{ij} = \langle\phi_i|X|\phi_j\rangle, b_{ij} = \langle\phi_i|Y|\phi_j\rangle$. It is easy to show that

$$|I_\rho|(X) = \frac{1}{2}\sum_{i,j}(\lambda_i^{1/2} - \lambda_j^{1/2})^2 |a_{ji}|^2,$$

$$|J_\rho|(X) = \frac{1}{2}\sum_{i,j}(\lambda_i^{1/2} + \lambda_j^{1/2})^2 |a_{ji}|^2,$$

$$\begin{aligned} |Corr_\rho^0|(X,Y) = \ & \frac{1}{2}\sum_{i,j}(\lambda_i - \lambda_i^{1/2}\lambda_j^{1/2})\overline{a_{ji}}b_{ji} \\ & + \frac{1}{2}\sum_{i,j}(\lambda_j - \lambda_j^{1/2}\lambda_i^{1/2}) a_{ji}\overline{b_{ji}}. \end{aligned}$$

Since

$$||Corr_\rho^0|(X,Y)| \leq \frac{1}{2}\sum_{i,j}(\lambda_i^{1/2} + \lambda_j^{1/2})|\lambda_i^{1/2} - \lambda_j^{1/2}||a_{ji}||b_{ji}|,$$

we have

$$\begin{aligned} & ||Corr_\rho^0|(X,Y)|^2 \\ \leq \ & \frac{1}{4}\{\sum_{i,j}(\lambda_i^{1/2} + \lambda_j^{1/2})|\lambda_i^{1/2} - \lambda_j^{1/2}||a_{ji}||b_{ji}|\}^2 \\ \leq \ & \frac{1}{4}\{\sum_{i,j}(\lambda_i^{1/2} - \lambda_j^{1/2})^2 |a_{ji}|^2\}\{\sum_{i,j}(\lambda_i^{1/2} + \lambda_j^{1/2})^2 |b_{ji}|^2\} \\ = \ & |I_\rho|(X) \cdot |J_\rho|(Y) \end{aligned}$$

by using Schwarz's inequality. Similarly

$$||Corr_\rho^0|(X,Y)|^2 \leq |I_\rho|(Y) \cdot |J_\rho|(X).$$

Then we have the result.

III. Non-Hermitian Extension II

Definition 3.1: For $X \in M_n(\mathbb{C})$, $\alpha \in [0,1]$ and $\rho \in M_{n,+,1}(\mathbb{C})$, non-hermitian Wigner-Yanase-Dyson skew information is defined by

$$\begin{aligned} & |I_{\rho,\alpha}|(X) \\ = \ & \frac{1}{2}Tr[(i[\rho^\alpha, X^*])(i[\rho^{1-\alpha}, X])] \\ = \ & \frac{1}{2}Tr[\rho X^* X] + \frac{1}{2}Tr[\rho X X^*] \\ & -\frac{1}{2}Tr[\rho^\alpha X^* \rho^{1-\alpha} X] - \frac{1}{2}Tr[\rho^{1-\alpha} X^* \rho^\alpha X]. \end{aligned}$$

We remark that $|I_{\rho,\alpha}|(X) = |I_{\rho,\alpha}|(X_0)$, where $X_0 = X - Tr[\rho X]I$. We also define

$$\begin{aligned} & |J_{\rho,\alpha}|(X) \\ = \ & \frac{1}{2}Tr[\{\rho^\alpha, X_0^*\}\{\rho^{1-\alpha}, X_0\}] \\ = \ & \frac{1}{2}Tr[\rho X_0^* X_0] + \frac{1}{2}Tr[\rho X_0 X_0^*] \\ & +\frac{1}{2}Tr[\rho^\alpha X_0^* \rho^{1-\alpha} X_0] + \frac{1}{2}Tr[\rho^{1-\alpha} X_0^* \rho^\alpha X_0]. \end{aligned}$$

If we define

$$|U_{\rho,\alpha}|(X) = \sqrt{|I_{\rho,\alpha}|(X) \cdot |J_{\rho,\alpha}|(X)},$$

then we have

$$0 \leq |I_{\rho,\alpha}|(X) \leq |U_{\rho,\alpha}|(X) \leq |V_\rho|(X) \leq |J_{\rho,\alpha}|(X).$$

And also we define for $X, Y \in M_n(\mathbb{C})$

$$\begin{aligned} & |Corr_{\rho,\alpha}|(X,Y) \\ = \ & Tr[\rho X_0^* Y_0] - Tr[\rho^\alpha X_0^* \rho^{1-\alpha} Y_0], \end{aligned}$$

and

$$\begin{aligned} & |Corr_{\rho,\alpha}^0|(X,Y) \\ = \ & \frac{1}{2}|Corr_{\rho,\alpha}|(X,Y) + \frac{1}{2}|Corr_{\rho,\alpha}|(X^*,Y^*). \end{aligned}$$

Now we have the following theorem which is non-hermitian extension of Theorem 2.1 given by Furuichi-Yanagi [4].

Theorem 3.1: For $X, Y \in M_n(\mathbb{C})$ and $\rho \in M_{n,+,1}(\mathbb{C})$, if $1/2 \leq \alpha \leq 1$, then

$$|U_{\rho,\alpha}|(X) \cdot |U_{\rho,\alpha}|(Y) \geq 4\alpha(1-\alpha)||Corr_{\rho,\alpha}^0|(X,Y)|^2.$$

Proof. It is easy to show that

$$|I_{\rho,\alpha}|(X) = \frac{1}{2}\sum_{i,j}(\lambda_i^\alpha - \lambda_j^\alpha)(\lambda_i^{1-\alpha} - \lambda_j^{1-\alpha})|a_{ji}|^2,$$

$$|J_{\rho,\alpha}|(X) = \frac{1}{2}\sum_{i,j}(\lambda_i^\alpha + \lambda_j^\alpha)(\lambda_i^{1-\alpha} + \lambda_j^{1-\alpha})|a_{ji}|^2,$$

$$\begin{aligned} |Corr_{\rho,\alpha}^0|(X,Y) = \ & \frac{1}{2}\sum_{i,j}(\lambda_i - \lambda_i^\alpha\lambda_j^{1-\alpha})\overline{a_{ji}}b_{ji} \\ & + \frac{1}{2}\sum_{i,j}(\lambda_j - \lambda_j^\alpha\lambda_i^{1-\alpha}) a_{ji}\overline{b_{ji}}. \end{aligned}$$

Since

$$||Corr^0_{\rho,\alpha}|(X,Y)| \leq \frac{1}{2}\sum_{i,j}(\lambda_i^\alpha+\lambda_j^\alpha)|\lambda_i^{1-\alpha}-\lambda_j^{1-\alpha}||a_{ji}||b_{ji}|,$$

we have

$$\begin{aligned} & 4\alpha(1-\alpha)||Corr^0_{\rho,\alpha}|(X,Y)|^2 \\ \leq\ & \alpha(1-\alpha)\{\sum_{i,j}(\lambda_i^\alpha+\lambda_j^\alpha)|\lambda_i^{1-\alpha}-\lambda_j^{1-\alpha}||a_{ji}||b_{ji}|\}^2 \\ \leq\ & \{\sum_{i,j}\sqrt{\alpha(1-\alpha)}|\lambda_i^\alpha-\lambda_j^\alpha||a_{ji}||b_{ji}|\}^2 \\ \leq\ & \frac{1}{4}\{\sum_{i,j}(\Delta^-_\alpha\Delta^-_{1-\alpha}\Delta^+_\alpha\Delta^+_{1-\alpha})^{1/2}|a_{ji}||b_{ji}|\}^2 \\ \leq\ & \frac{1}{4}\{\sum_{i,j}\Delta^-_\alpha\Delta^-_{1-\alpha}|a_{ji}|^2\}\{\sum_{i,j}\Delta^+_\alpha\Delta^+_{1-\alpha}|b_{ji}|^2\} \\ =\ & |I_{\rho,\alpha}|(X)\cdot|J_{\rho,\alpha}|(Y) \end{aligned}$$

by using Schwarz's inequality, where

$$\Delta^\pm_\alpha = \lambda_i^\alpha \pm \lambda_j^\alpha, \quad \Delta^\pm_{1-\alpha} = \lambda_i^{1-\alpha} \pm \lambda_j^{1-\alpha}.$$

Similarly

$$4\alpha(1-\alpha)||Corr^0_{\rho,\alpha}|(X,Y)|^2 \leq |I_{\rho,\alpha}|(Y)\cdot|J_{\rho,\alpha}|(X).$$

Then we have the result.

IV. Operator Monotone Functions

A function $f:(0,+\infty)\to\mathbb{R}$ is said operator monotone if, for any $n\in\mathbb{N}$, and $A,B\in M_{n,+}(\mathbb{C})$ such that $0\leq A\leq B$, the inequalities $0\leq f(A)\leq f(B)$ hold. An operator monotone function is said symmetric if $f(x)=xf(x^{-1})$ and normalized if $f(1)=1$.

Definition 4.1: $\mathcal{F}_{op}$ is the class of functions $f:(0,+\infty)\to(0,+\infty)$ such that

1) $f(1)=1$,
2) $tf(t^{-1})=f(t)$,
3) f is operator monotone.

Example 4.1: Examples of elements of $\mathcal{F}_{op}$ are given by the following list

$$f_{RLD}(x)=\frac{2x}{x+1},\quad f_{WY}(x)=\left(\frac{\sqrt{x}+1}{2}\right)^2,$$

$$f_{BKM}(x)=\frac{x-1}{\log x},\quad f_{SLD}(x)=\frac{x+1}{2},$$

$$f_{WYD}(x)=\alpha(1-\alpha)\frac{(x-1)^2}{(x^\alpha-1)(x^{1-\alpha}-1)},\ \alpha\in(0,1).$$

Remark 4.1: Any $f\in\mathcal{F}_{op}$ satisfies

$$\frac{2x}{x+1}\leq f(x)\leq\frac{x+1}{2},\quad x>0.$$

For $f\in\mathcal{F}_{op}$ define $f(0)=\lim_{x\to 0}f(x)$. We introduce the sets of regular and non-regular functions

$$\mathcal{F}^r_{op}=\{f\in\mathcal{F}_{op}|f(0)\neq 0\},\quad \mathcal{F}^n_{op}=\{f\in\mathcal{F}_{op}|f(0)=0\}$$

and notice that trivially $\mathcal{F}_{op}=\mathcal{F}^r_{op}\cup\mathcal{F}^n_{op}$.

Definition 4.2: For $f\in\mathcal{F}^r_{op}$ we set

$$\tilde{f}(x)=\frac{1}{2}\left[(x+1)-(x-1)^2\frac{f(0)}{f(x)}\right],\quad x>0.$$

Theorem 4.1 ([6], [7]): The correspondence $f\to\tilde{f}$ is a bijection between $\mathcal{F}^r_{op}$ and $\mathcal{F}^n_{op}$.

V. Non-Hermitian Extension III

In Kubo-Ando theory [11] of matrix means one associates a mean to each operator monotone function $f\in\mathcal{F}_{op}$ by the formula

$$m_f(A,B)=A^{1/2}f(A^{-1/2}BA^{-1/2})A^{1/2},$$

where $A,B\in M_{n,sa}(\mathbb{C})$. Using the notion of matrix means one may define the class of monotone metrics (also said quantum Fisher informtions) by the following formula

$$\langle X,Y\rangle_{\rho,f}=Tr(X^*\cdot m_f(L_\rho,R_\rho)^{-1}(Y)),$$

where $X,Y\in M_n(\mathbb{C}), L_\rho(X)=\rho X, R_\rho(X)=X\rho$. In the hermitian case one has to think of $A,B\in M_{n,sa}(\mathbb{C})$ as tangent vectors to the manifold $M_{n,+,1}(\mathbb{C})$ at the point ρ (see [13], [7]).

Definition 5.1: For $X\in M_n(\mathbb{C})$ and $\rho\in M_{n,+,1}(\mathbb{C})$, we define the following quantities:

$$\begin{aligned}|I^f_\rho|(X) &= \frac{1}{2}Tr[\rho X^*X]+\frac{1}{2}Tr[\rho XX^*] \\ &\quad -Tr[X^*m_{\tilde{f}}(L_\rho,R_\rho)X].\end{aligned}$$

We remark that $|I^f_\rho|(X)=|I^f_\rho|(X_0)$, where $X_0=X-Tr[\rho X]I$. We also define

$$\begin{aligned}|J^f_\rho|(X) &= \frac{1}{2}Tr[\rho X_0^*X_0]+\frac{1}{2}Tr[\rho X_0X_0^*] \\ &\quad +Tr[X_0^*m_{\tilde{f}}(L_\rho,R_\rho)X_0].\end{aligned}$$

If we define

$$|U^f_\rho|(X)=\sqrt{|I^f_\rho|(X)\cdot|J^f_\rho|(X)},$$

then we have

$$0\leq|I^f_\rho|(X)\leq|U^f_\rho|(X)\leq|V_\rho|(X)\leq|J^f_\rho|(X).$$

And also we define for $X,Y\in M_n(\mathbb{C})$

$$\begin{aligned}& |Corr_{\rho,f}|(X,Y) \\ =\ & Tr[\rho X_0^*Y_0]-Tr[X_0^*m_{\tilde{f}}(L_\rho,R_\rho)Y_0],\end{aligned}$$

and

$$\begin{aligned}& |Corr^0_{\rho,f}|(X,Y) \\ =\ & \frac{1}{2}|Corr_{\rho,f}|(X,Y)+\frac{1}{2}|Corr_{\rho,f}|(X^*,Y^*).\end{aligned}$$

Now we have the following theorem which is non-hermitian extension of Theorem 3 given by Yanagi-Furuichi-Kuriyama [18].

Theorem 5.1: For $X,Y\in M_n(\mathbb{C})$ and $\rho\in M_{n,+,1}(\mathbb{C})$, if

$$\frac{x+1}{2}+\tilde{f}(x)\geq 2f(x),$$

then we have

$$|U_{\rho,f}|(X)\cdot|U_{\rho,f}|(Y)\geq 4f(0)||Corr^0_{\rho,f}|(X,Y)|^2.$$

Proof. It is easy to show that

$$|I^f_\rho|(X)=\sum_{i,j}\{\frac{\lambda_i+\lambda_j}{2}-m_{\tilde f}(\lambda_i,\lambda_j)\}\overline{a_{ji}}a_{ji},$$

$$|J^f_\rho|(X)=\sum_{i,j}\{\frac{\lambda_i+\lambda_j}{2}+m_{\tilde f}(\lambda_i,\lambda_j)\}\overline{a_{ji}}a_{ji},$$

$$\begin{aligned}|Corr^f_\rho|(X,Y) &= \frac{1}{2}\sum_{i,j}(\lambda_i-m_{\tilde f}(\lambda_i,\lambda_j))\overline{a_{ji}}b_{ji}\\ &\quad+\frac{1}{2}\sum_{i,j}(\lambda_j-m_{\tilde f}(\lambda_i,\lambda_j))a_{ji}\overline{b_{ji}}.\end{aligned}$$

Since

$$\begin{aligned}&||Corr^f_\rho|(X,Y)|\\ \leq\ & \frac{1}{2}\sum_{i,j}(|\lambda_i-m_{\tilde f}(\lambda_i,\lambda_j)|+|\lambda_j-m_{\tilde f}(\lambda_i,\lambda_j)|)|a_{ji}||b_{ji}|\\ =\ & \frac{1}{2}\sum_{i,j}|\lambda_i-\lambda_j||a_{ji}||b_{ji}|,\end{aligned}$$

we have

$$\begin{aligned}&f(0)||Corr^f_\rho(X,Y)|^2\\ \leq\ & \frac{1}{4}\left(\sum_{i,j}\{\frac{\lambda_i+\lambda_j}{2}-m_{\tilde f}(\lambda_i,\lambda_j)\}|a_{ji}|^2\right)\\ &\times\left(\sum_{i,j}\{\frac{\lambda_i+\lambda_j}{2}+m_{\tilde f}(\lambda_i,\lambda_j)\}|b_{ji}|^2\right)\\ =\ & \frac{1}{4}|I^f_\rho|(X)\cdot|J^f_\rho|'Y),\end{aligned}$$

by using Schwarz's inequality. Similarly

$$4f(0)||Corr^f_\rho(X,Y)|^2\leq|I^f_\rho|(Y)\cdot|J^f_\rho|(X).$$

Then we have the result.

Remark 5.1: In Theorem 5.1 we let $f(x)=f_{WYD}(x)$. Then we have the result in Theorem 3.1. In particular for $\alpha=\frac{1}{2}$ we have the result in Theorem 2.1.

VI. Non-Hermitian Extension IV

Definition 6.1: For $X\in M_n(\mathbb{C})$ and $\rho\in M_{n,+,1}(\mathbb{C})$, we define the following quantities:

$$\begin{aligned}|I^f_\rho|(X) &= \frac{f(0)}{2}\langle i[\rho,X],i[\rho,X]\rangle_{\rho,f}\\ &= \frac{1}{2}Tr[\rho X^*X]+\frac{1}{2}Tr[\rho XX^*]\\ &\quad -Tr[X^*m_{\tilde f}(L_\rho,R_\rho)X].\end{aligned}$$

We remark that $|I^f_\rho|(X)=|I^f_\rho|(X_0)$, where $X_0=X-Tr[\rho X]I$. And also we define for $X,Y\in M_n(\mathbb{C})$

$$|Corr_{\rho,f}|(X,Y)=\frac{f(0)}{2}\langle i[\rho,X],i[\rho,Y]\rangle_{\rho,f}.$$

Now we have the following theorem which is non-hermitian extension of Theorem 2 given by Yanagi-Furuichi-Kuriyama [18].

Theorem 6.1: For $X,Y\in M_n(\mathbb{C})$ and $\rho\in M_{n,+,1}(\mathbb{C})$,

$$|I^f_\rho|(X)\cdot|I^f_\rho|(Y)\geq||Corr_{\rho,f}|(X,Y)|^2.$$

Proof. $|Corr_{\rho,f}|(X,Y)$ is an inner product on $M_n(\mathbb{C})$. Then by Schwarz's inequality, we have

$$\begin{aligned}||Corr_{\rho,f}|(X,Y)|^2 &\leq |Corr_{\rho,f}|(X,X)\cdot|Corr_{\rho,f}|(Y,Y)\\ &= |I^f_\rho|(X)\cdot|J^f_\rho|(Y).\end{aligned}$$

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