Non-Hermitian Extension of Uncertainty Relation

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1 Introduction

Wigner-Yanase skew information

$$\begin{aligned} I_\rho(H) &= \frac{1}{2}Tr\left[\left(i\left[\rho^{1/2}, H\right]\right)^2\right] \\ &= Tr[\rho H^2] - Tr[\rho^{1/2}H\rho^{1/2}H] \end{aligned}$$

was defined in [11]. This quantity can be considered as a kind of the degree for non-commutativity between a quantum state ρ and an observable H. Here we denote the commutator by $[X, Y] = XY - YX$. This quantity was generalized by Dyson

$$\begin{aligned} I_{\rho,\alpha}(H) &= \frac{1}{2}Tr[(i[\rho^\alpha, H])(i[\rho^{1-\alpha}, H])] \\ &= Tr[\rho H^2] - Tr[\rho^\alpha H\rho^{1-\alpha}H], \alpha \in [0,1] \end{aligned}$$

which is known as the Wigner-Yanase-Dyson skew information. Recently it is shown that these skew informations are connected to special choices of quantum Fisher information in [3]. The family of all quantum Fisher informations is parametrized by a certain class of operator monotone functions $\mathcal{F}_{op}$ which were justified in [9]. The Wigner-Yanase skew information and Wigner-Yanase-Dyson skew information are given by the following operator monotone functions

$$f_{WY}(x) = \left(\frac{\sqrt{x}+1}{2}\right)^2,$$

$$f_{WYD}(x) = \alpha(1-\alpha)\frac{(x-1)^2}{(x^\alpha - 1)(x^{1-\alpha} - 1)}, \ \alpha \in (0,1),$$

respectively. In particular the operator monotonicity of the function f_{WYD} was proved in [10]. On the other hand the uncertainty relation related to Wigner-Yanase skew information was given by Luo [8] and the uncertainty relation related to Wigner-Yanase-Dyson skew information was given by Yanagi [12], respectively. Also these uncertainty relations were generalized to the uncertainty relations related to quantum Fisher informations by using (generalized) metric adjusted skew information or correlation measure in [13, 14, 15]. In this paper we don't assume that observables are hermitian. Then we give the corresponding uncertainty relations by using generalized quasi-metric adjusted skew informations and generalized quasi-adjusted correlation measures. In particular we show how is the corresponding variance represented.

2 Operator Monotone Functions

Let $M_n(\mathbb{C})$(resp. $M_{n,sa}(\mathbb{C})$) be the set of all $n \times n$ complex matrices (resp. all $n \times n$ self-adjoint matrices), endowed with the Hilbert-Schmidt scalar product $\langle A, B\rangle = Tr(A^*B)$. Let $M_{n,+}(\mathbb{C})$ be the set of strictly positive elements of $M_n(\mathbb{C})$ and $M_{n,+,1}(\mathbb{C})$ be the set of stricly positive density matrices, that is $M_{n,+,1}(\mathbb{C}) = \{\rho \in M_n(\mathbb{C}) | Tr\rho = 1, \rho > 0\}$. If it is not otherwise specified, from now on we shall treat the case of faithful states, that is $\rho > 0$.

A function $f : (0, +\infty) \to \mathbb{R}$ is said operator monotone if, for any $n \in \mathbb{N}$, and $A, B \in M_n$ such that $0 \leq A \leq B$, the inequalities $0 \leq f(A) \leq f(B)$ hold. An operator monotone function is said symmetric if $f(x) = xf(x^{-1})$ and normalized if $f(1) = 1$.

Definition 2.1 *$\mathcal{F}_{op}$ is the class of functions $f : (0, +\infty) \to (0, +\infty)$ such that*

(1) $f(1) = 1$,

(2) $tf(t^{-1}) = f(t)$,

(3) f is operator monotone.

Example 2.1 *Examples of elements of $\mathcal{F}_{op}$ are given by the following list*

$$f_{RLD}(x) = \frac{2x}{x+1}, \quad f_{WY}(x) = \left(\frac{\sqrt{x}+1}{2}\right)^2, \quad f_{BKM}(x) = \frac{x-1}{\log x},$$

$$f_{SLD}(x) = \frac{x+1}{2}, \quad f_{WYD}(x) = \alpha(1-\alpha)\frac{(x-1)^2}{(x^\alpha - 1)(x^{1-\alpha} - 1)}, \quad \alpha \in (0, 1).$$

Remark 2.1 *Any $f \in \mathcal{F}_{op}$ satisfies*

$$\frac{2x}{x+1} \leq f(x) \leq \frac{x+1}{2}, \quad x > 0.$$

For $f \in \mathcal{F}_{op}$ define $f(0) = \lim_{x\to 0} f(x)$. We introduce the sets of regular and non-regular functions

$$\mathcal{F}_{op}^r = \{f \in \mathcal{F}_{op} | f(0) \neq 0\}, \quad \mathcal{F}_{op}^n \{f \in \mathcal{F}_{op} | f(0) = 0\}$$

and notice that trivially $\mathcal{F}_{op} = \mathcal{F}_{op}^r \cup \mathcal{F}_{op}^n$.

Definition 2.2 *Let $g, f \in \mathcal{F}_{op}^r$ satisfy*

$$g(x) \geq k\frac{(x-1)^2}{f(x)} \tag{2.1}$$

for some $k > 0$. We define

$$\Delta_g^f(x) = g(x) - k\frac{(x-1)^2}{f(x)} \in \mathcal{F}_{op}$$

3 Generalized Quasi-Metric Adusted Skew Information and Correlation Measure

In Kubo-Ando theory of matrix means one associates a mean to each operator monotone function $f \in \mathcal{F}_{op}$ by the formula

$$m_f(A, B) = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2},$$

where $A, B \in M_{n,+}(\mathbb{C})$. Using the notion of matrix means one may define the class of monotone metrics (also said quantum Fisher informtions) by the following formula

$$\langle A, B \rangle_{\rho,f} = Tr(A^* \cdot m_f(L_\rho, R_\rho)^{-1}(B)),$$

where $A, B \in M_n(\mathbb{C}), L_\rho(A) = \rho A, R_\rho(A) = A\rho$.

Now we define generalized quasi-metric adjusted skew information and correlation measure for non-hermitian matrices $M_n(\mathbb{C})$.

Definition 3.1 *For $A, B \in M_n(\mathbb{C})$ and $\rho \in M_{n,+,1}(\mathbb{C})$, we define the following quantities:*

$$|Corr_\rho^{(g,f)}|(A, B) = k\langle i[\rho, A], i[\rho, B]\rangle_{\rho,f}, \quad |I_\rho^{(g,f)}|(A) = |Corr_\rho^{(g,f)}|(A, A),$$

$$|C_\rho^f|(A, B) = Tr[A^* m_f(L_\rho, R_\rho)B], \quad |C_\rho^f|(A) = |C_\rho^f|(A, A),$$

$$|U_\rho^{(g,f)}|(A) = \sqrt{(|C_\rho^g|(A) + |C_\rho^{\Delta_g^f}|(A))(|C_\rho^g|(A) - |C_\rho^{\Delta_g^f}|(A))},$$

The quantity $|I_\rho^{(g,f)}|(A)$ and $|Corr_\rho^{(g,f)}|(A, B)$ are said generalized quasi-metric adjusted skew information and generalized quasi-metric adjusted correlation measure, respectively.

Then we have the following proposition.

Proposition 3.1 *For $A, B \in M_n(\mathbb{C})$ and $\rho \in M_{n,+,1}(\mathbb{C})$, we have the following relations, where we put $A_0 = A - Tr[\rho A]I$ and $B_0 = B - Tr[\rho B]$.*

(1) $|I_\rho^{(g,f)}|(A) = |I_\rho^{(g,f)}|(A_0) = |C_\rho^g|(A_0) - |C_\rho^{\Delta_g^f}|(A_0)$,

(2) $|J_\rho^{(g,f)}|(A) = |C_\rho^g|(A_0) + |C_\rho^{\Delta_g^f}|(A_0)$,

(3) $|U_\rho^{(g,f)}|(A) = \sqrt{|I_\rho^{(g,,f)}|(A) \cdot |J_\rho^{(g,f)}|(A)}$.

(4) $|Corr_\rho^{(g,f)}|(A, B) = |Corr_\rho^{(g,f)}|(A_0, B_0)$.

Theorem 3.1 *For $f \in \mathcal{F}_{op}^r$, it holds*

$$|I_\rho^{(g,f)}|(A) \cdot |I_\rho^{(g,f)}|(B) \geq ||Corr_\rho^{(g,f)}|(A,B)|^2,$$

where $A, B \in M_n(\mathbb{C})$ and $\rho \in M_{n,+,1}(\mathbb{C})$.

Proof of Theorem 3.1. We define for $X, Y \in M_n(\mathbb{C})$

$$|Corr_\rho^{(g,f)}|(X,Y) = k\langle i[\rho, X], i[\rho, Y]\rangle_{\rho,f}.$$

Since

$$\begin{aligned} |Corr_\rho^{(g,f)}|(X,Y) &= kTr((i[\rho, X])^* m_f(L_\rho, R_\rho)^{-1} i[\rho, Y]) \\ &= kTr((i(L_\rho - R_\rho)X)^* m_f(L_\rho, R_\rho)^{-1} i(L_\rho - R_\rho)Y) \\ &= Tr(X^* m_g(L_\rho, R_\rho)Y) - Tr(X^* m_{\Delta_g^f}(L_\rho, R_\rho)Y), \end{aligned}$$

it is easy to show that $|Corr_\rho^{(g,f)}|(X,Y)$ is an inner product in $M_n(\mathbb{C})$. Then we can get the result by using Schwarz inequality. □

Theorem 3.2 *For $f \in \mathcal{F}_{op}^r$, if*

$$g(x) + \Delta_g^f(x) \geq \ell f(x) \tag{3.1}$$

for some $\ell > 0$, then it holds

$$|U_\rho^{(g,f)}|(A) \cdot |U_\rho^{(g,f)}|(B) \geq k\ell |Tr(\rho[A, B])|^2, \tag{3.2}$$

where $A, B \in M_n(\mathbb{C})$ and $\rho \in M_{n,+,1}(\mathbb{C})$.

In order to prove Theorem 3.2, we need the following lemmas

Lemma 3.1 *If (2.1) and (3.1) are satisfied, then we have the following inequality:*

$$m_g(x,y)^2 - m_{\Delta_g^f}(x,y)^2 \geq k\ell(x-y)^2.$$

Proof of Lemma 3.1: By (2.1) and (3.1), we have

$$m_{\Delta_g^f}(x,y) = m_g(x,y) - k\frac{(x-y)^2}{m_f(x,y)}. \tag{3.3}$$

$$m_g(x,y) + m_{\Delta_g^f}(x,y) \geq \ell m_f(x,y), \tag{3.4}$$

Therefore by (3.3), (3.4)

$$\begin{aligned} & m_g(x,y)^2 - m_{\Delta_g^f}(x,y)^2 \\ = & \left\{m_g(x,y) - m_{\Delta_g^f}(x,y)\right\}\left\{m_g(x,y) + m_{\Delta_g^f}(x,y)\right\} \\ \geq & k\frac{(x-y)^2}{m_f(x,y)}\ell m_f(x,y) \\ = & k\ell(x-y)^2. \end{aligned}$$

□

Lemma 3.2 *Let $\{|\phi_1\rangle, |\phi_2\rangle, \cdots, |\phi_n\rangle\}$ be a basis of eigenvectors of ρ, corresponding to the eigenvalues $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$. We put $a_{jk} = \langle\phi_j|A_0|\phi_k\rangle, b_{jk} = \langle\phi_j|B_0|\phi_k\rangle$, where $A_0 \equiv A - Tr[\rho A]I$ and $B_0 \equiv B - Tr[\rho B]I$ for $A, B \in M_n(\mathbb{C})$ and $\rho \in M_{n,+,1}(\mathbb{C})$. Then we have*

$$\begin{aligned} |I_\rho^{(g,f)}|(A) &= \sum_{j,k} m_g(\lambda_j,\lambda_k)|a_{jk}|^2 - \sum_{j,k} m_{\Delta_g^f}(\lambda_j,\lambda_k)|a_{jk}|^2 \\ &= 2\sum_{j<k}\left\{(m_g(\lambda_j,\lambda_k) - m_{\Delta_g^f}(\lambda_j,\lambda_k)\right\}|a_{jk}|^2, \end{aligned}$$

$$\begin{aligned} |J_\rho^{(g,f)}|(A) &= \sum_{j,k} m_g(\lambda_j,\lambda_k)|a_{jk}|^2 + \sum_{j,k} m_{\Delta_g^f}(\lambda_j,\lambda_k)|a_{jk}|^2 \\ &\geq 2\sum_{j<k}\left\{m_g(\lambda_j,\lambda_k) + m_{\Delta_g^f}(\lambda_j,\lambda_k)\right\}|a_{jk}|^2, \end{aligned}$$

$$|U_\rho^{(g,f)}|(A)^2 = \left(\sum_{j,k} m_g(\lambda_j,\lambda_k)|a_{jk}|^2\right)^2 - \left(\sum_{j,k} m_{\Delta_g^f}(\lambda_j,\lambda_k)|a_{jk}|^2\right)^2$$

and

$$\begin{aligned} & |Corr_\rho^{(g,f)}|(A,B) \\ = & \sum_{j,k} m_g(\lambda_j,\lambda_k)\overline{a_{jk}}b_{jk} - \sum_{j,k} m_{\Delta_g^f}(\lambda_j,\lambda_k)\overline{a_{jk}}b_{jk} \\ = & \sum_{j<k}\left(m_g(\lambda_j,\lambda_k) - m_{\Delta_g^f}(\lambda_j,\lambda_k)\right)\overline{a_{jk}}b_{jk} + \sum_{j<k}\left(m_g(\lambda_k,\lambda_j) - m_{\Delta_g^f}(\lambda_k,\lambda_j)\right)\overline{a_{kj}}b_{kj}. \end{aligned}$$

We are now in a position to prove Theorem 3.2.

Proof of Theorem 3.2: At first we prove (3.3). Since

$$Tr(\rho[A,B]) = \sum_{j,k}(\lambda_j - \lambda_k)a_{jk}b_{kj},$$

$$|Tr(\rho[A,B])| \le \sum_{j,k}|\lambda_j - \lambda_k||a_{jk}||b_{kj}|.$$

Then by Lemma 3.1, we have

$$\begin{aligned}
& k\ell|Tr(\rho[A,B])|^2 \\
\le\ & \left\{\sum_{j,k}\sqrt{k\ell}|\lambda_j - \lambda_k||a_{jk}||b_{kj}|\right\}^2 \\
\le\ & \left\{\sum_{j,k}\left(m_g(\lambda_j,\lambda_k)^2 - m_{\Delta_g^f}(\lambda_j,\lambda_k)^2\right)^{1/2}|a_{jk}||b_{kj}|\right\}^2 \\
\le\ & \left\{\sum_{j,k}\left(m_g(\lambda_j,\lambda_k) - m_{\Delta_g^f}(\lambda_j u,\lambda_k)\right)|a_{jk}|^2\right\}\left\{\sum_{j,k}\left(m_g(\lambda_j,\lambda_k) + m_{\Delta_g^f}(\lambda_j,\lambda_k)\right)|b_{kj}|^2\right\} \\
=\ & |I_\rho^{(g,f)}|(A)|J_\rho^{(g,f)}|(B).
\end{aligned}$$

By the similar way, we also have

$$|I_\rho^{(g,f)}|(B)|J_\rho^{(g,f)}|(A) \ge k\ell|Tr(\rho[A,B])|^2.$$

Hence we have the desired inequality (3.2). □

4 Examples

Example 4.1 *When*

$$g(x) = \frac{x+1}{2}, \quad f(x) = \alpha(1-\alpha)\frac{(x-1)^2}{(x^\alpha - 1)(x^{1-\alpha} - 1)}, \quad k = \frac{f(0)}{2}, \quad \ell = 2,$$

and $A, B \in M_n(\mathbb{C})$, *we give the following:*

$$\Delta_g^f(x) = g(x) - k\frac{(x-1)^2}{f(x)} = \frac{1}{2}(x^\alpha + x^{1-\alpha}) \ge 0.$$

$$\begin{aligned}
& g(x) + \Delta_g^f(x) - \ell f(x) \\
=\ & \frac{1}{2(x^\alpha - 1)(x^{1-\alpha} - 1)}\{(x^{2\alpha} - 1)(x^{2(1-\alpha)} - 1) - 4\alpha(1-\alpha)(x-1)^2\} \ge 0.
\end{aligned}$$

Then

$$\begin{aligned} & |I_\rho^{(f,g)}|(A) = |I_\rho^{(f,g)}|(A_0) \\ = & \frac{1}{2}Tr[\rho A_0 A_0^*] + \frac{1}{2}Tr[\rho A_0^* A_0] - \frac{1}{2}Tr[\rho^\alpha A_0 \rho^{1-\alpha} A_0^*] - \frac{1}{2}Tr[\rho^\alpha A_0^* \rho^{1-\alpha} A_0]. \end{aligned}$$

In particular for $\alpha = 1/2$,

$$|I_\rho^{(f,g)}|(A) = |I_\rho^{(f,g)}|(A_0) = \frac{1}{2}Tr[\rho A_0 A_0^*] + \frac{1}{2}Tr[\rho A_0^* A_0] - Tr[\rho^{1/2} A_0 \rho^{1/2} A_0^*].$$

Then the corresponding variance is given by

$$|V_\rho|(A) = \frac{1}{2}Tr[\rho(|A_0|^2 + |A_0^*|^2)].$$

Example 4.2 *When*

$$g(x) = \left(\frac{\sqrt{x}+1}{2}\right)^2, \quad f(x) = \alpha(1-\alpha)\frac{(x-1)^2}{(x^\alpha - 1)(x^{1-\alpha} - 1)}.$$

and $A, B \in M_n(\mathbb{C})$, *we assume* $k = f(0)/8$ *and* $\ell = 3/2$, *then we have the following.*

$$\begin{aligned} \Delta_g^f(x) &= g(x) - k\frac{(x-1)^2}{f(x)} = \left(\frac{1+\sqrt{x}}{2}\right)^2 - \frac{1}{8}(x^\alpha - 1)(x^{1-\alpha} - 1) \\ &= \frac{1}{8}\{(1+\sqrt{x})^2 + (x^{\alpha/2} + x^{(1-\alpha)/2})^2\} \geq 0. \end{aligned}$$

$$\begin{aligned} & g(x) + \Delta_g^f(x) - \ell f(x) \\ = & \ 2g(x) - \frac{1}{8}(x^\alpha - 1)(x^{1-\alpha} - 1) - \frac{3}{2}f(x) \\ \geq & \ \frac{1}{2}g(x) - \frac{1}{8}(x^\alpha - 1)(x^{1-\alpha} - 1) \\ = & \ \frac{1}{8}(x^{\alpha/2} + x^{(1-\alpha)/2})^2 \geq 0. \end{aligned}$$

Example 4.3 *When*

$$g(x) = \left(\frac{x^\gamma + 1}{2}\right)^{1/\gamma} \quad (\frac{3}{4} \leq \gamma \leq 1), \quad f(x) = \left(\frac{\sqrt{x}+1}{2}\right)^2,$$

$$k = \frac{f(0)}{4}, \quad \ell = 2,$$

and $A, B \in M_n(\mathbb{C})$, *we give the following: Let*

$$F(x,r) = \left(\frac{1+x^r}{2}\right)^{1/r}.$$

Since $F(x,r)$ *is concave in* $r \in [1/2, 1]$ *(see [15]),*

$$F(t, \frac{3}{4}) \geq \frac{1}{2}F(t,1) + \frac{1}{2}F(t, \frac{1}{2}).$$

Then

$$2F(x,r) \geq 2F(x, \frac{3}{4}) \geq F(x,1) + F(x, \frac{1}{2}),$$

That is

$$2\left(\frac{1+x^r}{2}\right)^{1/r} - \left(\frac{\sqrt{x}-1}{2}\right)^2 > 2\left(\frac{\sqrt{x}+1}{2}\right)^2.$$

Then since

$$\Delta_g^f(x) = \left(\frac{1+x^r}{2}\right)^{1/r} - \left(\frac{\sqrt{x}-1}{2}\right)^2,$$

we have

$$g(x) + \Delta_g^f(x) \geq 2f(x).$$

Example 4.4 *When*

$$g(x) = \left(\frac{1+x^r}{2}\right)^{1/r}, \ (\frac{5}{8} \leq r \leq 1)$$

$$f(x) = \left(\frac{1+\sqrt{x}}{2}\right)^2, \quad k = \frac{f(0)}{8} = \frac{1}{32}, \quad \ell = 2.$$

we give the following. Since $F(x,r)$ *is concave in* $r \in [1/2, 3/4]$ *(see [15]),*

$$F(x, \frac{5}{8}) \geq \frac{1}{2}F(x, \frac{1}{2}) + \frac{1}{2}F(x, \frac{3}{4}).$$

Then

$$\begin{aligned}
2F(x,r) \geq 2F(x, \frac{5}{8}) &\geq F(x, \frac{1}{2}) + F(x, \frac{3}{4}) \\
&\geq F(x, \frac{1}{2}) + \frac{1}{2}\left\{\frac{x+1}{2} + \left(\frac{\sqrt{x}+1}{2}\right)^2\right\} \\
&= \frac{3}{2}\left(\frac{\sqrt{x}+1}{2}\right)^2 + \frac{1}{2}\frac{x+1}{2} \\
&= \frac{3}{2}\left(\frac{\sqrt{x}+1}{2}\right)^2 + \frac{1}{2}\left\{\left(\frac{\sqrt{x}-1}{2}\right)^2 + \left(\frac{\sqrt{x}+1}{2}\right)^2\right\} \\
&= 2\left(\frac{\sqrt{x}+1}{2}\right)^2 + \frac{1}{2}\left(\frac{\sqrt{x}-1}{2}\right)^2
\end{aligned}$$

Thus we have

$$g(x) + \Delta_g^f(x) \geq 2f(x).$$

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