On the Correspondence of Dual Orbits Related to Some Representation, I

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The purpose of this paper is to investigate the dual orbits correspondence of the non-regular simple prehomogeneous representation $(GL(1)^2 \times Sp(2), \Lambda_2 + \Lambda_1, V(5) \oplus V(4))$ (see Proposition 2).

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Preliminaries. In the following, we denote by G the group $GL(1)^2 \times Sp(2)$ and by ρ the representation $\Lambda_2 + \Lambda_1$ of $Sp(2)$ with scalar multiplications.

We define an element e_i of $\mathbb{C}^4$ by $e_1 = {}^t(1,0,0,0)$, $e_2 = {}^t(0,1,0,0)$, $e_3 = {}^t(0,0,1,0)$, $e_4 = {}^t(0,0,0,1)$. Put $u_1 = \frac{1}{2}(e_1 \wedge e_3 - e_2 \wedge e_4), u_2 = e_1 \wedge e_2, u_3 = e_1 \wedge e_4, u_4 = e_2 \wedge e_3, u_5 = e_3 \wedge e_4$. The representation space of ρ is identified with

$$V = \{\tilde{x} = (x_1, x_2); x_1 \in V_1, x_2 \in V_2\},$$

where $V_1 = \{x_1 = \sum_{i=1}^{5} x_{i1} u_i \in \bigwedge^2 \mathbb{C}^4; x_{i1} \in \mathbb{C}\ (1 \leqq i \leqq 5)\}$ and $V_2 = \{x_2 = \sum_{i=1}^{4} x_{i2} e_i \in \mathbb{C}^4; x_{i2} \in \mathbb{C}\ (1 \leqq i \leqq 4)\}$. Then the action ρ is given by

$$\rho(\tilde{g})\tilde{x} = (\alpha \rho_2(g) x_1, \beta g x_2)$$

for $\tilde{g} = (\alpha, \beta; g) \in G = GL(1)^2 \times Sp(2)$ and $\tilde{x} = (x_1, x_2) \in V$, where $\rho_2(g)(e_j \wedge e_k) = (g e_j) \wedge (g e_k)$.

Proposition 1. The triplet (G, ρ, V) has eight orbits $\rho(G)\tilde{x}_i$ $(1 \leqq i \leqq 8)$ where the representative points $\tilde{x}_i$ $(1 \leqq i \leqq 8)$ are given as follows:

Representative point	Codimension
(1) $\tilde{x}_1 = (e_1 \wedge e_2 + e_3 \wedge e_4, e_1)$	0
(2) $\tilde{x}_2 = (e_1 \wedge e_2, e_3)$	1
(3) $\tilde{x}_3 = (e_1 \wedge e_2 + e_3 \wedge e_4, e_1 + e_4)$	2
(4) $\tilde{x}_4 = (e_1 \wedge e_2, e_1)$	3
(5) $\tilde{x}_5 = (e_1 \wedge e_2 + e_3 \wedge e_4, 0)$	4
(6) $\tilde{x}_6 = (e_1 \wedge e_2, 0)$,	5
(7) $\tilde{x}_7 = (0, e_1)$	5
(8) $\tilde{x}_8 = (0, 0)$.	9

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Let Λ be the conormal bundle of an orbit S in V and Λ^* that of an orbit S^* in V^*. When $\Lambda = \Lambda^*$, we say that S and S^* are the dual orbits of each other.

Since G is reductive, we have $(G, \rho^*, V^*) \cong (G, \rho, V)$ and hence the dual space V^* has also eight G-orbits. We identify V and V^* as usual.

Proposition 2. The dual orbits correspondence of $(GL(1)^2 \times Sp(2), \Lambda_2 + \Lambda_1)$ is given as follows:

Representative point	Point of the dual orbit
$\tilde{x}_1 = (e_1 \wedge e_2 + e_3 \wedge e_4, e_1)$	$\tilde{x}_8$
$\tilde{x}_2 = (e_1 \wedge e_2, e_3)$	$\tilde{x}_6$
$\tilde{x}_3 = (e_1 \wedge e_2 + e_3 \wedge e_4, e_1 + e_4)$	$\tilde{x}_4$
$\tilde{x}_4 = (e_1 \wedge e_2, e_1)$	$\tilde{x}_3$
$\tilde{x}_5 = (e_1 \wedge e_2 + e_3 \wedge e_4, 0)$	$\tilde{x}_7$
$\tilde{x}_6 = (e_1 \wedge e_2, 0)$	$\tilde{x}_2$
$\tilde{x}_7 = (0, e_1)$	$\tilde{x}_5$
$\tilde{x}_8 = (0, 0)$	$\tilde{x}_1$

For a point $\tilde{x}$ of V, we denote by $G_{\tilde{x}} = \{\tilde{g} \in G; \rho(g)\tilde{x} = \tilde{x}\}$ the isotropy subgroup of G at $\tilde{x}$. Let $\mathfrak{G}_{\tilde{x}}$ (resp.$\mathfrak{G}$) be the Lie algebra of $G_{\tilde{x}}$ (resp.G), and $d\rho_{\tilde{x}}$ (resp.$d\rho$) the infinitesimal representation of $\rho_{\tilde{x}} = \rho^* |_{G_{\tilde{x}}}$ (resp.ρ).

We identify the cotangent bundle T^*V of V with $V \times V^*$. Let $\tilde{x}$ be a point of V. The conornal vector space $V^*_{\tilde{x}}$ is defined by

$$V^*_{\tilde{x}} = (d\rho(\mathfrak{G})\tilde{x})^{\perp} = \{y \in V^*; < d\rho(A)\tilde{x}, y >= 0 \ for \ all \ A \in \mathfrak{G}\}.$$

Since $V^*_{\rho(G)\tilde{x}} = \rho^*(g)V^*_{\tilde{x}}$, the isotropy subgroup $G_{\tilde{x}}$ at $\tilde{x}$ acts on $V^*_{\tilde{x}}$ by $\rho_{\tilde{x}} = \rho^* |_{G_{\tilde{x}}}$, and hence we obtain the triplet $(G_{\tilde{x}}, \rho_{\tilde{x}}, V^*_{\tilde{x}})$.

The conormal bundle $T(\rho(G)\tilde{x})^{\perp}$ of an orbit $\rho(G)\tilde{x}$ is, by definition, the Zariski-closure of $\{(v,w)\in V\times V^*; v\in\rho(G)\tilde{x}, w\in V_v^*\}$. The group G acts on $T(\rho(G)\tilde{x})^{\perp}$ by $(x,y)\mapsto(\rho(g)x,\rho^*(g)y)$ for $g\in G$. Then G acts on $T(\rho(G)\tilde{x})^{\perp}$ prehomogenously if and only if the triplet $(G_{\tilde{x}},\rho_{\tilde{x}},V_{\tilde{x}}^*)$ is a prehomogenous representation.

If the triplet $(G_{\tilde{x}},\rho_{\tilde{x}},V_{\tilde{x}}^*)$ is a prehomogeneous representation, then we denote by $\tilde{y}_0$ its generic point. Moreover, if there is one one-codimensional orbit, then $\tilde{y}_1$ denotes a point of that orbit.

Put

$$\mathfrak{sp}(2)=\left\{A=\left(\begin{array}{cc|cc} a_1 & a_{12} & b_1 & b_{12} \\ a_{21} & a_2 & b_{12} & b_2 \\ \hline c_1 & c_{12} & -a_1 & -a_{21} \\ c_{12} & c_2 & -a_{12} & -a_2 \end{array}\right)\in\mathfrak{sl}(4)\right\}.$$

For $A\in\mathfrak{sp}(2)$, we have the following:

$$d\rho_2(A)(u_1,u_2,u_3,u_4,u_5)$$

$$=(u_1,u_2,u_3,u_4,u_5)\left(\begin{array}{c|cc|cc} 0 & c_{12} & -a_{21} & a_{12} & -b_{12} \\ \hline b_{12} & a_1+a_2 & b_2 & -b_1 & 0 \\ -a_{12} & c_2 & a_1-a_2 & 0 & b_1 \\ \hline a_{21} & -c_1 & 0 & a_2-a_1 & -b_2 \\ -c_{12} & 0 & c_1 & -c_2 & -a_1-a_2 \end{array}\right).$$

(1) The case of $\tilde{x}_1=(e_1\wedge e_2+e_3\wedge e_4, e_1)$.

$$\mathfrak{G}_{\tilde{x}_1}=\{(0,-a_1;\left(\begin{array}{cc|cc} a_1 & a_{12} & b_1 & 0 \\ 0 & -a_1 & 0 & 0 \\ \hline 0 & 0 & -a_1 & 0 \\ 0 & -b_1 & -a_{12} & a_1 \end{array}\right))\in\mathfrak{G}\}.$$

$V_{\tilde{x}_1}^*=\{\ (0,0)\ \}$. $\tilde{y}_0=(0,0)\in\rho^*(G)\tilde{x}_8$.

(2) The case of $\tilde{x}_2=(e_1\wedge e_2, e_3)$.

$$\mathfrak{G}_{\tilde{x}_2}=\{\tilde{A}=(-a_1-a_2,a_1;\left(\begin{array}{cc|cc} a_1 & 0 & 0 & 0 \\ a_{21} & a_2 & 0 & b_2 \\ \hline 0 & 0 & -a_1 & -a_{21} \\ 0 & 0 & 0 & -a_2 \end{array}\right))\in\mathfrak{G}\}.$$

$V^*_{\tilde{x}_2} = \mathbb{C}\langle\, (u_5, 0)\, \rangle$. $d\rho_{\tilde{x}_2}(\tilde{A})(u_5, 0) = 2(a_1 + a_2)(u_5, 0)$.
$\tilde{y}_0 = (u_5, 0) \in \rho^*(G)\tilde{x}_6$ and $\tilde{y}_1 = 0$.

(3) The case of $\tilde{x}_3 = (e_1 \wedge e_2 + e_3 \wedge e_4, e_1 + e_4)$.

$$\mathfrak{G}_{\tilde{x}_3} = \{\tilde{A} = (0, -a_1 - b_{12}; \left(\begin{array}{cc|cc} a_1 & a_{12} & b_1 & b_{12} \\ a_{21} & -a_1 & b_{12} & -a_{21} \\ \hline a_{21} & b_{12} & -a_1 & -a_{21} \\ b_{12} & -b_1 & -a_{12} & a_1 \end{array}\right)) \in \mathfrak{G}\}.$$

$V^*_{\tilde{x}_3} = \mathbb{C}\langle\, v_1, v_2\, \rangle$ where $v_1 = (-2u_1 + u_2 - u_5, -2e_1 + 2e_4)$ and $v_2 = (u_4, e_2 + e_3)$.

$d\rho_{\tilde{x}_3}(\tilde{A})(v_1, v_2) = (v_1, v_2) \begin{pmatrix} -2b_{12} & -a_{21} \\ -2(b_1 + a_{12}) & -2a_1 \end{pmatrix}$.
$\tilde{y}_0 = v_2 \in \rho^*(G)\tilde{x}_4$.

(4) The case of $\tilde{x}_4 = (e_1 \wedge e_2, e_1)$.

$$\mathfrak{G}_{\tilde{x}_4} = \{\tilde{A} = (-a_1 - a_2, -a_1; \left(\begin{array}{cc|cc} a_1 & a_{12} & b_1 & b_{12} \\ 0 & a_2 & b_{12} & b_2 \\ \hline 0 & 0 & -a_1 & 0 \\ 0 & 0 & -a_{12} & -a_2 \end{array}\right)) \in \mathfrak{G}\}.$$

$V^*_{\tilde{x}_4} = \mathbb{C}\langle\, v_1, v_2, v_3\, \rangle$ where $v_1 = (u_1, -e_4)$, $v_2 = (u_4, e_3)$ and $v_3 = (u_5, 0)$.

$d\rho_{\tilde{x}_4}(\tilde{A})(v_1, v_2, v_3) = (v_1, v_2, v_3) \begin{pmatrix} a_1 + a_2 & 0 & 0 \\ -a_{12} & 2a_1 & 0 \\ b_{12} & b_2 & 2(a_1 + a_2) \end{pmatrix}$.
$\tilde{y}_0 = v_1 \in \rho^*(G)\tilde{x}_3$ and $\tilde{y}_1 = v_2 \in \rho^*(G)\tilde{x}_4$.

(5) The case of $\tilde{x}_5 = (e_1 \wedge e_2 + e_3 \wedge e_4, 0)$.

$$\mathfrak{G}_{\tilde{x}_5} = \{\tilde{A} = (0, \beta; A = \left(\begin{array}{cc|cc} a_1 & a_{12} & b_1 & b_{12} \\ a_{21} & -a_1 & b_{12} & b_2 \\ \hline -b_2 & b_{12} & -a_1 & -a_{21} \\ b_{12} & -b_1 & -a_{12} & a_1 \end{array}\right)) \in \mathfrak{G}\}.$$

$V^*_{\tilde{x}_5} = \mathbb{C}\langle\, v_1, v_2, v_3, v_4\, \rangle$ where $v_1 = (0, e_1)$, $v_2 = (0, e_2)$, $v_3 = (0, e_3)$ and $v_4 = (0, e_4)$.

$d\rho_{\tilde{x}_5}(\tilde{A})(0, y) = (0, -{}^tAy - \beta y)$ for $(0, y) \in V^*_{\tilde{x}_5}$.
$\tilde{y}_0 = v_1 = (0, e_1) \in \rho^*(G)\tilde{x}_7$.

(6) The case of $\tilde{x}_6 = (e_1 \wedge e_2, 0)$.

$$\mathfrak{G}_{\tilde{x}_6} = \{\tilde{A} = (-a_1 - a_2, \beta; A = \left(\begin{array}{cc|cc} a_1 & a_{12} & b_1 & b_{12} \\ a_{21} & a_2 & b_{12} & b_2 \\ \hline 0 & 0 & -a_1 & -a_{21} \\ 0 & 0 & -a_{12} & -a_2 \end{array}\right)) \in \mathfrak{G}\}.$$

$V^*_{\tilde{x}_6} = \mathbb{C}\langle v_1, v_2, v_3, v_4, v_5 \rangle$ where $v_1 = (u_5, 0)$, $v_2 = (0, e_1)$, $v_3 = (0, e_2)$, $v_4 = (0, e_3)$ and $v_5 = (0, e_4)$.

$d\rho_{\tilde{x}_6}(\tilde{A})(x_5 u_5, y) = (2(a_1 + a_2)x_5 u_5, -{}^t A y - \beta y)$ for $(x_5 u_5, y) \in V^*_{\tilde{x}_6}$.

$\tilde{y}_0 = v_1 + v_2 = (u_5, e_1) \in \rho^*(G)\tilde{x}_2$ and $\tilde{y}_1 = v_2 \in \rho^*(G)\tilde{x}_7$.

(7) The case of $\tilde{x}_7 = (0, e_1)$.

$$\mathfrak{G}_{\tilde{x}_7} = \{\tilde{A} = (\alpha, -a_1; A = \left(\begin{array}{cc|cc} a_1 & a_{12} & b_1 & b_{12} \\ 0 & a_2 & b_{12} & b_2 \\ \hline 0 & 0 & -a_1 & 0 \\ 0 & c_2 & -a_{12} & -a_2 \end{array}\right)) \in \mathfrak{G}\}.$$

$V^*_{\tilde{x}_7} = \mathbb{C}\langle v_1, v_2, v_3, v_4, v_5 \rangle$ where $v_1 = (u_1, 0)$, $v_2 = (u_2, 0)$, $v_3 = (u_3, 0)$, $v_4 = (u_4, 0)$ and $v_5 = (u_5, 0)$.

$$d\rho_{\tilde{x}_7}(\tilde{A})(x, 0) = (\begin{pmatrix} 0 & -b_{12} & a_{12} & 0 & 0 \\ 0 & -a_1 - a_2 & -c_2 & 0 & 0 \\ 0 & -b_2 & -a_1 + a_2 & 0 & 0 \\ -a_{12} & b_1 & 0 & a_1 - a_2 & c_2 \\ b_{12} & 0 & -b_1 & b_2 & a_1 + a_2 \end{pmatrix} x - \alpha x, 0)$$

for $(x, 0) \in V^*_{\tilde{x}_7}$.

$\tilde{y}_0 = v_2 + v_5 = (u_2 + u_5, 0) \in \rho^*(G)\tilde{x}_5$ and $\tilde{y}_1 = v_2 \in \rho^*(G)\tilde{x}_6$.

(8) The case of $\tilde{x}_8 = (0, 0)$.

In this case, we have $(G_{x_8}, \rho_{x_8}, V^*_{x_8}) \cong (G, \rho, V)$. $\tilde{y}_0 = \tilde{x}_1$, $\tilde{y}_1 = \tilde{x}_2$.

References

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