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Uncertainty relations for generalized metric adjusted skew information and generalized metric adjusted correlation measure

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Abstract
In this paper, we give a Heisenberg-type or a Schrödinger-type uncertainty relation for generalized metric adjusted skew information or generalized metric adjusted correlation measure. These results generalize the previous result of Furuichi and Yanagi (J. Math. Anal. Appl. 388:1147-1156, 2012).

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Introduction

We start from the Heisenberg uncertainty relation [1]:

$$V_\rho(A)V_\rho(B) \geq \frac{1}{4}|Tr\,[\rho[A,B]]\,|^2$$

for a quantum state (density operator) ρ and two observables (self-adjoint operators) A and B. The further stronger result was given by Schrödinger in [2,3]:

$$V_\rho(A)V_\rho(B) - |Re\left\{\mathrm{Cov}_\rho(A,B)\right\}|^2 \geq \frac{1}{4}|Tr\,[\rho[A,B]]\,|^2,$$

where the covariance is defined by $\mathrm{Cov}_\rho(A,B) \equiv Tr\,[\rho\,(A - Tr\,[\rho A]\,I)\,(B - Tr\,[\rho B]\,I)]\,.$

The Wigner-Yanase skew information represents a measure for non-commutativity between a quantum state ρ and an observable H. Luo introduced the quantity $U_\rho(H)$ representing a quantum uncertainty excluding the classical mixture [4]:

$$U_\rho(H) \equiv \sqrt{V_\rho(H)^2 - \left(V_\rho(H) - I_\rho(H)\right)^2},$$

with the Wigner-Yanase skew information [5]:

$$I_\rho(H) \equiv \frac{1}{2}Tr\left[(i[\rho^{1/2}, H_0])^2\right] = Tr\,[\rho H_0^2] - Tr\,[\rho^{1/2}H_0\rho^{1/2}H_0]\,, \quad H_0 \equiv H - Tr\,[\rho H]\,I,$$

and then he successfully showed a new Heisenberg-type uncertainty relation on $U_\rho(H)$ in [4]:

$$U_\rho(A)U_\rho(B) \geq \frac{1}{4}|Tr\,[\rho[A,B]]|^2. \quad (1)$$

As stated in [4], the physical meaning of the quantity $U_\rho(H)$ can be interpreted as follows. For a mixed state ρ, the variance $V_\rho(H)$ has both classical mixture and quantum uncertainty. Also, the Wigner-Yanase skew information $I_\rho(H)$ represents a kind of quantum uncertainty [6,7]. Thus, the difference $V_\rho(H) - I_\rho(H)$ has a classical mixture so that we can regard that the quantity $U_\rho(H)$ has a quantum uncertainty excluding a classical mixture. Therefore, it is meaningful and suitable to study an uncertainty relation for a mixed state by the use of the quantity $U_\rho(H)$.

Recently, a one-parameter extension of the inequality (1) was given in [8]:

$$U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) \geq \alpha(1-\alpha)|Tr\,[\rho[A,B]]|^2,$$

where

$$U_{\rho,\alpha}(H) \equiv \sqrt{V_\rho(H)^2 - \left(V_\rho(H) - I_{\rho,\alpha}(H)\right)^2},$$

with the Wigner-Yanase-Dyson skew information $I_{\rho,\alpha}(H)$ defined by

$$I_{\rho,\alpha}(H) \equiv \frac{1}{2} Tr\left[(i[\rho^\alpha, H_0])(i[\rho^{1-\alpha}, H_0])\right] = Tr\,[\rho H_0^2] - Tr\,[\rho^\alpha H_0 \rho^{1-\alpha} H_0]\,.$$

It is notable that the convexity of $I_{\rho,\alpha}(H)$ with respect to ρ was successfully proven by Lieb in [9]. The further generalization of the Heisenberg-type uncertainty relation on $U_\rho(H)$ has been given in [10] using the generalized Wigner-Yanase-Dyson skew information introduced in [11]. Recently, it is shown that these skew informations are connected to special choices of quantum Fisher information in [12]. The family of all quantum Fisher informations is parametrized by a certain class of operator monotone functions $\mathcal{F}_{op}$ which were justified in [13]. The Wigner-Yanase skew information and Wigner-Yanase-Dyson skew information are given by the following operator monotone functions:

$$f_{\mathrm{WY}}(x) = \left(\frac{\sqrt{x}+1}{2}\right)^2,$$

$$f_{\mathrm{WYD}}(x) = \alpha(1-\alpha)\frac{(x-1)^2}{(x^\alpha - 1)(x^{1-\alpha} - 1)},\ \alpha \in (0,1),$$

respectively. In particular, the operator monotonicity of the function f_{WYD} was proved in [14] (see also [15]). On the other hand, the uncertainty relation related to the Wigner-Yanase skew information was given by Luo [4], and the uncertainty relation related to the Wigner-Yanase-Dyson skew information was given by Yanagi [8]. In this paper, we generalize these uncertainty relations to the uncertainty relations related to quantum Fisher informations by using (generalized) metric adjusted skew information or correlation measure.

Operator monotone functions

Let $M_n(\mathbb{C})$ (respectively $M_{n,sa}(\mathbb{C})$) be the set of all $n \times n$ complex matrices (respectively all $n \times n$ self-adjoint matrices), endowed with the Hilbert-Schmidt scalar product $\langle A, B\rangle = Tr(A^*B)$. Let $M_{n,+}(\mathbb{C})$ be the set of strictly positive elements of $M_n(\mathbb{C})$ and $M_{n,+,1}(\mathbb{C})$ be the set of stricly positive density matrices, that is $M_{n,+,1}(\mathbb{C}) = \{\rho \in M_n(\mathbb{C}) | Tr\rho = 1, \rho > 0\}$. If it is not otherwise specified, from now on, we shall treat the case of faithful states, that is $\rho > 0$.

A function $f : (0, +\infty) \to \mathbb{R}$ is said to be operator monotone if, for any $n \in \mathbb{N}$ and $A, B \in M_{n,+}(\mathbb{C})$ such that $0 \leq A \leq B$, the inequalities $0 \leq f(A) \leq f(B)$ hold. An operator monotone function is said to be symmetric if $f(x) = xf(x^{-1})$ and normalized if $f(1) = 1$.

Definition 1. $\mathcal{F}_{op}$ is the class of functions $f : (0, +\infty) \to (0, +\infty)$ such that

1. $f(1) = 1$,
2. $tf(t^{-1}) = f(t)$,
3. f is operator monotone.

Example 1. Examples of elements of $\mathcal{F}_{op}$ are given by the following list:

$$f_{\mathrm{RLD}}(x) = \frac{2x}{x+1}, \ f_{\mathrm{WY}}(x) = \left(\frac{\sqrt{x}+1}{2}\right)^2,$$
$$f_{\mathrm{BKM}}(x) = \frac{x-1}{\log x}, \ f_{\mathrm{SLD}}(x) = \frac{x+1}{2},$$
$$f_{\mathrm{WYD}}(x) = \alpha(1-\alpha)\frac{(x-1)^2}{(x^\alpha - 1)(x^{1-\alpha}-1)}, \ \alpha \in (0,1).$$

Remark 1. Any $f \in \mathcal{F}_{op}$ satisfies

$$\frac{2x}{x+1} \le f(x) \le \frac{x+1}{2}, \quad x > 0.$$

For $f \in \mathcal{F}_{op}$, define $f(0) = \lim_{x\to 0} f(x)$. We introduce the sets of regular and non-regular functions

$$\mathcal{F}_{op}^r = \left\{ f \in \mathcal{F}_{op} | f(0) \neq 0 \right\}, \ \mathcal{F}_{op}^n = \left\{ f \in \mathcal{F}_{op} | f(0) = 0 \right\}$$

and notice that trivially $\mathcal{F}_{op} = \mathcal{F}_{op}^r \cup \mathcal{F}_{op}^n$.

Definition 2. For $f \in \mathcal{F}_{op}^r$, we set

$$\tilde{f}(x) = \frac{1}{2}\left[(x+1) - (x-1)^2 \frac{f(0)}{f(x)}\right], \quad x > 0.$$

Theorem 1. ([12,16,17]) *The correspondence $f \to \tilde{f}$ is a bijection between $\mathcal{F}_{op}^r$ and $\mathcal{F}_{op}^n$.*

Metric adjusted skew information and correlation measure

In the Kubo-Ando theory of matrix means, one associates a mean to each operator monotone function $f \in \mathcal{F}_{op}$ by the formula

$$m_f(A,B) = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2},$$

where $A, B \in M_{n,+}(\mathbb{C})$. Using the notion of matrix means, one may define the class of monotone metrics (also called quantum Fisher informations) by the following formula:

$$\langle A, B \rangle_{\rho,f} = Tr(A \cdot m_f(L_\rho, R_\rho)^{-1}(B)),$$

where $L_\rho(A) = \rho A, R_\rho(A) = A\rho$. In this case, one has to think of A, B as tangent vectors to the manifold $M_{n,+,1}(\mathbb{C})$ at the point ρ (see [12,13]).

Definition 3. For $A, B \in M_{n,sa}(\mathbb{C})$ and $\rho \in M_{n,+,1}(\mathbb{C})$, we define the following quantities:

$$\mathrm{Corr}_\rho^f(A,B) = Tr\left[\rho AB\right] - Tr\left[A \cdot m_{\tilde{f}}(L_\rho, R_\rho)B\right],$$

$$\mathrm{Corr}_\rho^{s(f)}(A,B) = \frac{f(0)}{2}\langle i[\rho,A], i[\rho,B]\rangle_{\rho,f},$$
$$I^f_\rho(A) = \mathrm{Corr}^f_\rho(A,A),$$
$$C^f_\rho(A,B) = Tr\left[A \cdot m_f(L_\rho,R_\rho)B\right],$$
$$C^f_\rho(A) = C^f_\rho(A,A),$$
$$U^f_\rho(A) = \sqrt{V_\rho(A)^2 - (V_\rho(A) - I^f_\rho(A))^2},$$

The quantity $I^f_\rho(A)$ is known as metric adjusted skew information [18], and the metric adjusted correlation measure $\mathrm{Corr}^f_\rho(A,B)$ was also previously defined in [18].

Then we have the following proposition.

Proposition 1. ([16,19]) *For $A,B \in M_{n,sa}(\mathbb{C})$ and $\rho \in M_{n,+,1}(\mathbb{C})$, we have the following relations, where we put $A_0 = A - Tr[\rho A]I$ and $B_0 = B - Tr[\rho B]I$:*

1. $I^f_\rho(A) = I^f_\rho(A_0) = Tr[\rho A_0^2] - Tr[A_0 \cdot m_{\tilde f}(L_\rho,R_\rho)A_0] = V_\rho(A) - C^{\tilde f}_\rho(A_0),$
2. $J^f_\rho(A) = Tr[\rho A_0^2] + Tr[A_0 \cdot m_{\tilde f}(L_\rho,R_\rho)A_0] = V_\rho(A) + C^{\tilde f}_\rho(A_0),$
3. $0 \le I^f_\rho(A) \le U^f_\rho(A) \le V_\rho(A),$
4. $U^f_\rho(A) = \sqrt{I^f_\rho(A)\cdot J^f_\rho(A)},$
5. $Corr^f_\rho(A,B) = Corr^f_\rho(A_0,B_0) = Tr[\rho A_0 B_0] - Tr[A_0 \cdot m_{\tilde f}(L_\rho,R_\rho)B_0],$
6. $$\begin{aligned} Corr^{s(f)}_\rho(A,B) = Corr^{s(f)}_\rho(A_0,B_0) &= \tfrac{1}{2}Tr[\rho A_0 B_0] + \tfrac{1}{2}Tr[\rho B_0 A_0] - Tr[A_0 \cdot m_{\tilde f}(L_\rho,R_\rho)B_0] \\ &= \tfrac{1}{2}Tr[\rho A_0 B_0] + \tfrac{1}{2}Tr[\rho B_0 A_0] - C^{\tilde f}_\rho(A_0,B_0). \end{aligned}$$

Now we modify the uncertainty relation given by [20].

Theorem 2. *For $f \in \mathcal{F}^r_{op}$, it holds*

$$I^f_\rho(A)\cdot I^f_\rho(B) \ge |Corr^{s(f)}_\rho(A,B)|^2,$$

where $A,B \in M_{n,sa}(\mathbb{C})$ and $\rho \in M_{n,+,1}(\mathbb{C})$.

Remark 2. Since Theorem 2 is easily given by using the Schwarz inequality, we omit the proof. In [20] we gave the uncertainty relation

$$U^f_\rho(A)\cdot U^f_\rho(B) \ge 4f(0)|\mathrm{Corr}^{s(f)}_\rho(A,B)|^2.$$

But since $4f(0) \le 1$ and $I^f_\rho(A) \le U^f_\rho(A)$, it is easily given by Theorem 2.

Theorem 3. ([20,21]) *For $f \in \mathcal{F}^r_{op}$, if*

$$\frac{x+1}{2} + \tilde f(x) \ge 2f(x), \tag{2}$$

then it holds

$$U^f_\rho(A)\cdot U^f_\rho(B) \ge f(0)|Tr(\rho[A,B])|^2, \tag{3}$$

$$U^f_\rho(A)\cdot U^f_\rho(B) \ge 4f(0)|Corr^f_\rho(A,B)|^2, \tag{4}$$

where $A, B \in M_{n,sa}(\mathbb{C})$ *and* $\rho \in M_{n,+,1}(\mathbb{C})$*.*

Remark 3. Though we cannot use the Schwarz inequality, we can get (4) in Theorem 3 by modifying the proof given by [20].

By putting

$$f_{\mathrm{WYD}}(x) = \alpha(1-\alpha)\frac{(x-1)^2}{(x^\alpha - 1)(x^{1-\alpha} - 1)}, \quad \alpha \in (0,1),$$

we obtain the following uncertainty relation.

Corollary 1. *For* $A, B \in M_{n,sa}(\mathbb{C})$ *and* $\rho \in M_{n,+,1}(\mathbb{C})$*,*

$$U_\rho^{f_{WYD}}(A) \cdot U_\rho^{f_{WYD}}(B) \geq \alpha(1-\alpha)|Tr\,(\rho[A,B])|^2,$$

$$U_\rho^{f_{WYD}}(A) \cdot U_\rho^{f_{WYD}}(B) \geq 4\alpha(1-\alpha)|Corr_\rho^{f_{WYD}}(A,B)|^2,$$

where

$$Corr_\rho^{f_{WYD}}(A,B) = Tr\,[\rho A_0 B_0] - \frac{1}{2} Tr\,[\rho^\alpha A_0 \rho^{1-\alpha} B_0] - \frac{1}{2} Tr\,[\rho^\alpha B_0 \rho^{1-\alpha} A_0]\,.$$

Remark 4. Even if (2) does not necessarily hold, then

$$U_\rho^f(A) \cdot U_\rho^f(B) \geq f(0)^2 |Tr\,[(\rho[A,B])|^2, \tag{5}$$

$$U_\rho^f(A) \cdot U_\rho^f(B) \geq 4 f(0)^2 |\mathrm{Corr}_\rho^f(A,B)|^2, \tag{6}$$

where $A, B \in M_{n,sa}(\mathbb{C})$ and $\rho \in M_{n,+,1}(\mathbb{C})$. Since $f(0) < 1$, it is easy to show that (5) and (6) are weaker than (3) and (4), respectively.

Generalized metric adjusted skew information and correlation measure

We give some generalizations of Heisenberg or Schrd̈inger uncertainty relations which include Theorem 3 as corollary.

Definition 4. ([22]) Let $g, f \in \mathcal{F}_{op}^r$ satisfy

$$g(x) \geq k\frac{(x-1)^2}{f(x)}$$

for some $k > 0$. We define

$$\Delta_g^f(x) = g(x) - k\frac{(x-1)^2}{f(x)} \in \mathcal{F}_{op}. \tag{7}$$

Definition 5. For $A, B \in M_{n,sa}(\mathbb{C})$ and $\rho \in M_{n,+,1}(\mathbb{C})$, we define the following quantities:

$$\mathrm{Corr}_\rho^{s(g,f)}(A,B) = k\langle i[\rho, A]\,, i[\rho, B]\rangle_{\rho,f},$$

$$I_\rho^{(g,f)}(A) = \mathrm{Corr}_\rho^{s(g,f)}(A,A),$$

$$C_\rho^f(A,B) = Tr\,[A \cdot m_f(L_\rho, R_\rho)B]\,,$$

$$C_\rho^f(A) = C_\rho^f(A,A),$$

$$U_\rho^{(g,f)}(A) = \sqrt{(C_\rho^g(A) + C_\rho^{\Delta_g^f}(A))(C_\rho^g(A) - C_\rho^{\Delta_g^f}(A))}.$$

The quantity $I_\rho^{(g,f)}(A)$ and $\mathrm{Corr}_\rho^{s(g,f)}(A,B)$ are said to be generalized metric adjusted skew information and generalized metric adjusted correlation measure, respectively.

Then we have the following proposition.

Proposition 2. *For $A, B \in M_{n,sa}(\mathbb{C})$ and $\rho \in M_{n,+,1}(\mathbb{C})$, we have the following relations, where we put $A_0 = A - Tr\,[\rho A]\,I$ and $B_0 = B - Tr\,[\rho B]\,I$:*

1. $I_\rho^{(g,f)}(A) = I_\rho^{(g,f)}(A_0) = C_\rho^g(A_0) - C_\rho^{\Delta_g^f}(A_0),$
2. $J_\rho^{(g,f)}(A) = C_\rho^g(A_0) + C_\rho^{\Delta_g^f}(A_0),$
3. $U_\rho^{(g,f)}(A) = \sqrt{I_\rho^{(g,f)}(A) \cdot J_\rho^{(g,f)}(A)},$
4. $Corr_\rho^{s(g,f)}(A,B) = Corr_\rho^{s(g,f)}(A_0,B_0).$

Theorem 4. *For $f \in \mathcal{F}_{op}^r$, it holds*

$$I_\rho^{(g,f)}(A) \cdot I_\rho^{(g,f)}(B) \geq |Corr_\rho^{s(g,f)}(A,B)|^2,$$

where $A, B \in M_{n,sa}(C)$ and $\rho \in M_{n,+,1}(\mathbb{C})$.

Proof of Theorem 4. We define for $X, Y \in M_n(\mathbb{C})$

$$\mathrm{Corr}_\rho^{s(g,f)}(X,Y) = k\langle i[\rho,X]\,, i[\rho,Y]\rangle_{\rho,f}.$$

Since

$$\begin{aligned}\mathrm{Corr}_\rho^{s(g,f)}(X,Y) &= kTr((i[\rho,X])^* m_f(L_\rho,R_\rho)^{-1} i[\rho,Y]) \\ &= kTr((i(L_\rho - R_\rho)X)^* m_f(L_\rho,R_\rho)^{-1} i(L_\rho - R_\rho)Y) \\ &= Tr(X^* m_g(L_\rho,R_\rho)Y) - Tr(X^* m_{\Delta_g^f}(L_\rho,R_\rho)Y),\end{aligned}$$

it is easy to show that $\mathrm{Corr}_\rho^{s(g,f)}(X,Y)$ is an inner product in $M_n(\mathbb{C})$. Then we can get the result by using the Schwarz inequality.

□

Theorem 5. *For $f \in \mathcal{F}_{op}^r$, if*

$$g(x) + \Delta_g^f(x) \geq \ell f(x) \tag{8}$$

for some $\ell > 0$, then it holds

$$U_\rho^{(g,f)}(A) \cdot U_\rho^{(g,f)}(B) \geq k\ell |Tr(\rho[A,B])|^2, \tag{9}$$

where $A, B \in M_{n,sa}(\mathbb{C})$ and $\rho \in M_{n,+,1}(\mathbb{C})$.

In order to prove Theorem 5, we need the following lemmas.

Lemma 1. *If (7) and (8) are satisfied, then we have the following inequality:*

$$m_g(x,y)^2 - m_{\Delta_g^f}(x,y)^2 \geq k\ell(x-y)^2.$$

Proof of Lemma 1. By (7) and (8), we have

$$m_{\Delta_g^f}(x,y) = m_g(x,y) - k\frac{(x-y)^2}{m_f(x,y)}, \tag{10}$$

$$m_g(x,y) + m_{\Delta_g^f}(x,y) \geq \ell m_f(x,y). \tag{11}$$

Therefore, by (10) and (11),

$$\begin{aligned} & m_g(x,y)^2 - m_{\Delta_g^f}(x,y)^2 \\ &= \left\{ m_g(x,y) - m_{\Delta_g^f}(x,y) \right\} \left\{ m_g(x,y) + m_{\Delta_g^f}(x,y) \right\} \\ &\geq k\frac{(x-y)^2}{m_f(x,y)} \ell m_f(x,y) \\ &= k\ell(x-y)^2. \end{aligned}$$

We have the following expressions for the quantities $I_\rho^{(g,f)}(A)$, $J_\rho^{(g,f)}(A)$, $U_\rho^{(g,f)}(A)$, and $\mathrm{Corr}_\rho^{s(g,f)}(A,B)$ by using Proposition 2 and a mean $m_{\Delta_g^f}$.

Lemma 2. *Let $\{|\phi_1\rangle, |\phi_2\rangle, \ldots, |\phi_n\rangle\}$ be a basis of eigenvectors of ρ, corresponding to the eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. We put $a_{jk} = \langle \phi_j | A_0 | \phi_k \rangle$, $b_{jk} = \langle \phi_j | B_0 | \phi_k \rangle$, where $A_0 \equiv A - Tr[\rho A]I$ and $B_0 \equiv B - Tr[\rho B]I$ for $A, B \in M_{n,sa}(\mathbb{C})$ and $\rho \in M_{n,+,1}(\mathbb{C})$. Then we have*

$$\begin{aligned} I_\rho^{(g,f)}(A) &= \sum_{j,k} m_g(\lambda_j,\lambda_k) a_{jk} a_{kj} - \sum_{j,k} m_{\Delta_g^f}(\lambda_j,\lambda_k) a_{jk} a_{kj} \\ &= 2\sum_{j<k} \left\{ (m_g(\lambda_j,\lambda_k) - m_{\Delta_g^f}(\lambda_j,\lambda_k) \right\} |a_{jk}|^2, \end{aligned}$$

$$\begin{aligned} J_\rho^{(g,f)}(A) &= \sum_{j,k} m_g(\lambda_j,\lambda_k) a_{jk} a_{kj} + \sum_{j,k} m_{\Delta_g^f}(\lambda_j,\lambda_k) a_{jk} a_{kj} \\ &\geq 2\sum_{j<k} \left\{ m_g(\lambda_j,\lambda_k) + m_{\Delta_g^f}(\lambda_j,\lambda_k) \right\} |a_{jk}|^2, \end{aligned}$$

$$U_\rho^{(g,f)}(A)^2 = \left(\sum_{j,k} m_g(\lambda_j,\lambda_k) |a_{jk}|^2 \right)^2 - \left(\sum_{j,k} m_{\Delta_g^f}(\lambda_j,\lambda_k) |a_{jk}|^2 \right)^2,$$

and

$$\begin{aligned} & Corr_\rho^{s(g,f)}(A,B) \\ &= \sum_{j,k} m_g(\lambda_j,\lambda_k) a_{jk} b_{kj} - \sum_{j,k} m_{\Delta_g^f}(\lambda_j,\lambda_k) a_{jk} b_{kj} \\ &= \sum_{j<k} \left(m_g(\lambda_j,\lambda_k) - m_{\Delta_g^f}(\lambda_j,\lambda_k) \right) a_{jk} b_{kj} + \sum_{j<k} \left(m_g(\lambda_k,\lambda_j) - m_{\Delta_g^f}(\lambda_k,\lambda_j) \right) a_{kj} b_{jk}. \end{aligned}$$

We are now in a position to prove Theorem 5.

Proof of Theorem 5. At first we prove (9). Since

$$Tr(\rho[A,B]) = \sum_{j,k} (\lambda_j - \lambda_k) a_{jk} b_{kj},$$

$$|Tr(\rho[A,B])| \le \sum_{j,k} |\lambda_j - \lambda_k||a_{jk}||b_{kj}|.$$

Then by Lemma 1, we have

$$\begin{aligned}
&k\ell|Tr(\rho[A,B])|^2 \\
&\le \left\{ \sum_{j,k} \sqrt{k\ell}|\lambda_j - \lambda_k||a_{jk}||b_{kj}| \right\}^2 \\
&\le \left\{ \sum_{j,k} \left(m_g(\lambda_j,\lambda_k)^2 - m_{\Delta_g^f}(\lambda_j,\lambda_k)^2 \right)^{1/2} |a_{jk}||b_{kj}| \right\}^2 \\
&\le \left\{ \sum_{j,k} \left(m_g(\lambda_j,\lambda_k) - m_{\Delta_g^f}(\lambda_j u,\lambda_k) \right) |a_{jk}|^2 \right\} \left\{ \sum_{j,k} \left(m_g(\lambda_j,\lambda_k) + m_{\Delta_g^f}(\lambda_j,\lambda_k) \right) |b_{kj}|^2 \right\} \\
&= I_\rho^{(g,f)}(A) J_\rho^{(g,f)}(B).
\end{aligned}$$

By a similar way, we also have

$$I_\rho^{(g,f)}(B) J_\rho^{(g,f)}(A) \ge k\ell|Tr(\rho[A,B])|^2.$$

Hence, we have the desired inequality (9). □

We give some examples satisfying the condition (8).

Example 2. Let

$$g(x) = \frac{x+1}{2},$$

$$f(x) = \alpha(1-\alpha)\frac{(x-1)^2}{(x^\alpha - 1)(x^{1-\alpha} - 1)}, \ \alpha \in (0,1),$$

$$k = \frac{f(0)}{2} = \frac{\alpha(1-\alpha)}{2}, \ \ell = 2.$$

Then

$$g(x) + \Delta_g^f(x) \ge 2f(x).$$

Proof of Example 2. In [10,21] we give

$$(x^{2\alpha} - 1)(x^{2(1-\alpha)} - 1) \ge 4\alpha(1-\alpha)(x-1)^2$$

for $x > 0$ and $0 \le \alpha \le 1$. Then we have

$$g(x) + \Delta_g^f(x) \ge 2f(x).$$

□

Example 3. Let

$$g(x) = \left(\frac{\sqrt{x}+1}{2} \right)^2,$$

$$f(x) = \alpha(1-\alpha)\frac{(x-1)^2}{(x^\alpha - 1)(x^{1-\alpha} - 1)}, \ \alpha \in (0,1),$$

$$k = \frac{f(0)}{8} = \frac{\alpha(1-\alpha)}{8}, \quad \ell = \frac{3}{2}.$$

Then

$$g(x) + \Delta_g^f(x) \geq \frac{3}{2} f(x)$$

holds for $0 < \alpha < 1$.

Proof of Example 3. Since

$$\begin{aligned}
&\frac{1}{2}\left(\frac{1+\sqrt{x}}{2}\right)^2 - \frac{1}{8}(x^\alpha - 1)(x^{1-\alpha} - 1) \\
&= \frac{1}{8}(x + 2\sqrt{x} + 1 - x - 1 + x^\alpha + x^{1-\alpha}) \\
&= \frac{1}{8}(2\sqrt{x} + x^\alpha + x^{1-\alpha}) \\
&= \frac{1}{8}(x^{\alpha/2} + x^{(1-\alpha)/2})^2 \geq 0,
\end{aligned}$$

we have

$$2\left(\frac{1+\sqrt{x}}{2}\right)^2 \geq \frac{1}{8}(x^\alpha - 1)(x^{1-\alpha} - 1) + \frac{3}{2}\left(\frac{1+\sqrt{x}}{2}\right)^2.$$

Since

$$\alpha(1-\alpha)\frac{(x-1)^2}{(x^\alpha - 1)(x^{1-\alpha} - 1)} \leq \left(\frac{1+\sqrt{x}}{2}\right)^2,$$

we have

$$2\left(\frac{1+\sqrt{x}}{2}\right)^2 \geq \frac{1}{8}(x^\alpha - 1)(x^{1-\alpha} - 1) + \frac{3}{2}\alpha(1-\alpha)\frac{(x-1)^2}{(x^\alpha - 1)(x^{1-\alpha} - 1)}.$$

Then we have

$$g(x) + \Delta_g^f(x) \geq \frac{3}{2} f(x)$$

□

Example 4. Let

$$g(x) = \left(\frac{x^\gamma + 1}{2}\right)^{1/\gamma} \quad (\frac{3}{4} \leq \gamma \leq 1),$$

$$f(x) = \left(\frac{\sqrt{x} + 1}{2}\right)^2,$$

$$k = \frac{f(0)}{4} = \frac{1}{16}, \quad \ell = 2.$$

Then $g(x) + \Delta_g^f(x) \geq 2f(x)$.

In order to prove Example 4, we need the following lemma.

Lemma 3. *For $x > 0$, we set the function of y as*

$$F(y) \equiv \left(\frac{1 + x^y}{2}\right)^{1/y}.$$

Then $F(y)$ has following properties:

1. *$F(y)$ is monotone increasing for $y \in \mathbb{R}$.*

2. *$F(y)$ is convex for $y < 0$.*
3. *$F(y)$ is concave for $y \geq 1/2$.*

We give the proof of Lemma 3 in the Appendix.

Proof of Example 4. By Lemma 3,

$$2\left(\frac{1+x^{3/4}}{2}\right)^{4/3} \geq \frac{1+x}{2} + \left(\frac{1+\sqrt{x}}{2}\right)^2.$$

It follows from the monotonicity that

$$\left(\frac{1+x^y}{2}\right)^{1/y} \geq \left(\frac{1+x^{3/4}}{2}\right)^{4/3}$$

for $y \in [3/4, 1]$. Then

$$2\left(\frac{1+x^y}{2}\right)^{1/y} \geq \frac{1+x}{2} + \left(\frac{1+\sqrt{x}}{2}\right)^2$$

for $y \in [3/4, 1]$. Therefore, we have

$$2\left(\frac{1+x^y}{2}\right)^{1/y} - \left(\frac{\sqrt{x}-1}{2}\right)^2 \geq 2\left(\frac{\sqrt{x}+1}{2}\right)^2.$$

Hence, we have

$$g(x) + \Delta_g^f(x) \geq 2f(x).$$

□

Appendix

Proof of Lemma 3.

(i) Since $F(y) > 0$ for $x > 0$ and $t \in \mathbb{R}$, it is sufficient to prove $\frac{d}{dy}\log F(y) > 0$ for the proof of $F'(y) > 0$. We have

$$\frac{d}{dy}\log F(y) = \frac{1}{y^2}\left(\log 2 + \frac{x^y \log x^y}{1+x^y} - \log\left(1+x^y\right)\right).$$

Then we put

$$G(r) \equiv (r+1)\log 2 + r\log r - (r+1)\log(r+1), (r > 0),$$

where we put $x^y \equiv r > 0$. From elementary calculations, we have $G(r) \geq G(1) = 0$ which implies $\frac{d}{dy}\log F(y) > 0$.

(ii) We firstly set $f(y) \equiv \log F(y)$. Since $F(y) > 0$, we have only to prove $f''(y) > 0$ for the proof of $F''(y) > 0$. We set again $g(y) \equiv \frac{1+x^y}{2}, (x > 0, y < 0)$. Then we have $\frac{d^2}{dy^2}\log g(y) \equiv \frac{x^y(\log x)^2}{(1+x^y)^2} > 0$. In addition, by $f(y) = \frac{1}{y}\log g(y)$, we have

$$f'(y) = \frac{1}{y}\frac{g'(y)}{g(y)} - \frac{1}{y^2}\log g(y) > 0.$$

By $\frac{d^2}{dy^2}\log g(y) = \frac{g(y)g''(y) - \{g'(y)\}^2}{g(y)^2}$, we have

$$f''(y) = \frac{1}{y}\frac{g(y)g''(y) - \left\{g'(y)\right\}^2}{g(y)^2} - \frac{2}{y^2}\frac{g'(y)}{g(y)} + \frac{2}{y^3}\log g(y) = \frac{1}{y}\frac{d^2}{dy^2}\log g(y) - \frac{2}{y}f'(y).$$

We prove $f''(y) > 0$ for $y < 0$. We calculate

$$f''(y) = \frac{1}{y}\frac{x^y(\log x)^2}{(1+x^y)^2} - \frac{2}{y}\frac{1}{y^2}\left(\log 2 + \frac{x^y \log x^y}{1+x^y} - \log(1+x^y)\right)$$
$$= \frac{1}{y^3(1+x^y)^2}\left\{-2x^y(1+x^y)\log x^y + x^y(\log x^y)^2 + 2(1+x^y)^2 \log\frac{1+x^y}{2}\right\}.$$

Thus, if we put

$$h(y) \equiv -2x^y(1+x^y)\log x^y + x^y(\log x^y)^2 + 2(1+x^y)^2\log\frac{1+x^y}{2},$$

then we have only to prove $h(y) < 0$ for $y < 0$. Since we have $h(0) = 0$, we have only to prove $h'(y) > 0$ for $y < 0$. Here we have

$$h'(y) = -x^y \log x\left\{4x^y \log x^y - (\log x^y)^2 - 4(1+x^y)\log\frac{1+x^y}{2}\right\}.$$

If we set again

$$l(t) \equiv 4t\log t - (\log t)^2 - 4(t+1)\log\frac{t+1}{2},$$

where we put $x^y \equiv t > 0$, then we prove the following cases:

(a) If $x < 1$ (i.e., $t > 1$), then $l(t) > 0$.
(b) If $x > 1$ (i.e., $0 < t < 1$), then $l(t) < 0$.

For case (a), we calculate

$$l'(t) = \frac{1}{t}\left(4t\log 2 + (4t-2)\log t - 4t\log(t+1)\right)$$

and

$$l''(t) = \frac{2\left\{(t+1)\log t + t - 1\right\}}{t^2(t+1)} > 0, (t > 1).$$

Thus, we have $l'(t) \geq l'(1) = 0$, and then we have $l(t) \geq l(1) = 0$. For case (b), we easily find that

$$l''(t) = \frac{2\left\{(t+1)\log t + t - 1\right\}}{t^2(t+1)} < 0, (0 < t < 1).$$

Thus, we have $l'(t) \geq l'(1) = 0$, and then we have $l(t) \leq l(1) = 0$.

(iii) We calculate

$$\frac{d^2}{dy^2}F(y) = \frac{1}{y^4}\left(\frac{1+x^y}{2}\right)^{1/y} h(x,y),$$

where

$$h(x,y) = (\log 2 - 2y)\log 2 + \frac{2\log 2}{1+x^y}\{x^y \log x^y - (1+x^y)\log(1+x^y)\}$$
$$+\frac{1}{(1+x^y)^2}\{x^y y^2(x^y+y)(\log x)^2\}$$
$$-\frac{1}{(1+x^y)^2}\{2x^y(1+x^y)(y+\log(1+x^y))\log x^y\}$$
$$+\{2y+\log(1+x^y)\}\log(1+x^y).$$

We prove $h(x,y) \leq 0$ for $x > 0$ and $y \geq 1/2$. Then we have

$$\frac{dh(x,y)}{dx} = -\frac{x^{-1+y}y^2\log x}{(1+x^y)^3}\left\{(x^y(y-2)-y)\log x^y + 2(1+x^y)\log\left(\frac{1+x^y}{2}\right)\right\}.$$

Here we note that $\frac{dh(1,y)}{dx} = 0$. We also put

$$g(x,y) = \left\{x^y(-2+y) - y\right\} \log x^y + 2(1+x^y) \log\left(\frac{1+x^y}{2}\right).$$

If we have $g(x,y) \geq 0$ for $x > 0$ and $y \geq 1/2$, then we have $\frac{dh(x,y)}{dx} \geq 0$ for $0 < x \leq 1$ and $\frac{dh(x,y)}{dx} \leq 0$ for $x \geq 1$. Thus, we then obtain $h(x,y) \leq h(1,y) = 0$ for $y \geq 1/2$, due to $\frac{dh(1,y)}{dx} = 0$. Therefore, we have only to prove $g(x,y) \geq 0$ for $x > 0$ and $y \geq 1/2$.

(a) For the case $0 < x \leq 1$, we have

$$\frac{dg(x,y)}{dx} = \frac{y}{x}\left\{y(x^y - 1) + (y-2)x^y \log x^y + 2x^y \log\left(\frac{x^y+1}{2}\right)\right\}.$$

Since $g(1,y) = 0$, if we prove $\frac{dg(x,y)}{dx} \leq 0$, then we can prove $g(x,y) \geq g(1,y) = 0$ for $y \geq 1/2$ and $0 < x \leq 1$. Since we have the relations

$$\frac{x-1}{\sqrt{x}} \leq \log x \leq \frac{2(x-1)}{x+1} \leq 0$$

for $0 < x \leq 1$, we calculate

$$\begin{aligned} & y\left(x^y - 1\right) + (y-2)\, x^y \log x^y + 2x^y \log\left(\frac{x^y+1}{2}\right) \\ & \leq y\left(x^y - 1\right) + (y-2)\, x^y \frac{(x^y - 1)}{x^{y/2}} + 2x^y \frac{2\left(\frac{x^y+1}{2} - 1\right)}{\frac{x^y+1}{2} + 1} \\ & = \frac{x^y - 1}{x^y + 3}\left\{3(y-2)\, x^{y/2} + (y-2)\, x^{3y/2} + 3y + (y+4)\, x^y\right\}. \end{aligned}$$

Thus, we have only to prove

$$k(y) \equiv 3(y-2)\, x^{y/2} + (y-2)\, x^{3y/2} + 3y + (y+4)\, x^y \geq 0$$

for $0 < x \leq 1$ and $y \geq 1/2$. Since it is trivial $k(y) \geq 0$ for $y \geq 2$, we assume $1/2 \leq y < 2$ from here. To this end, we prove that $k_1(y) \equiv 3(y-2)\, x^{y/2} + (y-2)\, x^{3y/2}$ is monotone increasing for $1/2 \leq y < 2$ and $k_2(y) \equiv 3y + (y+4)\, x^y$ is also monotone increasing for $1/2 \leq y < 2$. We easily find that

$$\frac{dk_1(y)}{dy} = \frac{1}{2} x^{y/2}\left\{2(x^y+3) + 3(x^y+1)(y-2)\log x\right\} > 0,$$

for $0 < x \leq 1$ and $1/2 \leq y < 2$.

We also have

$$\frac{dk_2(y)}{dy} = x^y + 3 + (y+4)x^y \log x.$$

Here we prove $\frac{dk_2(y)}{dy} \geq 0$ for $0 < x \leq 1$ and $1/2 \leq y < 2$. We put again

$$k_3(x) \equiv x^y + 3 + (y+4)x^y \log x,$$

then we have

$$\frac{dk_3(x)}{dx} = x^{-1+y}\left\{2(y+2) + y(y+4)\log x\right\}.$$

Thus, we have

$$\frac{dk_3(x)}{dx} = 0 \Leftrightarrow x = \mathrm{e}^{-\frac{2(y+2)}{y(y+4)}} \equiv \alpha_y.$$

Since $\frac{dk_3(x)}{dx} < 0$ for $0 < x < \alpha_y$ and $\frac{dk_3(x)}{dx} > 0$ for $\alpha_y < x \leq 1$, we have

$$k_3(x) \geq k_3(\alpha_y) = 3 - \frac{(y+4)e^{-\frac{2(y+2)}{y+4}}}{y} \equiv k_4(y).$$

Since we have $\frac{dk_4(y)}{dy} = \frac{8(y+2)e^{-\frac{2(y+2)}{y+4}}}{y^2(y+4)} > 0$, the function $k_4(y)$ is monotone increasing for y. Thus, we have

$$k_3(x) \geq k_3(\alpha_y) = 3 - \frac{(y+4)e^{-\frac{2(y+2)}{y+4}}}{y} \equiv k_4(y) \geq k_4(1/2) = 3 - \frac{9}{e^{10/9}} > 0$$

since $e^{10/9} \simeq 3.03773$. Therefore, $k_2(y)$ is also a monotone increasing function of y for $0 < x \leq 1$ and $1/2 \leq y < 2$. Thus, $k(y)$ is monotone increasing for $y \geq 1/2$, and then we have

$$k(y) \geq k(1/2) = -\frac{3}{2}\left(x^{1/4} - 1\right)^3 \geq 0.$$

(b) For the case $x \geq 1$, we firstly calculate

$$\frac{dg(x,y)}{dy} = \left(x^y - 1\right)\log x^y + \left\{y\left(x^y - 1\right) + (y-2)\,x^y \log x^y + 2x^y \log\left(\frac{1+x^y}{2}\right)\right\}\log x.$$

We put

$$p(x,y) \equiv (x^y - 1)y + x^y(y-2)\log x^y + 2x^y \log\left(\frac{1+x^y}{2}\right).$$

Then we calculate

$$\frac{dp(x,y)}{dx} = \frac{y}{x + x^{1-y}}\left\{(1+x^y)(y-2)\log x^y + 2\left(y(1+x^y) - 1 + (1+x^y)\log\left(\frac{1+x^y}{2}\right)\right)\right\}.$$

Then we put

$$q(x,y) = (y-2)\log x^y + 2\log\left(\frac{1+x^y}{2}\right) + 2y - \frac{2}{1+x^y}.$$

We have

$$\frac{dq(x,y)}{dy} = \frac{((1+x^y)^2 y - 2)\log x + (1+x^y)^2(\log x^y + 2)}{(1+x^y)^2} > 0$$

and then

$$q(x,y) \geq q(x,1/2) = 1 - \frac{2}{\sqrt{x}+1} + 2\log\left(\frac{1+\sqrt{x}}{2}\right) - \frac{3}{4}\log x.$$

Since we find

$$\frac{dq(x,1/2)}{dx} = \frac{(\sqrt{x}+3)(\sqrt{x}-1)}{4x\left(\sqrt{x}+1\right)^2} \geq 0$$

for $x \geq 1$, we have $q(x,y) \geq q(x,1/2) \geq q(1,1/2) = 0$. Therefore, we have $\frac{dp(x,y)}{dx} \geq 0$, which implies $p(x,y) \geq p(1,y) = 0$. Thus, we have $\frac{dg(x,y)}{dy} \geq 0$, and then we have $g(x,y) \geq g(x,1/2)$, where

$$g(x,1/2) = -\frac{1}{2}(3x^{1/2}+1)\log x^{1/2} + 2(x^{1/2}+1)\log\left(\frac{x^{1/2}+1}{2}\right).$$

To prove $g(x,1/2) \geq 0$ for $x \geq 1$ and $y \geq 1/2$, we put $x^{1/2} \equiv z \geq 1$ and

$$r(z) \equiv -\frac{1}{2}(3z+1)\log z + 2(z+1)\log\left(\frac{z+1}{2}\right).$$

Since we have $r''(z) = \frac{(z-1)^2}{2z^2(z+1)} \geq 0$ and

$$r'(z) = \frac{1}{2z}\left\{z - 1 - 3z\log z + 4z\log\left(\frac{z+1}{2}\right)\right\},$$

we have $r'(1) = 0$ and then we have $r'(z) \geq 0$ for $z \geq 1$. Thus, we have $r(z) \geq 0$ for $z \geq 1$ by $r(1) = 0$. Finally, we have $g(x,y) \geq g(x,1/2) \geq 0$, for $x \geq 1$ and$y \geq 1/2$.

□

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