Higher-order nonlinear evolution equation for interfacial waves in a two-layer fluid system

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An alternative type of nonlinear evolution equation is derived that describes interfacial waves in a two-layer fluid system. The equation presented here is a higher-order version of the Benjamin-Ono (BO) equation. A solitary-wave solution of the equation is obtained by means of a singular perturbation method. The characteristics of the solution are discussed in comparison with those for a higher-order BO equation of the Lax type.

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Nonlinear waves on a fluid interface have received a great deal of attention due to their practical importance [1]. In contrast to surface gravity waves [2–5], relatively little study has been done on interfacial waves. Some investigators have been concerned with interfacial waves in two-layer fluid systems where each fluid has a constant density [6–8]. They derived nonlinear evolution equations (NEE's) such as the Korteweg–de Vries (KdV) and Benjamin-Ono (BO) equations that describe the time evolution of the interfacial elevation. In this respect, it can be remarked that a unified theory of interfacial waves has been developed recently which is applicable to wave phenomena on fluid of arbitrary depth [9]. Although these equations usually incorporate the lowest-order nonlinearity in wave amplitude, there appear to be few works taking into account of higher-order nonlinearities [10–12]. Almost all the works mentioned above were made under the assumption that the depth of both fluids is short compared with a typical wavelength of the wave. The resulting NEE's are called the higher-order KdV equations. These equations have been used extensively in dealing with the large amplitude interfacial waves. As for the BO equation, however, the corresponding higher-order equations have not been obtained as yet.

The main purpose of this paper is to derive a higher-order BO equation in a two-layer fluid system. A solitary-wave solution of the equation is also presented by employing a singular perturbation method. The results are compared with those for a higher-order BO equation of the Lax type.

We propose two-dimensional systems in which a layer of a light fluid overlies a layer of a heavier one resting on a flat bottom. The upper boundary is assumed to be rigid. For the sake of generality, we first consider the case where the thickness of the two layers is arbitrary, and then take appropriate limits to match with the physical configuration leading to a higher-order BO equation. Under the assumption of the irrotational flow of an incompressible and inviscid fluid, the equations governing the fluid motion and the boundary conditions are written in dimensionless form as follows [9]:

(i) Continuity equations,

$$\delta_1^2\phi_{1,xx}+\phi_{1,yy}=0(-\infty<x<\infty,\alpha_1\eta<y<1)\ , \tag{1}$$

$$\delta_1^2\phi_{2,xx}+\phi_{2,yy}=0(-\infty<x<\infty,-\delta_2/\delta_1<y<\alpha_1\eta)\ . \tag{2}$$

(ii) Kinematic boundary conditions at the fluid interface,

$$\eta_t+\kappa\epsilon\phi_{1,x}\eta_x=\frac{\kappa}{\delta_1}\phi_{1,y}\quad(y=\alpha_1\eta)\ , \tag{3}$$

$$\eta_t+\kappa\epsilon\phi_{2,x}\eta_x=\frac{\kappa}{\delta_1}\phi_{2,y}\quad(y=\alpha_1\eta)\ . \tag{4}$$

(iii) Dynamic boundary condition at the fluid interface,

$$\Delta\left[\phi_{1,t}+\frac{\kappa\epsilon}{2\delta_1^2}(\delta_1^2\phi_{1,x}^2+\phi_{1,y}^2)+\eta-\eta_0\right]=\left[\phi_{2,t}+\frac{\kappa\epsilon}{2\delta_1^2}(\delta_1^2\phi_{2,x}^2+\phi_{2,y}^2)+\eta-\eta_0\right]\quad(y=\alpha_1\eta)\ . \tag{5}$$

(iv) Upper and lower boundary conditions,

$$\phi_{1,y}=0\quad(y=1)\ , \tag{6}$$

$$\phi_{2,y}=0\quad(y=-\delta_2/\delta_1)\ . \tag{7}$$

Here $\phi_1=\phi_1(x,y,t)$ and $\phi_2=\phi_2(x,y,t)$ are the velocity potentials for the upper and lower fluids, respectively, $\eta=\eta(x,t)$ is the interfacial elevation, η_0 is a constant; and $\Delta=\rho_1/\rho_2(<1)$ is the density ratio. The subscripts x, y, and t appended to $\phi_j(j=1,2)$ and η denote partial differentiation. This notation will be used throughout the paper.

The dimensional quantities, with tildes, are related to the corresponding dimensional ones by the relations

$$\begin{gathered}\tilde{x}=lx\ ,\quad \tilde{y}=h_1y\ ,\quad \tilde{t}=(l/c_0)t\ ,\\ \tilde{\phi}_j=(gla/c_0)\phi_j\quad(j=1,2)\ ,\quad \tilde{\eta}=a\eta\ ,\end{gathered} \tag{8}$$

and the dimensionless parameters ϵ, α_j, and δ_j are defined by

$$\epsilon=\frac{a}{l}\ ,\quad \alpha_j=\frac{a}{h_j}\quad(j=1,2)\ ,\quad \delta_j=\frac{h_j}{l}\quad(j=1,2)\ , \tag{9}$$

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where l, a, and c_0 are characteristic scales of length (wavelength in periodic wave), amplitude, and velocity of the wave, respectively; h_1 and $-h_2$ are the vertical coordinates of the upper and lower boundaries, respectively; and g is the acceleration due to gravity. A characteristic velocity c_0 is given by $c_0=\sqrt{gl/\kappa}$, where κ is a parameter depending on Δ and $\delta_j (j=1,2)$ and is chosen appropriately according to various limits of the thickness ratio of the two layers. ϵ is called the steepness parameter, which measures the magnitude of the nonlinearity.

First of all, we derive the equation for the general configuration described above. In the following discussion, we shall restrict our consideration to higher-order NEE's with cubic nonlinearities. Although it is possible in principle to incorporate nonlinearities of arbitrary order, the resulting NEE's would be difficult to manage, particularly in an analytical treatment. We first note that the solutions of (1) and (2) satisfying the boundary conditions (6) and (7) have been obtained in the form of an integral representation [9]. Expanding the potentials at the fluid interface in powers of ϵ, and retaining the terms up to $O(\epsilon^2)$, we obtain the first three terms of the expansion to be

$$\phi_{j,x}|_{y=\alpha_1\eta}=-T_jf_{j,x}-(-1)^j\epsilon\eta f_{j,xx}+\tfrac{1}{2}\epsilon^2\eta^2T_jf_{j,xxx}+O(\epsilon^3) \quad (j=1,2)\,, \tag{10}$$

$$\phi_{j,y}|_{y=\alpha_1\eta}=(-1)^{j+1}\delta_1[f_{j,x}-(-1)^j\epsilon\eta T_jf_{j,xx}-\tfrac{1}{2}\epsilon^2\eta^2f_{j,xxx}+O(\epsilon^3)] \quad (j=1,2)\,, \tag{11}$$

$$\phi_{j,t}|_{y=\alpha_1\eta}=-T_jf_{j,t}-(-1)^j\epsilon\eta f_{j,xt}+\tfrac{1}{2}\epsilon^2\eta^2T_jf_{j,xxt}+O(\epsilon^3) \quad (j=1,2)\,, \tag{12}$$

where $f_j (j=1,2)$ are arbitrary real functions, and T_j $(j=1,2)$ are integral operators defined by

$$T_jf(x,t)=\frac{1}{2\delta_j}\mathrm{P}\int_{-\infty}^{\infty}\coth[\pi(y-x)/2\delta_j]f(y,t)dy \quad (j=1,2)\,. \tag{13}$$

At this stage, we introduce the horizontal components of the interfacial velocities by

$$u_j=\phi_{j,x}|_{y=\alpha_1\eta} \quad (j=1,2)\,, \tag{14}$$

and solve (10) iteratively in $f_{j,x}$ to express them in terms of η and u_j as

$$f_{j,x}=-\tilde{T}_ju_j+(-1)^j\epsilon\tilde{T}_j(\eta\tilde{T}_ju_{j,x})-\epsilon^2[\tilde{T}_j\{\eta\tilde{T}_j(\eta\tilde{T}_ju_{j,x})_x\}+\tfrac{1}{2}\tilde{T}_j(\eta^2u_{j,xx})]+O(\epsilon^3) \quad (j=1,2)\,, \tag{15}$$

where $\tilde{T}_j$ $(j=1,2)$ are the inverse operators of T_j, i.e., $T_j\tilde{T}_j=\tilde{T}_jT_j=I$ (I is the identity operator), and given explicitly by

$$\tilde{T}_jf(x,t)=-\frac{1}{2\delta_f}\mathrm{P}\int_{-\infty}^{\infty}\frac{f(y,t)}{\sinh[\pi(y-x)/2\delta_j]}dy \quad (j=1,2)\,. \tag{16}$$

Insertion of (15) into (11) and the x derivative of (12) yields

$$\begin{aligned}\phi_{j,y}|_{y=\alpha_1\eta}=\delta_1[(-1)^j\tilde{T}_ju_j-\epsilon\{\eta u_{j,x}+\tilde{T}_j(\eta\tilde{T}_ju_{j,x})\}\\+(-1)^j\epsilon^2[\tilde{T}_j\{\eta\tilde{T}_j\{\eta\tilde{T}_j(\eta\tilde{T}_ju_{j,x})_x\}+\tfrac{1}{2}\tilde{T}_j(\eta^2u_{j,xx})+\tfrac{1}{2}\eta^2\tilde{T}_ju_{j,xx}+\eta\eta_x\tilde{T}_ju_{j,x}]+O(\epsilon^3)] \quad (j=1,2)\,,\end{aligned} \tag{17}$$

$$\begin{aligned}(\phi_{j,t}|_{y=\alpha_1\eta})_x=u_{j,t}-(-1)^j\epsilon(\eta_t\tilde{T}_ju_{j,x}-\eta_x\tilde{T}_ju_{j,t})\\+\epsilon^2[\eta_t\tilde{T}_j(\eta\tilde{T}_ju_{j,x})_x-\eta_x\tilde{T}_j(\eta\tilde{T}_ju_{j,x})_t+\eta\eta_tu_{j,xx}-\eta\eta_xu_{j,xt}]+O(\epsilon^3) \quad (j=1,2)\,.\end{aligned} \tag{18}$$

Finally, by substituting (14), (17), and (18) into (3), (4), and (5), we obtain a closed system of NEE's for η, u_1, and u_2 as follows:

$$\begin{aligned}\eta_t+(-1)^{j+1}\kappa\tilde{T}_ju_j+\kappa\epsilon[(u_j\eta)_x+\tilde{T}_j(\eta\tilde{T}_ju_{j,x})]+(-1)^{j+1}\kappa\epsilon^2[\tilde{T}_j\{\eta\tilde{T}_j(\eta\tilde{T}_ju_{j,x})_x\}\\+\tfrac{1}{2}\tilde{T}_j(\eta^2u_{j,xx})+\tfrac{1}{2}\eta^2\tilde{T}_ju_{j,xx}+\eta\eta_x\tilde{T}_ju_{j,xx}]+O(\epsilon^3)=0 \quad (j=1,2)\,,\end{aligned} \tag{19}$$

$$\begin{aligned}\Delta\Big[u_{1,t}+\epsilon(\eta_t\tilde{T}_1u_{1,x}-\eta_x\tilde{T}_1u_{1,t})+\epsilon^2[\eta_t\tilde{T}_1(\eta\tilde{T}_1u_{1,x})_x-\eta_x\tilde{T}_1(\eta\tilde{T}_1u_{1,x})_t+\eta\eta_tu_{1,xx}-\eta\eta_xu_{1,xt}]\\+\frac{\kappa\epsilon}{2}[u_1^2+(\tilde{T}_1u_1)^2+2\epsilon\{\eta u_{1,x}+\tilde{T}_1(\eta\tilde{T}_1u_{1,x})\}\tilde{T}_1u_1]_x+\eta_x+O(\epsilon^3)\Big]\\=u_{2,t}-\epsilon(\eta_t\tilde{T}_2u_{2,x}-\eta_x\tilde{T}_2u_{2,t})+\epsilon^2[\eta_t\tilde{T}_2(\eta\tilde{T}_2u_{2,x})_x-\eta_x\tilde{T}_2(\eta\tilde{T}_2u_{2,x})_t+\eta\eta_tu_{2,xx}-\eta\eta_xu_{2,xt}]\\+\frac{\kappa\epsilon}{2}[u_2^2+(\tilde{T}_2u_2)^2-2\epsilon\{\eta u_{2,x}+\tilde{T}_2(\eta\tilde{T}_2u_{2,x})\}\tilde{T}_2u_2]_x+\eta_x+O(\epsilon^3)\,.\end{aligned} \tag{20}$$

If we express u_j in terms of η by using (19) and then substitute the resultant expressions into (20), we can obtain a single equation for η. The equation is, however, too complicated to write down here.

Let us now derive a higher-order BO equation. For the purpose, we specify the configuration such that the upper layer is infinitely deep ($\delta_1 \to \infty$), and the depth of the lower layer is very small compared with a typical wavelength of the wave ($\delta_2 \ll 1$). As a result, there remain only the two small parameters α_2 and δ_2 in the system under consideration (note the relation $\epsilon = \alpha_2\delta_2$). Furthermore, we assume that α_2 has a magnitude comparable to δ_2 i.e., $\alpha_2 = O(\delta_2)$. In the long-wave approximation, the linear dispersion relation for infinitesimal waves relevant to the present problem is given by $\omega^2 = \kappa(1-\Delta)\delta_2 k^2 + O(k^4)$, so that it is appropriate to choose κ as

$$\kappa = [(1-\Delta)\delta_2]^{-1} , \tag{21}$$

to normalize the phase velocity ($=\omega/k$) to unity. Under this situation, (19) and (20) are simplified considerably as shown below. The operators T_1 and $\tilde{T}_1$ are now reduced to

$$T_1 f \sim Hf , \tag{22a}$$

$$\tilde{T}_1 f \sim -Hf , \tag{22b}$$

while T_2 and $\tilde{T}_2$ are expanded in power of δ_2 as

$$T_2 f = \frac{1}{2\delta_2} Sf + \frac{\delta_2}{3} f_x + O(\delta_2^3) , \tag{22c}$$

$$\tilde{T}_2 f = -\delta_2 f_x - \frac{\delta_2^3}{3} f_{xxx} + O(\delta_2^5) , \tag{22d}$$

where H and S are the integral operator defined by

$$Hf(x,t) = \frac{1}{\pi} \mathrm{P} \int_{-\infty}^{\infty} \frac{f(y,t)}{y-x} dy , \tag{23}$$

$$Sf(x,t) = \int_{-\infty}^{\infty} \mathrm{sgn}(y-x) f(y,t) dy . \tag{24}$$

Introducing (21) and (22) into (19) and (20), and retaining the terms up to $O(\alpha_2^2)$, one obtains the following equations:

$$u_1 = -(1-\Delta)\delta_2 H\eta_t + O(\alpha_2^3) , \tag{25}$$

$$u_2 = \tfrac{1}{2}(1-\Delta) \left[S\eta_t - \alpha_2 \eta S\eta_t + \frac{2\delta_2^2}{3} \eta_{xt} + \alpha_2^2 \eta^2 S\eta_t + O(\alpha_2^3) \right] , \tag{26}$$

$$\Delta \left[u_{1,t} + \alpha_2\delta_2(-\eta_t H u_{1,x} + \eta_x H u_{1,t}) + \frac{\alpha_2}{2(1-\Delta)} \{ u_1^2 + (Hu_1)^2 \}_x + \eta_x + O(\alpha_2^3) \right] = u_{2,t} + \frac{\alpha_2}{2(1-\Delta)} (u_2^2)_x + \eta_x + O(\alpha_2^3) . \tag{27}$$

In deriving (25) and (26), we have operated T_j on both sides of (19) and taken the limits. If we substitute (25) and (26) into (27) and eliminate η_{tt} in higher-order terms by the first-order equation $\eta_{tt} = \eta_{xx} - \alpha_2(\eta_t S\eta_t - \eta\eta_x)_x + \Delta\delta_2 H\eta_{xxx}$, we finally arrive at a single NEE for the interfacial elevation as follows:

$$\eta_{tt} - \eta_{xx} + \alpha_2(\eta_t S\eta_t - \eta\eta_x)_x - \Delta\delta_2 H\eta_{xxx} + \frac{\alpha_2^2}{4} [\eta_x (S\eta_t)^2 - 4\eta\eta_t S\eta_t]_x + \Delta\alpha_2\delta_2 [-H(\eta\eta_x)_x + H(\eta_t S\eta_t)_x - \eta H\eta_{xx}]_x + (\Delta^2 - \tfrac{1}{3})\delta_2^2 \eta_{xxxx} + O(\alpha_2^3) = 0 . \tag{28}$$

Equation (28) is suitable to describe waves propagating to both right and left directions, and hence it can treat head-on collisions of various wave structures. To obtain a NEE describing a unidirectional motion to the right, for instance, we follow the standard procedure [13]. It then turns out from (28) that the right running waves evolve according to the equation

$$\eta_t + \eta_x + \tfrac{3}{2}\alpha_2 \eta\eta_x + \frac{\Delta\delta_2}{2} H\eta_{xx} - \tfrac{3}{8}\alpha_2^2 \eta^2 \eta_x + \frac{\Delta\alpha_2\delta_2}{2} [\tfrac{5}{4}\eta H\eta_{xx} + \tfrac{9}{4} H(\eta\eta_x)_x + \eta_x H\eta_x] - \tfrac{3}{8}(\Delta^2 - \tfrac{4}{9})\delta_2^2 \eta_{xxx} + O(\alpha_2^3) = 0 . \tag{29}$$

If we neglect the terms of $O(\alpha_2^2)$, (29) reduces to the well-known BO equation [14–16]. Therefore, we call it the higher-order BO equation.

There exists another type of higher-order BO equation which is the first higher-order equation of the Lax hierarchy [17–19]. Explicitly, it may be written in the form

$$u_t + 4uu_x + Hu_{xx} - \epsilon[u^3 + \tfrac{3}{4}uHu_x + \tfrac{3}{4}H(uu_x) - \tfrac{1}{4}u_{xx}]_x = 0 . \tag{30}$$

Equation (30) exhibits multisoliton solutions [17], and it can be shown to be completely integrable. In order to compare (29) with (30), we first transform (29) into a reference frame moving with the phase velocity of the wave, which is unity in the present case, and after that rescale the variables η, t, and x according to au, bt, and cx, respectively, where $a = 3\epsilon/\alpha_2$, $b = 32\Delta\delta_2/81\epsilon^2$, and $c = 4\Delta\delta_2/9\epsilon$. Equation (29) then becomes

$$u_t + 4uu_x + Hu_{xx} - \epsilon \left[3u^2 u_x - \tfrac{15}{4} uHu_{xx} - \tfrac{27}{4} H(uu_x)_x - 3u_x Hu_x + \frac{27}{16\Delta^2} (\Delta^2 - \tfrac{4}{9}) u_{xxx} \right] = 0 . \tag{31}$$

Hence, we see that (29) cannot be reduced to (30) by any scale transformation. This fact would imply that Eq. (29) may not be integrable. To confirm this statement, we have tried to obtain exact solutions, though the attempt has not succeeded as yet. Therefore, we rely on approximate methods.

We shall now seek a solution of solitary-wave type by means of a singular perturbation method based on multiple time scales [19]. To begin with, we write (31) as

$$u_t+4uu_x+Hu_{xx}=\epsilon R[u] \ , \tag{32}$$

where $R[u]$ represents the expression in brackets in Eq. (31) and expand u in powers of the small parameter ϵ as $u=u_0+\epsilon u_1+\cdots$. At the same time, we introduce the multiple times by $t_j=\epsilon^j t$ $(j=0,1,\ldots)$, so that the time derivative is replaced by $\partial/\partial t=\partial/\partial t_0+\epsilon\partial/\partial t_1+\cdots$. Substituting these expressions into (32) and equating the coefficients of like powers of ϵ, we obtain a hierarchy of equations for u_j $(j=0,1,\ldots)$. In the lowest order of the approximation, (32) reduces to the BO equation for u_0 whereas u_j for $j\geq 1$ satisfy inhomogeneous linear evolution equations. The BO equation exhibits an explicit solution of the form [14,16,20]

$$u_0=\frac{a}{z^2+1}[z=a(x-at_0-\xi_0)\equiv a(x-\xi)] \ , \tag{33}$$

where a and ξ_0 are the amplitude and phase of the solitary wave, respectively. Owing to the action of the perturbation ϵR, however, the constant parameters a and ξ_0 are modulated slowly on the time scale t_1. The time evolution of a and ξ is then governed by the following system of ordinary differential equations [19]:

$$\frac{\partial a}{\partial t_1}=\frac{4}{\pi}\int_{-\infty}^{\infty}\frac{R[u_0(z)]}{z^2+1}dz \ , \tag{34a}$$

$$\frac{\partial \xi}{\partial t_1}=\frac{4}{\pi a^2}\int_{-\infty}^{\infty}\frac{z}{z^2+1}R[u_0(z)]dz \ . \tag{34b}$$

Performing the integrals on the right-hand side of (34), one finds that

$$\frac{\partial a}{\partial t_1}=0 \ , \tag{35a}$$

$$\frac{\partial \xi}{\partial t_1}=-\frac{3a^2}{16\Delta^2}(31\Delta^2+6) \ , \tag{35b}$$

which can readily be integrated, and the results are expressed in terms of the original time variable as

$$a=a(0) \ , \tag{36a}$$

$$\xi=\xi_0+\left[a(0)-\frac{3\epsilon a^2(0)}{16\Delta^2}(31\Delta^2+6)\right]t \ . \tag{36b}$$

Therefore, the amplitude of the solitary wave remains constant while the velocity is decreased by a quantity $[3\epsilon a^2(0)/16\Delta^2](31\Delta^2+6)$. It should be remembered that these results are valid only over the time interval $0\leq t\leq \epsilon^{-1}$. Beyond the upper limit of the interval, one must take into account the higher-order modulation effects. With (33) and (36), we are now ready to solve the equation for u_1. Unfortunately, an analytical solution for u_1 has not been found as yet, and so we must use numerical procedure. However, this problem is beyond the scope of the present paper.

On the other hand, applying the same perturbation analysis to Eq. (30) leads to the expressions corresponding to (36) as follows:

$$a=a(0) \ , \tag{37a}$$

$$\xi=\xi_0+\left[a(0)-\frac{3\epsilon a^2(0)}{16}\right]t \ . \tag{37b}$$

In contrast to (36), the above results are seen to provide an *exact* solution of (30). Indeed, one can show that by virtue of (33) and (37) the hierarchy of linear equations for u_j $(j\geq 1)$ is satisfied by $u_j=0$ $(j\geq 1)$. This is a common nature of the Lax hierarchy of the BO equation; the functional form of the soliton solutions is the same for all members of the hierarchy, the only difference being the velocity of each soliton [19].

In this paper, we have derived a higher-order BO equation in a two-layer fluid system. The equation has been investigated by means of a singular perturbation method to obtain the solitary-wave solution. The results have been compared with those for the first higher-order BO equation of the Lax type. The present analysis suggests strongly the nonintegrability of the equation. The BO equation has various characteristics in comparison with these for other integrable NEE's such as the KdV and nonlinear Schrödinger equations [19]. One remarkable aspect is the phase shift which usually occurs during the interaction of solitons. It is known that the BO solitons exhibit no phase shift [20,21], and this property also holds for all members of the Lax hierarchy of the BO equation. However, the addition of the perturbation indicated in (31) may change the characteristic drastically. Therefore, it seems to be quite interesting to study the interaction process of solitary waves on the basis of Eq. (31). This problem will be considered in a future work.

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