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Pulse formation in a dissipative nonlinear system

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A nonlocal nonlinear evolution equation is proposed that describes pulse formation in a dissipative system. A novel feature of the equation is that it can be solved exactly through a linearization procedure. The solutions are constructed under appropriate initial and boundary conditions and their properties are investigated in detail. Of particular interest is pulse formation, which is caused by a balance between nonlinearity and dissipation. The asymptotic behavior of the solution for large time is then represented by a train of moving pulses with equal amplitudes. The corresponding position of each pulse is shown to be characterized by the zero of the Hermite polynomial, irrespective of initial conditions.

I. INTRODUCTION

The study of dissipative nonlinear systems is current interest in physics.[1] The systems considered are usually described by nonlinear evolution equations (NEE's) with small dissipative perturbations. Of particular importance is the situation where in the absence of dissipations the systems become completely integrable, namely they are characterized by soliton solutions, infinite number of conservation laws, Bäcklund transformations, and so on. In general, however, the presence of dissiapations prevents exact analytical treatment of the problems. One must resort to approximate methods. For this purpose various perturbation theories have been developed,[1-3] but their applicability is limited to the cases where perturbations are small. In addition, these methods seem to be far from completed from a mathematical point of view. In this sense solvable model equations are of great value, even if they have no direct background in physics. Motivated by these facts, we have recently found a few solvable dissipative NEE's.[4,5]

In this paper we present a novel example, which can be solved exactly by a linearization procedure. The equation that we propose here is written in the form

$$\theta_t = -\mu \sin \theta + \epsilon H\theta_x, \quad \theta=\theta(x,t), \tag{1.1a}$$

where the operator H is the Hilbert transform defined by

$$H\theta(x,t) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\theta(y,t)}{y-x}\, dy, \tag{1.1b}$$

the subscripts t and x denote partial differentiation and μ and ϵ are real parameters. Equation (1.1) may be interpreted as a nonlinear diffusion equation for positive ϵ. Indeed, if $\epsilon > 0$, then the second term on the right-hand side of Eq. (1.1) produces a short wavelength dissipative effect. To see the contribution of the term to the time evolution of θ, we consider the equation $\theta_t = \epsilon H\theta_x$ and substitute a solution of the form $\theta \propto e^{ikx+\sigma t}$. We then obtain the dispersion relation $\sigma = -\epsilon |k|$. Obviously, the term with positive ϵ exhibits the dissipative nature. Under certain conditions this will result in stationary solutions, which stem from a balance between nonlinearity and dissipation. One notes that Eq. (1.1) has similar structures to those of the NEE's of the Fisher type, in which there are no convective terms.[6,7] In this respect the famous Burgers equation in one-dimensional gasdynamics belongs to a different class of nonlinear diffusion equations.[6] Although Eq. (1.1) has been derived on a mathematical aspect, it is worthwhile to remark that in the static case $\theta_t = 0$, it reduces to the Peierls equation in the theory of dislocations.[8,9] In what follows, we solve Eq. (1.1) under the boundary conditions $\theta_x \to 0$ as $|x| \to \infty$.

In Sec. II, Eq. (1.1) is solved by means of a linearization procedure. In Sec. III, the properties of solutions are investigated in detail. Three types of solutions, that is, stationary, blowup, and decaying ones are found according to the sign of μ and ϵ. In particular, in the case of positive μ and ϵ, the asymptotic form of the solution $u \equiv \theta_x$ for large time is shown to be represented by a train of pulses with equal amplitudes, and their positions are characterized by the zeros of the Hermite polynomial.

II. EXACT METHOD

A. Linearization

In this section we show that Eq. (1.1) can be linearized through an appropriate dependent variable transformation. First of all, we introduce a new dependent variable f by

$$\theta = i \ln(f^*/f), \tag{2.1a}$$

with

$$f = \prod_{j=1}^{N} [x - x_j(t)], \tag{2.1b}$$

where x_j $(j=1,2,...,N)$ are complex functions of t whose imaginary parts are all positive, and the asterisk denotes a complex conjugate. The above transformation has the same form as that used in the linearization of the sine-Hilbert equation.[10–12] Substituting (2.1) into (1.1) and using the relation $H\theta_x = -(\ln f^* f)_x$, which stems from the analytical conditions $\mathrm{Im}\, x_j > 0$ $(j=1,2,...,N)$, Eq. (1.1) is transformed into the following bilinear equation for f and f^*:

$$i(f_t^* f - f^* f_t) = -\frac{\mu}{2i}(f^2 - f^{*2}) - \epsilon (f^* f)_x. \qquad (2.2)$$

Further reduction is possible if we modify (2.2) in the form

$$f\left(if_t^* + \frac{\mu}{2i}(f - f^*) + \epsilon f_x^*\right)$$

$$+ f^*\left(-if_t - \frac{\mu}{2i}(f^* - f) + \epsilon f_x\right) = 0. \qquad (2.3)$$

Equation (2.3) is then satisfied automatically, provided that the following linear equation for f holds:

$$f_t = -i\mu\, \mathrm{Im}\, f - i\epsilon f_x. \qquad (2.4)$$

Equation (2.4) is a linearization of Eq. (1.1) and is a main result in the present paper. Before solving Eq. (2.4), we derive the equation of motion for the pole x_j. To do so, we divide (2.4) by f, substitute (2.1), and then compare the coefficients of $(x-x_j)^{-1}$ on both sides. The resultant expression reads in the form

$$\dot{x}_j = -\frac{\mu}{2}\frac{\Pi_{k=1}^N (x_j - x_k^*)}{\Pi_{k=1(k\neq j)}^N (x_j - x_k)} + i\epsilon, \qquad (2.5)$$

where an overdot denotes differentiation with respect to t. By taking the imaginary part of Eq. (2.5), we have

$$\mathrm{Im}\, \dot{x}_j = -G_j\, \mathrm{Im}\, x_j + \epsilon, \qquad (2.6a)$$

where

$$G_j = \mathrm{Re}\left(\mu \prod_{\substack{k=1\\(k\neq j)}}^{N} \frac{x_j - x_k^*}{x_j - x_k}\right). \qquad (2.6b)$$

Integration of Eq. (2.6) yields an important relation:

$$\mathrm{Im}\, x_j(t) = \left[\mathrm{Im}\, x_j(0) + \epsilon \int_0^t dt' \exp\left(\int_0^{t'} G_j(t'')dt''\right)\right] \times \exp\left(-\int_0^t G_j(t')dt'\right). \qquad (2.7)$$

In the case of $\epsilon > 0$, (2.7) would ensure that if the condition $\mathrm{Im}\, x_j(0) > 0$ is satisfied, then the same inequality holds for later time. On the other hand, if $\epsilon < 0$, $\mathrm{Im}\, x_j(t)$ becomes zero at finite t under the appropriate initial condition and that leads to blowup of the solution.

B. Solutions

Let us now solve Eq. (2.4). We first expand f as

$$f = \sum_{j=0}^{N} (-1)^j s_j x^{N-j}, \qquad (2.8)$$

where $s_0 = 1$ and s_j $(j=1,2,...,N)$ are elementary symmetric functions of $x_1, x_2, ..., x_N$,

$$s_1 = \sum_{j=1}^{N} x_j, \quad s_2 = \sum_{j<k}^{N} x_j x_k, ..., s_N = \prod_{j=1}^{N} x_j. \qquad (2.9)$$

Substituting (2.8) into Eq. (2.4) and comparing the coefficients of x^{N-j} on both sides, we obtain the following system of linear differential equations for s_j:

$$\dot{s}_j = -i\mu\, \mathrm{Im}\, s_j + i\epsilon(N-j+1)s_{j-1} \quad (j=1,2,...,N). \qquad (2.10)$$

In order to solve Eq. (2.10), we specify the initial conditions for s_j as

$$s_j(0) = b_j + ia_j \quad (j=1,2,...,N), \qquad (2.11)$$

and those for x_j as

$$x_j(0) = \tilde{b}_j + i\tilde{a}_j, \quad \tilde{a}_j > 0 \quad (j=1,2,...,N). \qquad (2.12)$$

In view of (2.9) these parameters are linked to each other by the relations

$$a_1 = \sum_{j=1}^{N} \tilde{a}_j, \quad b_1 = \sum_{j=1}^{N} \tilde{b}_j, \qquad (2.13a)$$

$$a_2=\sum_{j<k}^{N}(\tilde{a}_j\tilde{b}_k+\tilde{a}_k\tilde{b}_j),\quad b_2=\sum_{j<k}^{N}(-\tilde{a}_j\tilde{a}_k+\tilde{b}_j\tilde{b}_k),\tag{2.13b}$$

$$\vdots$$

$$a_N=\mathrm{Im}\left(\prod_{j=1}^{N}(\tilde{b}_j+i\tilde{a}_j)\right),\quad b_N=\mathrm{Re}\left(\prod_{j=1}^{N}(\tilde{b}_j+i\tilde{a}_j)\right).\tag{2.13c}$$

It now follows from the real and imaginary parts of Eq. (2.10) that

$$\mathrm{Re}\,\dot{s}_j=-\epsilon(N-j+1)\mathrm{Im}\,s_{j-1},\tag{2.14a}$$

$$\mathrm{Im}\,\dot{s}_j=-\mu\,\mathrm{Im}\,s_j+\epsilon(N-j+1)\mathrm{Re}\,s_{j-1}.\tag{2.14b}$$

Integration of Eq. (2.14a) with the initial condition (2.11) gives

$$\mathrm{Re}\,s_j=-\epsilon(N-j+1)\int_0^t \mathrm{Im}\,s_{j-1}\,dt+b_j.\tag{2.15}$$

Substituting (2.15) into Eq. (2.14b) and integrating with the initial condition (2.11), we obtain, after some manipulations, the following integral equation:

$$\mathrm{Im}\,s_j=p_j\left(\int_0^t \mathrm{Im}\,s_{j-2}\,dt-e^{-\mu t}\int_0^t e^{\mu t}\,\mathrm{Im}\,s_{j-2}\,dt\right)+q_j,\tag{2.16a}$$

where we have put

$$p_j=-(\epsilon^2/\mu)(N-j+1)(N-j+2),\tag{2.16b}$$

$$q_j=(\epsilon/\mu)(N-j+1)b_{j-1}(1-e^{-\mu t})+a_j\,e^{-\mu t},\tag{2.16c}$$

for simplicity. Furthermore, if we introduce a linear integral operater $\hat{I}_\mu$ defined by

$$\hat{I}_\mu g=\int_0^t g\,dt-e^{-\mu t}\int_0^t e^{\mu t}g\,dt,\quad g=g(t),\tag{2.17}$$

Eq. (2.16) is written compactly as

$$\mathrm{Im}\,s_j=p_j\hat{I}_\mu\,\mathrm{Im}\,s_{j-2}+q_j.\tag{2.18}$$

Iteration of (2.18) immediately yields the formal solution as follows:

$$\mathrm{Im}\,s_j=\sum_{k=1}^{[(j-1)/2]}\left(\prod_{s=0}^{k-1}p_{j-2s}\right)\hat{I}_\mu^k q_{j-2k}+q_j\quad(j\geqslant 3).\tag{2.19}$$

Here the notation $[x]$ means the maximum integer not exceeding x. If we take into account (2.16b) and (2.16c), we find that the problem under consideration reduces to evaluate the integrals $\hat{I}_\mu^n 1$ and $\hat{I}_\mu^n e^{-\mu t}$ $(n\geqslant 1)$. However, since the latter integral is derived from the former one by the formula

$$\hat{I}_\mu^n e^{-\mu t}=(-1)^n e^{-\mu t}\hat{I}_\mu^n 1\quad(n\geqslant 1),\tag{2.20}$$

it is sufficient to evaluate $\hat{I}_\mu^n 1$.

Now we can easily surmise that $J_n\equiv\hat{I}_\mu^n 1$ will be expressed in the form

$$J_n=\sum_{j=0}^{n}\alpha_j^{(n)}t^j+e^{-\mu t}\sum_{j=0}^{n-1}\beta_j^{(n)}t^j.\tag{2.21}$$

In order to determine the unknown coefficients $\alpha_j^{(n)}$ and $\beta_j^{(n)}$, we use the identity $J_n=\hat{I}_\mu J_{n-1}$. Substituing (2.21) into this expression and comparing the coefficients of t^j $(j=0,1,\ldots,n)$ and $e^{-\mu t}t^j$ $(j=0,1,\ldots,n-1)$ on both sides, we obtain the following recursion relations for $\alpha_j^{(n)}$ and $\beta_j^{(n)}$:

$$\alpha_n^{(n)}=\alpha_{n-1}^{(n-1)}/n,\tag{2.22a}$$

$$\alpha_j^{(n)}=\frac{\alpha_{j-1}^{(n-1)}}{j}+\sum_{r=j}^{n-1}\frac{\binom{r}{j}}{(-\mu)^{r-j+1}}\alpha_r^{(n-1)}\quad(j=1,2,\ldots,n-1),\tag{2.22b}$$

$$\alpha_0^{(n)}=\sum_{j=0}^{n-1}\frac{\alpha_j^{(n-1)}}{(-\mu)^{j+1}}+\sum_{j=0}^{n-2}\frac{\beta_j^{(n-1)}}{\mu^{j+1}},\tag{2.22c}$$

$$\beta_{n-1}^{(n)}=\frac{-\beta_{n-2}^{(n-1)}}{n-1},\tag{2.22d}$$

$$\beta_j^{(n)}=\frac{-\beta_{j-1}^{(n-1)}}{j}-\sum_{r=j}^{n-2}\frac{\binom{r}{j}}{\mu^{r-j+1}}\beta_r^{(n-1)}\quad(j=1,2,\ldots,n-2),\tag{2.22e}$$

$$\beta_0^{(n)}=-\sum_{j=0}^{n-2}\frac{\beta_j^{(n-1)}}{\mu^{j+1}}-\sum_{j=0}^{n-1}\frac{\alpha_j^{(n-1)}}{(-\mu)^{j+1}}.\tag{2.22f}$$

Here $\binom{r}{j}$ is a binomial coefficient. In principle, these relations can be solved successively, starting with $\alpha_0^{(1)}=-1/\mu$, $\alpha_1^{(1)}=1$, and $\beta_0^{(1)}=1/\mu$. Although the explicit solutions have not been found for general n, those of (2.22a) and (2.22d) are easily obtained. They read in the forms

$$\alpha_n^{(n)}=1/n!,\tag{2.23a}$$

$$\beta_{n-1}^{(n)}=\frac{1}{\mu}\frac{(-1)^{n-1}}{(n-1)!}, \tag{2.23b}$$

and will be used in Sec. III to derive the long time behavior of the solution. Once $\mathrm{Im}\, s_j$ have been obtained, the solutions for $\mathrm{Re}\, s_j$ follow immediately from (2.15) by a simple integration.

Thus the construction of solutions for Eq. (2.14) has been completed. We only quote the first two of them, which are given by

$$s_1=b_1+i\left[-\left(a_1-\frac{\epsilon}{\mu}N\right)(1-e^{-\mu t})+a_1\right], \tag{2.24a}$$

$$\begin{aligned} s_2=&-\frac{\epsilon^2}{\mu}N(N-1)t-\frac{\epsilon}{\mu}(N-1)\left(a_1-\frac{\epsilon}{\mu}N\right)\\ &\times(1-e^{-\mu t})+b_2+i\left[-\left(a_2-\frac{\epsilon(N-1)}{\mu}b_1\right)\right.\\ &\left.\times(1-e^{-\mu t})+a_2\right]. \end{aligned} \tag{2.24b}$$

III. PROPERTIES OF SOLUTIONS

In this section we shall investigate the properties of solutions. The solutions θ obtained in Sec. II may be called the kink solutions, in analogy with those of the sine-Gordon equation; this can be seen by rewriting (2.1) as $\theta=2\tan^{-1}(\mathrm{Im}\, f/\mathrm{Re}\, f)$. However, for the purpose of visualizing the solutions it is convenient to introduce the function $u\equiv\theta_x$. It then follows from (2.1) that

$$u=\sum_{j=1}^{N}\frac{2\,\mathrm{Im}\, x_j}{(x-\mathrm{Re}\, x_j)^2+(\mathrm{Im}\, x_j)^2}. \tag{3.1}$$

The expression (3.1) indicates that the solution u is represented by a superposition of N pulses. The jth pulse has a Lorentzian profile with the amplitude $2/\mathrm{Im}\, x_j$ and the center position $\mathrm{Re}\, x_j$. We now consider the cases $N=1$ and general N separately.

A. $N=1$

The expression (2.24a) with $N=1$ gives

$$x_1=s_1=b_1+i[-(a_1-\epsilon/\mu)(1-e^{-\mu t})+a_1]$$

$$(a_1>0), \tag{3.2}$$

and (3.1) becomes

$$u=\frac{2\,\mathrm{Im}\, x_1}{(x-\mathrm{Re}\, x_1)^2+(\mathrm{Im}\, x_1)^2}. \tag{3.3}$$

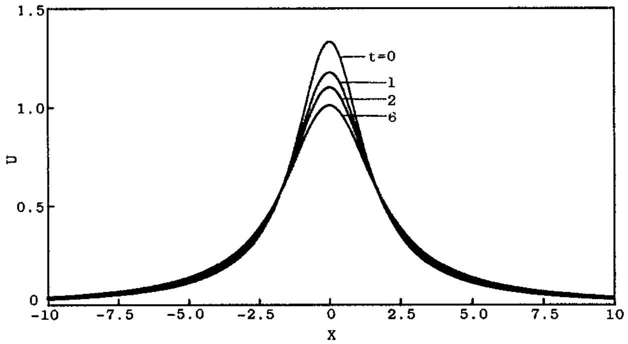

FIG. 1. Time evolution of u for case 1.

It is seen from (3.2) and (3.3) that the pulse does not propagate and only the amplitude changes. As already pointed out in the Introduction, Eq. (1.1) exhibits a dissipative nature only for positive ϵ. However, we also examine the characteristics of solutions for negative ϵ. Then three types of solutions, namely stationary, blowup, and decaying ones arise according to the sign of μ and ϵ:

1. $\mu>0$, $\epsilon>0$

Since $s_1\sim b_1+i(\epsilon/\mu)$ as $t\to\infty$, the solution approaches a stationary profile as time evolves. The asymptotic expression of u takes the form

$$u\sim\frac{2(\epsilon/\mu)}{(x-b_1)^2+(\epsilon/\mu)^2}\quad(t\to\infty). \tag{3.4}$$

The corresponding θ may be written as

$$\theta\sim\pi+2\tan^{-1}[(\mu/\epsilon)(x-b_1)], \tag{3.5}$$

where we have assumed the boundary condition $\theta\to 0$ as $x\to-\infty$. It then turns out that (3.5) represents a wave front connecting two constant states, $\theta=0$ and $\theta=2\pi$, which are exact solutions of Eq. (1.1). The theory of linear stability shows that these states are stable. Indeed, if we linearize Eq. (1.1) around $\theta=0,2\pi$ as

$$\theta_t=-\mu\theta+\epsilon H\theta_x, \tag{3.6}$$

and substitute the solution of the form $\theta\propto e^{ikx+\sigma t}$, we obtain the linear dispersion relation

$$\sigma=-\mu-\epsilon|k|. \tag{3.7}$$

Obviously, for positive μ and ϵ, (3.7) gives negative σ, implying that the two states are stable against small disturbances. A remarkable feature of (3.4) is that the amplitude does not depend on initial conditions. It is also interesting to remark that (3.5) has essentially the same functional form as that found by Peierls in his theory of dislocations.[7] Figure 1 represents the typical time evolu-

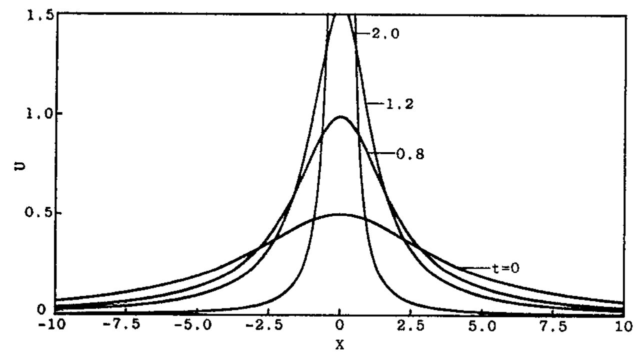

FIG. 2. Time evolution of u for case 2.

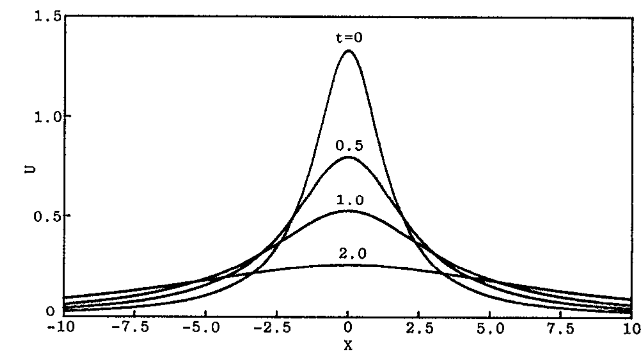

FIG. 3. Time evolution of u for case 3.

tion of u, where the parameters are chosen as $\mu=0.5$, $\epsilon=1.0$, $a_1=1.5$, and $b_1=0$.

2. $\mu>0$, $\epsilon<0$

As is easily confirmed, $\operatorname{Im} x_1$ becomes zero at $t=(1/\mu)\ln(1+\mu a_1/|\epsilon|)$, at which instant the solution blows up as

$$u \sim 2\pi\delta(x-b_1), \tag{3.8}$$

where δ is Dirac's delta function. In deriving (3.8) we have used the formula $\lim_{\epsilon\to 0}\epsilon/(x^2+\epsilon^2)=\pi\delta(x)$. The corresponding θ takes a form of step function with its discontinuity at $x=b_1$. To interpret these results we use (3.7). It now yields the linear dispersion relation $\sigma=-\mu+|\epsilon k|$. Thus, the short wavelength component of the disturbance becomes unstable. One notes that in addition to $\theta=0$, 2π another constant state $\theta=\pi$ exists for Eq. (1.1). However, it is not stable in the present situation, as seen from the corresponding linear dispersion relation $\sigma=\mu+|\epsilon k|$. Consequently, a singularity has been formed at $x=b_1$, as indicated by (3.8). The time evolution of u is shown in Fig. 2. The parameters are the same as those used in Fig. 1, except $\epsilon=-1.0$ and $a_1=4.0$. In this example the blowup time of the solution is estimated to be 2.197.

3. $\mu<0$, $\epsilon>0$

In this case $\operatorname{Im} x_1$ increases indefinitely so that the amplitude approaches zero as $t\to\infty$. The relation (3.7) is now written in the form $\sigma=|\mu|-\epsilon|k|$. This means that the constant states $\theta=0$, 2π become unstable for large wavelength disturbances. As a result, the system has evolved into another constant state $\theta=\pi$ (or equivalently $u=0$), which is seen to be stable in view of the corresponding linear dispersion relation $\sigma=-|\mu|-\epsilon|k|$. The time evolution of u is depicted in Fig. 3. The parameters are the same as those used in Fig. 1, except $\mu=-0.5$.

4. $\mu<0$, $\epsilon<0$

In this situation, the constant solutions $\theta=0$, 2π are unstable and two different asymptotic states are possible, depending on the initial condition. If $a_1>\epsilon/\mu$, then $\operatorname{Im} x_1\to\infty$ as $t\to\infty$, so that the behavior of the solution is similar to that of case 3. On the other hand, if $0<a_1<\epsilon/\mu$, the solution blows up like (3.8) when $t=(1/\mu)\ln(1-\mu a_1/\epsilon)$.

B. General N

As in the case of $N=1$, three types of solutions are found for general N. However, we restrict our considerations to case 1 in Sec. III A, namely $\mu>0$ and $\epsilon>0$. The main reason is that only in this situation pulses with finite amplitudes are formed that develop no singularities. Also, we shall be concerned with the asymptotic behavior of the solution for large t.

First, consider even N ($N=2n$, $n=1,2,\ldots$). Then (2.8) reduces to

$$f=\sum_{j=0}^{n} s_{2j}x^{2(n-j)}-\sum_{j=0}^{n-1} s_{2j+1}x^{2n-2j-1}. \tag{3.9}$$

The leading terms of the long time behaviors of s_{2j} and s_{2j+1} are readily derived from (2.15), (2.19), (2.21), and (2.23) as follows:

$$\operatorname{Re} s_{2j}\sim\left(\frac{-\epsilon^2}{\mu}\right)^j\frac{(2n)!}{j!(2n-2j)!}t^j, \tag{3.10a}$$

$$\operatorname{Im} s_{2j}\sim-\epsilon^{-1}\left(\frac{-\epsilon^2}{\mu}\right)^j b_1\frac{(2n-1)!}{(j-1)!(2n-2j)!}t^{j-1}, \tag{3.10b}$$

$$\operatorname{Re} s_{2j+1}\sim\left(\frac{-\epsilon^2}{\mu}\right)^j b_1\frac{(2n-1)!}{j!(2n-2j-1)!}t^j, \tag{3.10c}$$

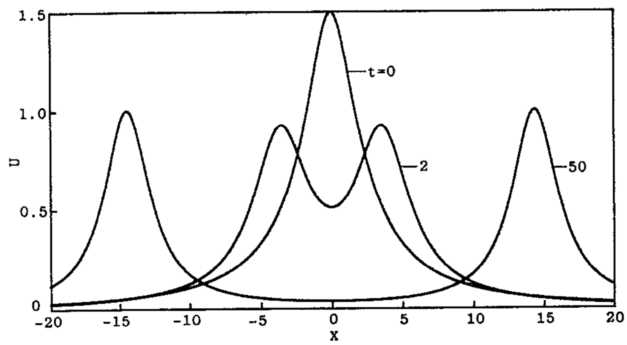

FIG. 4. Time evolution of u for $N=2$.

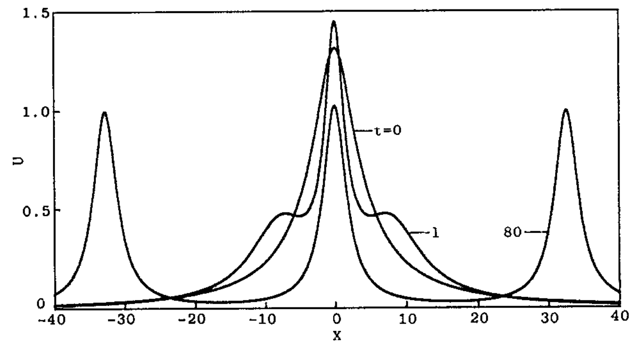

FIG. 5. Time evolution of u for $N=3$.

$$\mathrm{Im}\, s_{2j+1} \sim -\epsilon^{-1}\left(\frac{-\epsilon^2}{\mu}\right)^{j+1} \frac{(2n)!}{j!(2n-2j-1)!} t^j. \tag{3.10d}$$

To determine the locations of the poles, we must solve the algebraic equation $f=0$. A detailed inspection shows that the roots can be expressed in the leading order of the large time expansion as

$$x \sim \sqrt{(\epsilon^2/\mu)t}\,\alpha + i(\epsilon/\mu)\beta. \tag{3.11}$$

Substituting (3.10) and (3.11) into (3.9), we find that, αs are given by $2n$ roots of the algebraic equation,

$$\sum_{j=0}^{n} \frac{(-1)^j(2n)!}{j!(2n-2j)!}\alpha^{2(n-j)} = 0, \tag{3.12}$$

and β is determined to be 1. In the case of odd N ($N=2n+1$, $n=0,1,\ldots$), one can obtain the similar results. The equation corresponding to (3.12) now becomes

$$\sum_{j=0}^{n} \frac{(-1)^j(2n+1)!}{j!(2n+1-2j)!}\alpha^{2n+1-2j} = 0, \tag{3.13}$$

and $\beta=1$. Here we notice the definition of the Hermite polynomial

$$H_n(x) = \sum_{j=0}^{[n/2]} \frac{(-1)^j n!}{j!(n-2j)!}\frac{x^{n-2j}}{2^j}. \tag{3.14}$$

Let $x_{j,n}$ be the jth root of $H_n(x)$. Comparing (3.12) and (3.13) with (3.14), we see that the jth root of Eqs. (3.12) and (3.13) is related to $x_{j,N}$ by a simple formula

$$\alpha_j = \sqrt{2}x_{j,N} \quad (j=1,2,\ldots,N). \tag{3.15}$$

If we use (3.11), (3.15), and $\beta=1$, the asymptotic form of u for large t can be expressed in the form

$$u \sim \sum_{j=1}^{N} \frac{2(\epsilon/\mu)}{(x-\sqrt{(2\epsilon^2/\mu)t}x_{j,N})^2 + (\epsilon/\mu)^2}. \tag{3.16}$$

The profile of the jth pulse is the same as (3.4), but in the present case it propagates with the velocity $\sqrt{\epsilon^2/2\mu t}x_{j,N}$. Hence (3.16) represents a train of N moving pulses with equal amplitudes. However, due to the property of the root of H_n, we can put $x_{j,2n}=-x_{j+n,2n}$ ($j=1,2,\ldots,n$) for $N=2n$ and $x_{j,2n+1}=-x_{j+n,2n+1}$ ($j=1,2,\ldots,n$), $x_{2n+1,2n+1}=0$ for $N=2n+1$ without loss of generality. This implies that for even N the n pulses propagate to the right direction and the other ones to the left direction, while for odd N the same situation occurs, the only difference being that one pulse remains at the origin. It is interesting to observe that the asymptotic values of the amplitude and the position of each pulse do not depend on initial conditions in the leading order of the large time expansion. In terms of the original variable θ, the expression corresponding to (3.5) may be written in the form

$$\theta \sim N\pi + 2\sum_{j=1}^{N} \tan^{-1}[(\mu/\epsilon)(x-\sqrt{(2\epsilon^2/\mu)t}x_{j,N})]. \tag{3.17}$$

It represents N moving wave fronts connecting $N+1$ stable constant states $\theta=2n\pi$ ($n=0,1,\ldots,N$). The location of the jth front is given by $\sqrt{(2\epsilon^2/\mu)t}x_{j,N}$.

Typical time evolutions of u for $N=2$ and $N=3$ are illustrated in Figs. 4 and Fig. 5, respectively. The figures show the profiles of pulses emerged from a single pulse located initially at the origin. In Fig. 4 the parameters are chosen as $\mu=0.5$, $\epsilon=1.0$, $\tilde{a}_1=4.0$, $\tilde{a}_2=2.0$, $\tilde{b}_1=\tilde{b}_2=0$, and in Fig. 5 they are given by $\mu=0.5$, $\epsilon=1.0$, $\tilde{a}_1=8.0$, $\tilde{a}_2=5.0$, $\tilde{a}_3=3.0$, $\tilde{b}_1=\tilde{b}_2=\tilde{b}_3=0$.

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