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Y. Matsuno

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Two-dimensional dynamical system associated with Abel's nonlinear differential equation

Y. Matsuno
Department of Physics, Faculty of Liberal Arts, Yamaguchi University, Yamaguchi 753, Japan

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A two-dimensional dynamical system is proposed that is described by a pair of nonlinear ordinary differential equations (ODEs) with a complex parameter. It reduces to Abel's nonlinear ODE of the first kind by an appropriate transformation. Using this fact the properties of solutions are investigated in detail with the aid of numerical computations. It is found that various types of bifurcation phenomena occur depending on the values of the parameter. In particular, the solution is shown to blow up in finite time under certain conditions. In order to visualize the behaviors of dynamical motions the trajectories of solutions are depicted in the plane. Finally, a discussion is made on some generalizations of the proposed system.

I. INTRODUCTION

In this paper we shall study the following nonlinear ordinary differential equation (ODE) for a complex function z,

$$\dot{z}=\mathrm{Im}\, z+\lambda|z|^2 z, \quad z=x+iy, \tag{1.1}$$

where $x=x(t)$ and $y=y(t)$ are real functions of time t, λ is a complex parameter, and an overdot on z denotes differentiation with respect to t. If we put $\lambda=\alpha+i\beta(\alpha,\beta\text{:real})$ and equate the real and imaginary parts, respectively, Eq. (1.1) is decomposed into the system of nonlinear ODEs for x and y as follows:

$$\dot{x}=y+(\alpha x-\beta y)(x^2+y^2), \tag{1.2a}$$

$$\dot{y}=(\beta x+\alpha y)(x^2+y^2). \tag{1.2b}$$

Equation (1.2) describes dynamical motion of a point $z(=x+iy)$ in the (x,y) plane.

For $\lambda=0$ Eq. (1.1) becomes integrable, namely, it has an explicit solution of the form

$$z=y_0 t+x_0+iy_0, \tag{1.3}$$

where x_0 and y_0 are initial values of x and y, respectively. The trajectory (1.3) represents a straight line parallel to the x axis.

The purpose of this paper is to investigate the effect of the nonlinearity on the time evolution of z given by (1.3). In so doing we have taken into account a simplest nonlinearity $\lambda|z|^2 z$, as indicated by Eq. (1.1). It is seen that an addition of such a term changes drastically the properties of solutions; various types of bifurcation phenomena occur depending on the values of the complex parameter λ.

In Sec. II it is shown that Eq. (1.1) can be transformed into Abel's nonlinear ODE of the first kind by means of an appropriate dependent variable transformation. In Sec. III the two special cases $\alpha\neq 0$, $\beta=0$ and $\alpha=0$, $\beta\neq 0$ are analyzed in detail. For these cases the solutions are obtained explicitly by quadratures. The time dependence of z is then expressed in terms of rational functions in the former case and Jacobi's elliptic functions in the latter one. In Sec. IV we first discuss on the stationary solutions of Eq. (1.1) and investigate their linear stability. Then the properties of solutions for the general case $\alpha\beta\neq 0$ are studied in detail. Since in this case the explicit solutions cannot be obtained by quadratures, we have performed the numerical integrations using the Runge–Kutta–Gill method. The trajectories of solutions thus obtained are drawn graphically in the (x,y) plane. We find that for $\alpha>0$ the solutions blow up in finite time while for $\alpha<0$ they tend asymptotically to stationary solutions as the time goes to infinity. The blowup time is estimated both numerically and analytically. Section V is devoted to concluding remarks, where a few generalizations of Eq. (1.1) are proposed.

II. REDUCTION TO ABEL'S EQUATION

A. Abel's nonlinear ODE

Abel's nonlinear ODE of the first kind is written in the form[1–3]

$$\frac{dy}{dt}=a_0+a_1 y+a_2 y^2+a_3 y^3, \tag{2.1}$$

where a_j ($j=0$–3) are known functions of t. Equation (2.1) can be put into the standard form as

$$\frac{dz}{dx}=z^3+p(t), \tag{2.2}$$

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by introducing the following transformations:

$$y=a(t)z(x)+b(t), \tag{2.3a}$$

$$x=\int^t a^2a_3\,dt, \tag{2.3b}$$

with

$$a(t)=\exp\left[\int^t\left(a_1-\frac{a_2^2}{3a_3}\right)dt\right], \tag{2.4a}$$

$$b(t)=-a_2/3a_3. \tag{2.4b}$$

Here $p(t)$ in (2.2) has the form

$$p(t)=\left(a_0-\frac{1}{3}\frac{a_1a_2}{a_3}+\frac{2}{27}\frac{a_2^3}{a_3^2}+\frac{1}{3}\frac{d}{dt}\frac{a_2}{a_3}\right)(a^3a_3)^{-1}. \tag{2.5}$$

If $a_3=0$, Eq. (2.1) reduces to the famous Ricatti equation. For $a_0=0$, $a_1\neq0$ and either $a_2=0$, $a_3\neq0$ or $a_2\neq0$, $a_3=0$, it becomes the nonlinear ODE of Bernoulli type, which has an explicit general solution.[1–3] Although the solutions for Abel's nonlinear ODE cannot be obtained by quadratures, in general, certain nonlinear superposition formulas are already known.[4,5]

B. Reduction to Abel's ODE

In this section we show that Eq. (1.2) can be reducible to Abel's nonlinear ODE. For this purpose it is appropriate to introduce the polar coordinate (r,θ) by the relation

$$z=x+iy=re^{i\theta},\quad x=r\cos\theta,\quad y=r\sin\theta. \tag{2.6}$$

Equation (1.2) is then transformed into the form

$$\dot r=r\sin\theta\cos\theta+\alpha r^3, \tag{2.7a}$$

$$\dot\theta=-\sin^2\theta+\beta r^2. \tag{2.7b}$$

If we define a new dependent variable ρ by

$$\rho=r^2/(\beta r^2-\sin^2\theta), \tag{2.8}$$

then Eq. (2.7) can be recast into the form

$$\frac{d\rho}{d\theta}=\frac{\dot\rho}{\dot\theta}=2\rho(\beta\rho-1)[-(\alpha+\beta\cot\theta)\rho+2\cot\theta]. \tag{2.9}$$

Obviously Eq. (2.9) is a special case of Eq. (2.1) with

$$a_0=0, \tag{2.10a}$$

$$a_1=-4\cot\theta, \tag{2.10b}$$

$$a_2=2(\alpha+3\beta\cot\theta), \tag{2.10c}$$

$$a_3=-2\beta(\alpha+\beta\cot\theta). \tag{2.10d}$$

One can easily transform Eq. (2.9) into the standard form. The details are, however, omitted here.

Finally, we shall discuss the initial condition. In order to solve Eq. (2.7) the initial condition is specified as $r=r_0$, $\theta=\theta_0$ at $t=0$ for instance. However, as is easily seen, Eq. (2.7) is invariant under the scale transformations $r\to r_0r$, $\alpha\to\alpha/r_0^2$, and $\beta\to\beta/r_0^2$. Hence, we may put $r_0=1$ without loss of generality. Furthermore, we take $\theta_0=\pi/2$ for simplicity since the detailed investigation shows that the properties of solutions do not critically depend on the value of θ_0. The initial condition that we use in the following sections is therefore $r=1$, $\theta=\pi/2$, or, equivalently, $x=0$, $y=1$ due to (2.6).

III. SPECIAL CASE

A. $\alpha\neq0$, $\beta=0$

In this case, Eq. (2.9) becomes

$$\frac{d\rho}{d\theta}=2\rho(\alpha\rho-2\cot\theta). \tag{3.1}$$

Since Eq. (3.1) is a special case of the Bernoulli equation,[1–3]

$$\frac{dy}{dt}=ay^n+by,\quad a=a(t),\quad b=b(t), \tag{3.2}$$

it can be readily integrated. The result is expressed as follows:

$$\rho=(w^2+1)^2/[2\alpha(w^3/3+w)-1], \tag{3.3a}$$

$$w=\cot\theta. \tag{3.3b}$$

Here the initial condition $\rho_{\theta=\pi/2}=-1$ has been used. Substitution of (3.3) into the relation

$$r^2=\rho\sin^2\theta/(\beta\rho-1)=-\rho\sin^2\theta, \tag{3.4}$$

which stems from (2.8) and $\beta=0$, yields

$$r^2=(\cot^2\theta+1)/[1-2\alpha(\tfrac{1}{3}\cot^3\theta+\cot\theta)]. \tag{3.5}$$

The time evolution of θ is obtained by integrating (2.7b) with $\beta=0$ and using the initial condition $\theta_{t=0}=\pi/2$ as

$$\cot\theta=t. \tag{3.6}$$

It is seen from (3.5) and (3.6) that r^2 evolves as

$$r^2=(t^2+1)/[1-2\alpha(t^3/3+t)]. \tag{3.7}$$

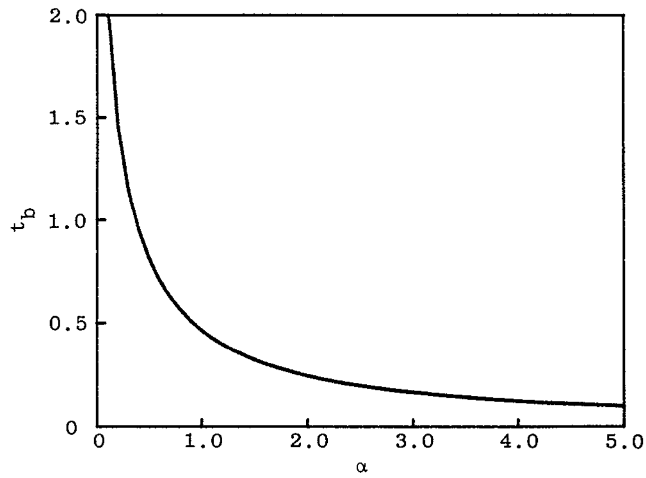

FIG. 1. The blowup time t_b as a function of α.

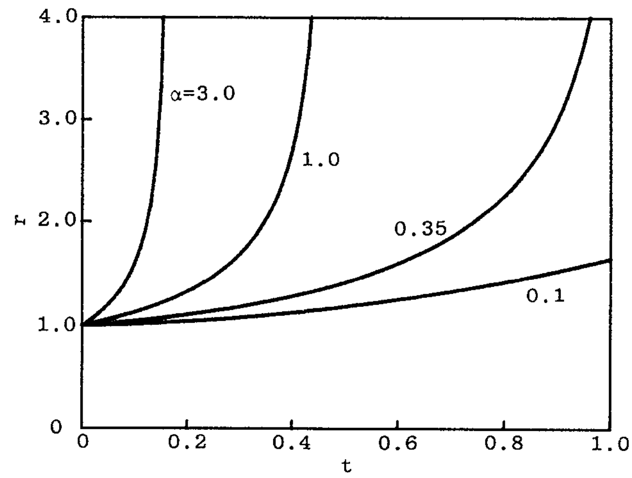

FIG. 2. The time evolution of r for positive α.

In the following we shall investigate the two cases $\alpha>0$ and $\alpha<0$ separately.

1. $\alpha>0$

In this case r is a monotonically increasing function of t and it blows up at a finite time t_b. The blowup time t_b is determined by solving the following algebraic equation of order three:

$$t^3+3t-3/2\alpha=0. \tag{3.8}$$

Explicitly, t_b is written in the form

$$t_b=\tau_1^{1/3}-\tau_2^{1/3}, \tag{3.9a}$$

with

$$\tau_1=(\sqrt{9/4\alpha^2+4}+3/2\alpha)/2, \tag{3.9b}$$

$$\tau_2=(\sqrt{9/4\alpha^2+4}-3/2\alpha)/2. \tag{3.9c}$$

The blowup time t_b as a function of α is plotted in Fig. 1. We see from (3.9) that $t_b \sim (3/2\alpha)^{1/3}$ for $\alpha \ll 1$ and $t_b \sim 1/2\alpha$ for $\alpha \gg 1$. Figure 2 shows the time evolution of r for various values of α.

2. $\alpha<0$

In this case r becomes a monotonically decreasing function of t for $|\alpha|>\frac{3}{4}$. On the other hand, for $|\alpha|<\frac{3}{4}$ it has two extremums at $t=t_1$ and $t=t_2$. Here t_1 and t_2 are positive roots of the algebraic equation $t^4-3t/|\alpha|+3=0$. In both cases r tends to zero as $t\to\infty$. Hence the trajectory in the (x,y) plane approaches the origin indefinitely as the time goes to infinity. Figure 3 depicts the time evolution of r for various values of α.

Now it follows from (2.7) with $\beta=0$ that

$$\frac{dy}{dx}=\frac{\dot r\sin\theta+r\dot\theta\cos\theta}{\dot r\cos\theta-r\dot\theta\sin\theta}=\frac{-|\alpha|r^2\sin^2\theta}{\sin\theta-|\alpha|r^2\cos\theta}, \tag{3.10}$$

so that $|dy/dx|\to\infty$ when the equation

$$\sin\theta-|\alpha|r^2\cos\theta=0 \tag{3.11}$$

is satisfied. Substituting (3.5) into (3.11) yields

$$\cot^3\theta-3\cot\theta-3/|\alpha|=0. \tag{3.12}$$

From this equation one can obtain the corresponding θ coordinate. The trajectory of the points at which $|dy/dx|$ becomes infinite is readily derived by eliminating α from (3.5) and (3.12) and it reads in the form

$$r^2=(\cot^2\theta-3)/3. \tag{3.13}$$

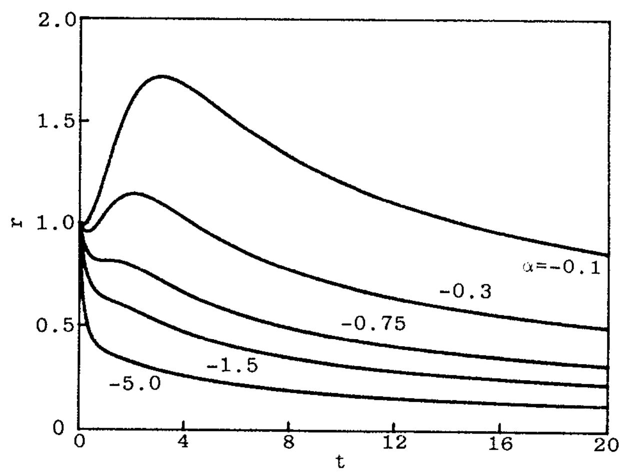

FIG. 3. The time evolution of r for negative α.

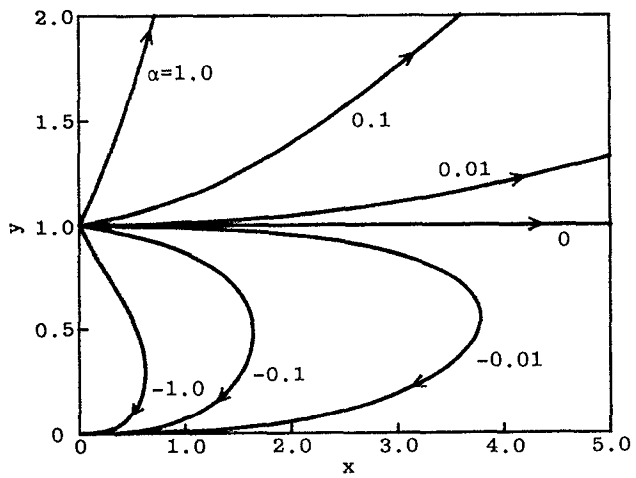

FIG. 4. The trajectories of solutions in the (x,y) plane for various values of α, where $\beta=0$.

Since $r^2>0$ the inequality $0<\theta<\pi/6$ must hold. When $\theta\sim 0$, $r^2\sim 1/(3\sin^2\theta)$. Therefore the trajectory approaches a straight line $y=1/\sqrt{3}$ as $\theta\to 0$. In Fig. 4 the trajectories of solutions are depicted in the (x,y) plane for various values of α. The arrow in the figure indicates the direction of the motion.

B. $\alpha=0$, $\beta\neq 0$

In this case Eq. (2.9) reduces to

$$\frac{d\rho}{d\theta}=2\rho(\beta\rho-1)(-\beta\rho+2)\cot\theta. \qquad (3.14)$$

Since the above equation is a separable ODE, the integration is readily performed. The result is written in the form

$$\rho(\beta\rho-2)/(\beta\rho-1)^2=c/\sin^4\theta, \qquad (3.15)$$

where c is an integration constant. Transforming back to r^2 by (2.8) and using the initial condition $r_{\theta=\pi/2}=1$, we obtain

$$r^4-(2/\beta)r^2\sin^2\theta-(\beta-2)/\beta=0, \qquad (3.16)$$

which represents the trajectories of solutions in the (r,θ) plane. Various possibilities arise according to the values of β, which we shall describe in order.

1. $\beta>2$

It follows from (3.16) that

$$r^2=[\sin^2\theta+\sqrt{\sin^4\theta+\beta(\beta-2)}]/\beta, \qquad (3.17)$$

which, combined with (2.7b), yields

$$\dot\theta=\sqrt{\sin^4\theta+\beta(\beta-2)}. \qquad (3.18)$$

Since $\dot\theta>0$, as seen from (3.18), the trajectories of solutions move counterclockwise. The time evolution of θ is now determined by the relation

$$t=\int_{\pi/2}^{\theta}\frac{d\phi}{\sqrt{\sin^4\phi+\beta(\beta-2)}}. \qquad (3.19)$$

After some calculations we find that the right-hand side of (3.19) can be reduced to an elliptic integral. The final result is expressed as follows:

$$\cot^2\theta=\frac{[\mathrm{cn}(\tau,k)+\mu\,\mathrm{sn}(\tau,k)]^2}{\eta[\mathrm{cn}(\tau,k)-\mu\,\mathrm{sn}(\tau,k)]^2}, \qquad (3.20a)$$

with

$$\tau=\gamma(t+t_0), \qquad (3.20b)$$

$$k=\sqrt{1-\mu^4}, \qquad (3.20c)$$

$$\gamma=[\sqrt{\beta(\beta-2)}/\sqrt{2}\mu]\sqrt{1+(\beta-1)/\sqrt{\beta(\beta-2)}}, \qquad (3.20d)$$

$$\mu=\sqrt{(\sqrt{2}-\sqrt{1-\eta})/(\sqrt{2}+\sqrt{1-\eta})}, \qquad (3.20e)$$

$$\eta=\sqrt{\beta(\beta-2)}/(\beta-1), \qquad (3.20f)$$

$$\mathrm{sn}(\gamma t_0,k)=-1/\sqrt{1+\mu^2}, \qquad (3.20g)$$

where $\mathrm{sn}(\tau,k)$ and $\mathrm{cn}(\tau,k)$ are Jacobi's elliptic functions and k is a modulus.[6] Hence, the trajectories of solutions are closed periodic orbits in the (x,y) plane and the period is given by $4K/\gamma$, where

$$K=\int_0^{\pi/2}\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} \qquad (3.21)$$

is the complete elliptic integral of the first kind. It should be noted that for $2<\beta\leqslant 1+\sqrt{2}$, $|dy/dx|$ becomes infinite when

$$\sin\theta=\sqrt{(-\beta^2+2\beta+1)/2}, \qquad (3.22)$$

in addition to $\theta=0$, π while for $\beta>1+\sqrt{2}$ the same situation occurs only at $\theta=0$, π.

2. $\beta=2$

In this special case (3.16) yields

$$r=\sin\theta, \qquad (3.23)$$

and (2.7b) reduces to

$$\dot\theta=\sin^2\theta. \qquad (3.24)$$

The time dependence of r and θ are readily obtained from (3.23) and (3.24) as follow:

$$r=1/\sqrt{t^2+1}, \tag{3.25}$$

$$\cot\theta=-t. \tag{3.26}$$

The trajectory is a semicircle with the radius $\frac{1}{2}$ and the center $(0,\frac{1}{2})$ in the (x,y) plane. One can observe that the trajectory approaches the origin as $t\to\infty$.

3. $0<\beta<2$

In this case it follows from (2.7) and (3.16) that

$$r^2=[\sin^2\theta\pm\sqrt{\sin^4\theta-\beta(2-\beta)}]/\beta, \tag{3.27}$$

$$\dot\theta=\pm\sqrt{\sin^4\theta-\beta(2-\beta)}, \tag{3.28}$$

where the plus(minus) sign in (3.28) corresponds to the plus(minus) sign in (3.27). These signs must be chosen appropriately to satisfy the initial condition. Since r^2 is a real quantity the solution exists only in the range

$$\theta_c\leqslant\theta\leqslant\pi-\theta_c, \tag{3.29a}$$

where

$$\theta_c=\sin^{-1}[\beta(2-\beta)]^{1/4}\quad(0<\theta_c<\pi/2). \tag{3.29b}$$

The $\dot\theta$ changes its sign at $\theta=\theta_c$ and $\theta=\pi-\theta_c$, where the corresponding value of r is given by

$$r=r_c=[(2-\beta)/\beta]^{1/4}. \tag{3.30}$$

The trajectory consists of the upper branch $r\geqslant r_c$ and the lower one $r<r_c$ and it is symmetric with respect to the y axis. The time dependence of θ is now obtained by integrating Eq. (3.28) as

$$t=\pm\int_{\theta_0}^{\theta}\frac{d\phi}{\sqrt{\sin^4\phi-\beta(2-\beta)}}, \tag{3.31}$$

where θ_0 is a constant determined by the initial condition. After some manipulations we obtain the following explicit form of $\cot^2\theta$ in terms of Jacobi's elliptic function

$$\cot^2\theta=\frac{(\epsilon^2-1)\mathrm{sn}^2[\sqrt{2/\epsilon}(\pm t+t_0),k]}{2\epsilon-(\epsilon-1)\mathrm{sn}^2[\sqrt{2/\epsilon}(\pm t+t_0),k]}, \tag{3.32a}$$

with

$$k=\sqrt{[1-\sqrt{\beta(2-\beta)}]/2}, \tag{3.32b}$$

$$\epsilon=1/\sqrt{\beta(2-\beta)}, \tag{3.32c}$$

where t_0 is a constant depending on θ_0.

As an example we consider the case where $1<\beta<2$. For the upper branch $r\geqslant r_c$ the solution reads from (3.27) and (3.32) as

$$r^2=[\sin^2\theta+\sqrt{\sin^4\theta-\beta(2-\beta)}]/\beta, \tag{3.33}$$

$$\cot^2\theta=\frac{(\epsilon^2-1)\mathrm{sn}^2(\sqrt{2/\epsilon}t,k)}{2\epsilon-(\epsilon-1)\mathrm{sn}^2(\sqrt{2/\epsilon}t,k)}. \tag{3.34}$$

For the lower branch $r<r_c$, on the other hand, it is written in the form

$$r^2=[\sin^2\theta-\sqrt{\sin^4\theta-\beta(2-\beta)}]/\beta, \tag{3.35}$$

$$\cot^2\theta=\frac{(\epsilon^2-1)\mathrm{sn}^2[\sqrt{2/\epsilon}(t+t_0),k]}{2\epsilon-(\epsilon-1)\mathrm{sn}^2[\sqrt{2/\epsilon}(t+t_0),k]}, \tag{3.36}$$

with $t_0=\sqrt{\epsilon/2}K$. Here K is the complete elliptic integral of the first kind with a modulus given by (3.32b). The trajectory is seen to be a closed path moving counterclockwise with the period $\sqrt{8\epsilon}K$. The similar results also hold for the β within the range $0<\beta<1$. The only difference is that the trajectories always lie in the region $y>1$.

4. $\beta<0$

In this final case r^2 and $\dot\theta$ are given, respectively, by

$$r^2=[-\sin^2\theta+\sqrt{\sin^4\theta+\beta(\beta-2)}]/|\beta|, \tag{3.37}$$

$$\dot\theta=-\sqrt{\sin^4\theta+\beta(\beta-2)}. \tag{3.38}$$

It turns out that the trajectories of solutions are closed paths moving clockwise because the sign of $\dot\theta$ is always negative by (3.38). One notes that $|dy/dx|$ becomes infinite only at $\theta=0$, π and for these values of θ, $r=(1+2/|\beta|)^{1/4}>1$. Therefore the existence region of the trajectories is outside a circle with the radius 1 and the center being the origin. The time dependence of θ is simply derived from (3.20) by replacing t by $-t$. The typical trajectories for the cases 1–4 mentioned above are depicted in Fig. 5.

IV. GENERAL CASE

A. Stationary solutions and their stability

In this section we shall seek stationary solutions of Eq. (1.1) and study their linear stability. It will be seen in B that they are closely related to the asymptotic values of the general solutions for large time. The stationary solution is simply obtained by setting $\dot r=\dot\theta=0$ in Eq. (2.7). The resultant solution reads in the form

$$r_s=\sqrt{\beta/(\alpha^2+\beta^2)}, \tag{4.1a}$$

$$\tan\theta_s=-\beta/\alpha. \tag{4.1b}$$

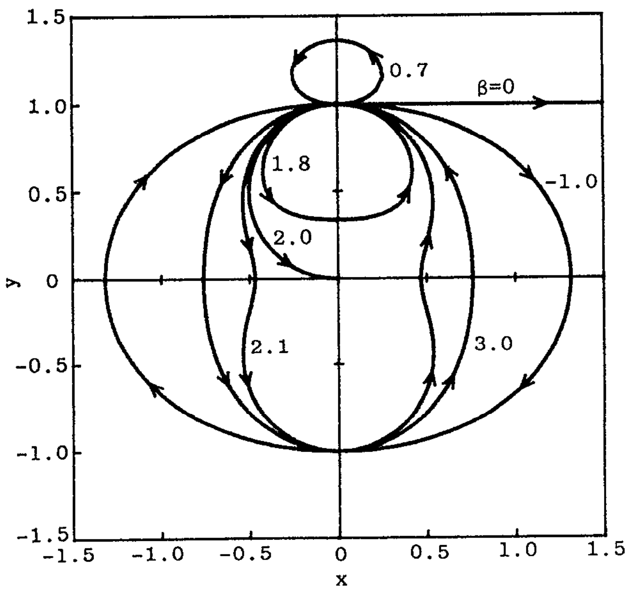

FIG. 5. The trajectories of solutions in the (x,y) plane for various values of β, where $\alpha=0$.

Obviously β must be positive to yield real r_s.

Now we investigate the linear stability of the solution (4.1). To do so, we put

$$r=r_s+\delta r, \tag{4.2a}$$

$$\theta=\theta_s+\delta\theta, \tag{4.2b}$$

and linearize Eq. (2.7) around r_s and θ_s. It then turns out that the following system of linear ODEs for δr and $\delta\theta$ arises:

$$\delta\dot{r}=(\tfrac{1}{2}\sin 2\theta_s+3\alpha r_s^2)\delta r+r_s\cos 2\theta_s\,\delta\theta, \tag{4.3a}$$

$$\delta\dot{\theta}=2\beta r_s\,\delta r-\sin 2\theta_s\,\delta\theta. \tag{4.3b}$$

Assuming that $\delta r\propto e^{\omega t}$, $\delta\theta\propto e^{\omega t}$, we obtain from (4.1) and (4.3) the eigenvalue equation for ω as follows:

$$\omega^2-[4\alpha\beta/(\alpha^2+\beta^2)]\omega+2\beta^2/(\alpha^2+\beta^2)=0. \tag{4.4}$$

The roots of Eq. (4.4) are given by

$$\omega=2\alpha\beta/(\alpha^2+\beta^2)\pm[\beta/(\alpha^2+\beta^2)]\sqrt{2(\alpha^2-\beta^2)}. \tag{4.5}$$

If $\alpha>0$, then $\operatorname{Re}\omega>0$. This means that the solution becomes unstable for infinitesimal disturbance. On the other hand, if $\alpha<0$, then $\operatorname{Re}\omega<0$ irrespective of the magnitude of $|\alpha|$ and the solution is stable. We note that in another stationary solution $r=0$, $\theta=0 (\mathrm{mod}\ \pi)$ exists. In this case it follows from (2.7) that if $r=0$ and $\theta=0 (\mathrm{mod}\ \pi)$ hold, then $d^n r/dt^n=d^n\theta/dt^n=0 (n\geqslant 1)$. In other words, when we start from $x=y=0$ at $t=0$, the solution remains at the origin forever.

B. General solutions for $\alpha\beta\neq 0$

We now consider the general case, namely $\alpha\neq 0$ and $\beta\neq 0$. In this case, however, we have not succeeded in integrating Eq. (2.9) in contrast to the two special cases described in Sec. III; it does not reduce to any of special ODEs where specific solution methods are applicable.[1–3] Hence, we have employed numerical integrations. Before entering into the detail, we shall investigate the generic properties of solutions. For this purpose, we use Eq. (2.7). If we put $u=1/r^2$, it is transformed into the system of ODEs,

$$\dot{u}=-u\sin 2\theta-2\alpha, \tag{4.6a}$$

$$\dot{\theta}=-\sin^2\theta+\beta/u. \tag{4.6b}$$

Since Eq. (4.6a) is a linear equation for u, it can be immediately integrated to yield the expression

$$r^2=\frac{1}{u}=\frac{f(t)}{1-2\alpha\int_0^t f(t')dt'}, \tag{4.7a}$$

where

$$f(t)=\exp\left(\int_0^t \sin 2\theta(t')dt'\right). \tag{4.7b}$$

If we define the function $F(t)$ by

$$F(t)=\int_0^t f(t')dt', \tag{4.8}$$

then $\dot{F}(t)=f(t)>0$ and $F(0)=0$. This implies that $F(t)$ is positive and a monotonically increasing function of t for $t>0$. It readily follows from this fact that $\lim_{t\to\infty}F(t)\to\infty$. Hence, there always exists a finite time t_b satisfying the relation

$$F(t_b)=1/2\alpha, \tag{4.9}$$

provided that $\alpha>0$. At this time the solution blows up as proved from (4.7). On the other hand, for $\alpha<0$, one can easily show that r remains a finite value for all time. It is worthwhile to remark that the above facts are generic properties of solutions independent of initial conditions. In the general case considered here the four combinations are possible according to the sign of α and β; (1) $\alpha>0$, $\beta>0$, (2) $\alpha>0$, $\beta<0$, (3) $\alpha<0$, $\beta>0$, and (4) $\alpha<0$, $\beta<0$. We now investigate these cases separately.

1. $\alpha>0$, $\beta>0$

As already noted, the solution always blows up for $\alpha>0$. To obtain an insight about the characteristic of the blowup time, we first consider the situation $\beta\gg 1$. In this case an inequality $\beta/u\gg 1$ holds and Eq. (4.6b) may be approximated by the equation

$$\dot{\theta}=\beta/u. \tag{4.10}$$

If we put $\psi=\dot{\theta}$, then (4.6a) and (4.10) yield the linear ODE for ψ as follows:

$$\frac{d\psi}{d\theta}=\frac{2\alpha}{\beta}\psi+\sin 2\theta. \tag{4.11}$$

Integrating Eq. (4.11) under the initial condition $\psi_{\theta=\pi/2}\sim(\beta/u)_{\theta=\pi/2}=\beta$, we obtain

$$r^2=[1-\beta/2(\alpha^2+\beta^2)]\exp[(2\alpha/\beta)(\theta-\pi/2)] - [1/2(\alpha^2+\beta^2)](\alpha\sin 2\theta+\beta\cos 2\theta). \tag{4.12}$$

Furthermore, if we assume $\theta-\pi/2\gg\beta/2\alpha$, which corresponds to the final stage of the time evolution of the solution, the second term on the right-hand side of (4.12) is negligible when it is compared with the first one, so that (4.12) can be approximated by the equation

$$r^2=\left(1-\frac{\beta}{2(\alpha^2+\beta^2)}\right)\exp\left[\frac{2\alpha}{\beta}\left(\theta-\frac{\pi}{2}\right)\right]. \tag{4.13}$$

Therefore

$$\dot{\theta}\sim\beta r^2=\tilde{\beta}\exp[(2\alpha/\beta)(\theta-\pi/2)], \tag{4.14a}$$

with

$$\tilde{\beta}=\beta[1-\beta/2(\alpha^2+\beta^2)]. \tag{4.14b}$$

Integration of Eq. (4.14) is readily done and yields the result

$$\theta=\pi/2-(\beta/2\alpha)\ln(1-2\tilde{\alpha}t), \tag{4.15a}$$

with

$$\tilde{\alpha}=\alpha[1-\beta/2(\alpha^2+\beta^2)], \tag{4.15b}$$

where the initial condition $\theta_{t=0}=\pi/2$ has been used. Substituting (4.15) into (4.13), we obtain the explicit time dependence of r as

$$r=\sqrt{\tilde{\beta}/\beta}/\sqrt{1-2\tilde{\alpha}t}. \tag{4.16}$$

Hence, the blowup time t_b is found to be as follows:

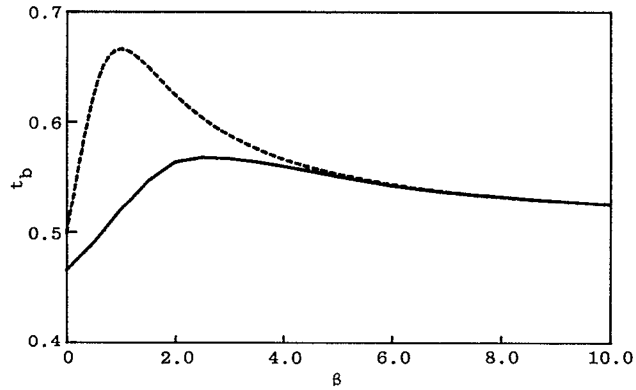

FIG. 6. The blowup time t_b as a function of β. Solid line: numerical values; broken line: analytical expression (4.17).

$$t_b=1/\{2\alpha[1-\beta/2(\alpha^2+\beta^2)]\}. \tag{4.17}$$

We have performed the numerical integration of Eq. (1.2) by using the Runge–Kutta–Gill method. Figure 6 illustrates the β dependence of t_b, where α has been taken to be 1. The solid line in the figure plots the numerical values while the broken line shows the analytical expression (4.17). The coincidence of both curves is excellent for $\beta\geqslant 5$.

For small β, t_b may be approximated by the expression (3.9). Indeed (3.9) with $\alpha=1$ gives $t_b = 2^{1/3} - 0.5^{1/3}=0.466$ while the numerical value for $\beta=0.01$ is 0.467. The t_b has a maximum value 0.568 when $\beta=2.5$. Figure 7 depicts the trajectories of solutions in the (x,y) plane for various values of β. For all cases we have taken

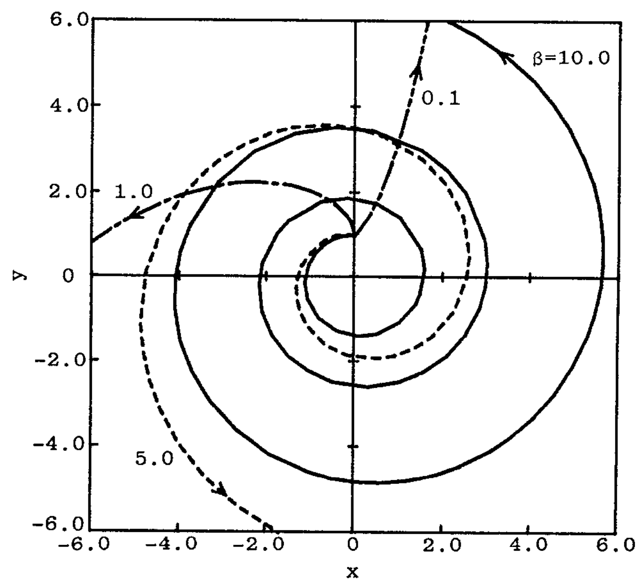

FIG. 7. The trajectories of solutions in the (x,y) plane for various values of positive β, where $\alpha=1$.

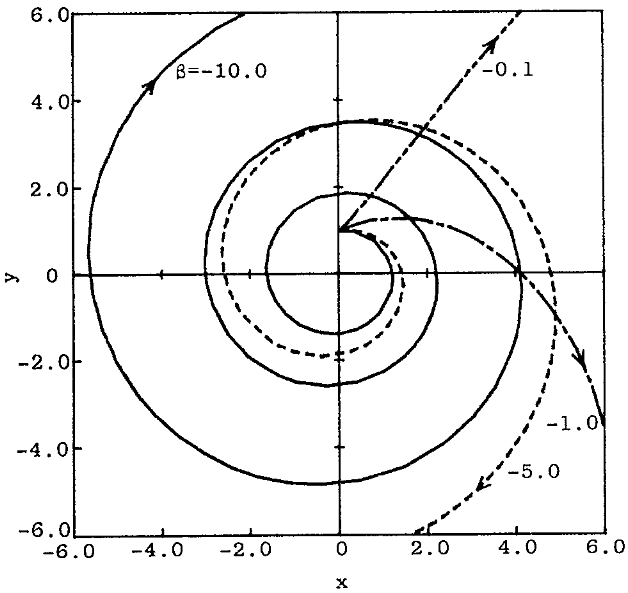

FIG. 8. The trajectories of solutions in the (x,y) plane for various values of negative β, where $\alpha = 1$.

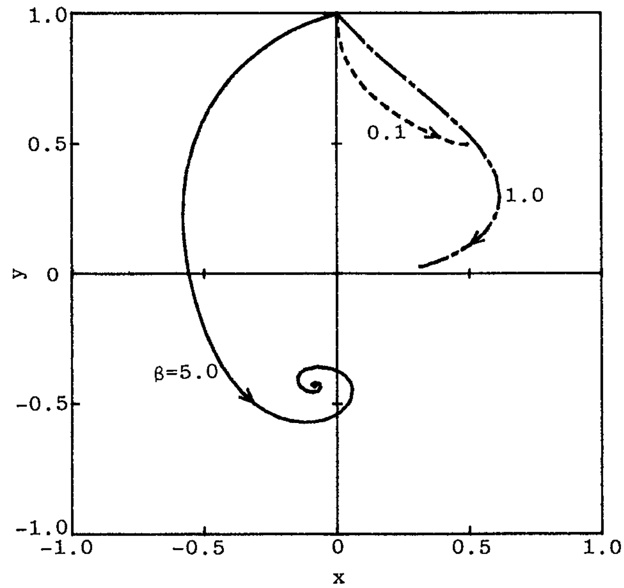

FIG. 9. The trajectories of solutions in the (x,y) plane for three typical values of positive β, where $\alpha = -1$.

$\alpha = 1$. The arrow in the figure indicates the direction of the motion. The trajectories exhibit spiral motions around the origin and they tend to infinity as the time approaches the blowup time t_b. It is interesting to note that the numerical results are well simulated by Eq. (4.13) for $\beta \gg 5$, even in the range $\theta - \pi/2 \gg \beta/2\alpha$.

2. $\alpha > 0$, $\beta < 0$

In this case the qualitative features of solutions are similar to those for the case 1; the behaviors in the (x,y) plane show divergent spiral motions. The only difference is that the direction of the motion is always clockwise irrespective of the magnitude of $|\beta|$. This fact is a direct consequence of Eq. (4.6b) with negative β, namely $\dot{\theta} < 0$. On the other hand, in the case 1, one can observe that the trajectory with small β moves clockwise in the initial stage of the time evolution and then changes its direction counterclockwise in the later time (see the trajectory for $\beta = 0.1$ in Fig. 7). Figure 8 shows the trajectories of solutions for various values of β with $\alpha = 1$. For $|\beta| \gg 5$ it is seen that these are well approximated by Eq. (4.13) when $|\theta - \pi/2| \gg |\beta|/2\alpha$. The blowup time as a function of β has a similar profile as that shown in Fig. 6 and hence it is not presented here. We only remark that the coincidence with an analytical expression (4.17) is excellent for $|\beta| \gg 5$.

3. $\alpha < 0$, $\beta > 0$

For negative α the denominator on the right-hand side of (4.7a) diverges when the time goes to infinity, as already noted in a sentence following (4.8). Therefore the two possibilities arise according to the limiting value of $f(t)$: (i) $\lim_{t\to\infty} f(t) \to \infty$, (ii) $\lim_{t\to\infty} f(t) \to$ finite value. In case (i) it follows from (4.7) that r^2 tends to finite value while in case (ii) it approaches zero. The case considered here corresponds to case (i) since the stationary solution exists only for positive β, as is evident from (4.1). Figure 9 represents the trajectories of solutions for typical values of β where α is taken to be -1. It is important to observe that each trajectory approaches the corresponding stationary point (4.1). As seen from (4.5) with $\alpha = -1$, the eigenvalue is real and negative in the range $0 < \beta \leqslant 1$. The corresponding stationary point then represents a stable node.[7] For $\beta > 1$, on the other hand, it is a stable focus since the eigenvalue appears as complex conjugate with negative real part.[7] In the latter case the trajectory exhibits a spiral motion around the stationary point (see the trajectory with $\beta = 5.0$ in Fig. 9).

4. $\alpha < 0$, $\beta < 0$

This corresponds to case (ii). Since in this case there exist no stationary solutions except for the origin, all trajectories approach the origin as $t \to \infty$. Figure 10 illustrates the situation where only two examples are exhibited.

V. CONCLUDING REMARKS

In this paper we investigated the nonlinear dynamical system described by Eq. (1.1) and showed that it is associated with Abel's nonlinear ODE of the first kind. The behaviors of solutions are found to depend critically on a

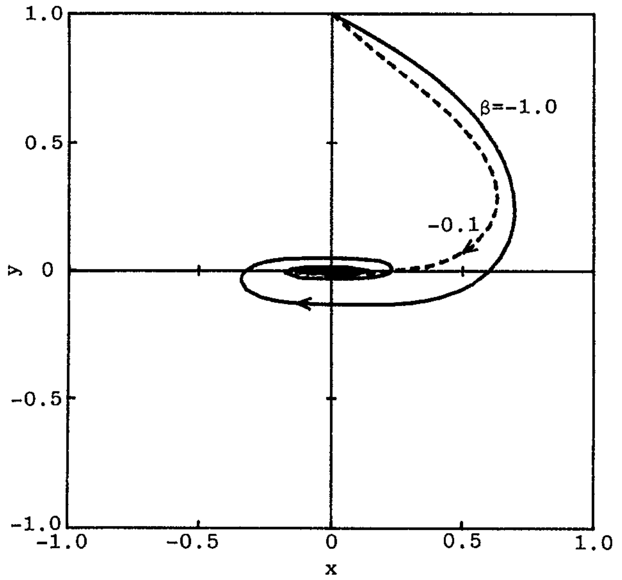

FIG. 10. The trajectories of solutions in the (x,y) plane for two typical values of negative β, where $\alpha = -1$.

complex parameter λ. The characteristics of the bifurcation of the solutions were studied in detail with the aid of the numerical analysis.

One can generalize Eq. (1.1) by incorporating a more generic nonlinear term as

$$\dot{z}=\text{Im}\, z + \lambda g(|z|^2)z, \tag{5.1}$$

where $g=g(\zeta)$ is an arbitrary real function of ζ. A simple choice of g is, for instance, $g = \zeta^\nu$ (ν: real constant). Introduction of a variable ρ by the relation

$$\rho=r^{2\nu}/(\beta r^{2\nu} - \sin^2\theta) \tag{5.2}$$

enables us to recast (5.1) into the following nonlinear ODE:

$$\frac{d\rho}{d\theta}=2\rho(\beta\rho-1)[-\nu(\alpha+\beta\cot\theta)\rho+(\nu+1)\cot\theta]. \tag{5.3}$$

Obviously for $\nu=1$, Eq. (5.3) reduces to Eq. (2.9). Since Eq. (5.3) is a special class of Abel's nonlinear ODE of the first kind, it cannot be integrated in general by quadratures. However, the two exceptional cases arise when $\nu = -1, 0$. In the case $\nu = -1$ Eq. (5.3) reduces to an ODE of the separable type,

$$\frac{d\rho}{d\theta}=2\rho^2(\beta\rho-1)(\alpha+\beta\cot\theta), \tag{5.4}$$

and integration of which yields an explicit solution

$$(\beta-\rho^{-1})e^{(\beta\rho)^{-1}}=ce^{2\alpha\theta/\beta}\sin^2\theta, \tag{5.5}$$

where c is an integration constant. The expression (5.5) can be rewritten in terms of the variables r and θ in the form

$$r^2 e^{\beta^{-1}(\beta - r^2\sin^2\theta)}=ce^{2\alpha\theta/\beta}, \tag{5.6}$$

which represents the trajectory of solution in the (r,θ) plane. In the case $\nu=0$ Eq. (5.3) is satisfied identically by (5.2). Equation (5.1) then becomes linear with respect to z and it can be trivially integrated.

Another generalization of special importance is to introduce spatial structure in z; $z=z(\xi,t)$, with ξ being a spatial variable. However, since Eq. (1.1) does not include a spatial derivative, ξ may be regarded as a parameter. The initial value problem for Eq. (1.1) is then solved by specifying the initial value $z(\xi,0)$ at $t=0$. It is quite interesting to note that Eq. (1.1) with pure imaginary λ satisfies the following bilinear equation for z:

$$(z^*z)_t=\text{Im}\, z^2. \tag{5.7}$$

Here the subscript t denotes partial differentiation with respect to t and * means complex conjugate. The above equation has already stemmed in the bilinearization of the sine-Hilbert equation,

$$H\theta_t=-\sin\theta, \quad \theta=\theta(\xi,t), \tag{5.8}$$

where H is the Hilbert transform operator.[8–10] Indeed, if we introduce a dependent variable transformation of the form

$$\theta(\xi,t)=i\ln[z^*(\xi,t)/z(\xi,t)], \tag{5.9}$$

where $z^*(z)$ is an analytic function with zeros in the lower (upper)-half ξ plane, then Eq. (5.8) is transformed into Eq. (5.7) due to the relation $H\theta_t = -(\ln z^*z)_t$.[8–10] Thus there arises a possibility that the solutions of Eq. (1.1) also solve Eq. (5.8) if the analytic condition imposed on z is fulfilled. This problem will be dealt with in the future work.

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