AIP Journal of Mathematical Physics

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Citation: Journal of Mathematical Physics **32**, 120 (1991); doi: 10.1063/1.529134
View online: http://dx.doi.org/10.1063/1.529134
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/32/1?ver=pdfcov
Published by the AIP Publishing

Linearization of novel nonlinear diffusion equations with the Hilbert kernel and their exact solutions

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(Received 28 November 1989; accepted for publication 5 September 1990)

Two novel nonlinear diffusion equations with the Hilbert kernel are proposed. The equations can be linearized by introducing appropriate dependent variable transformations. The initial value problems for the proposed equations are then solved exactly through the linearization and explicit nonperiodic and periodic solutions are constructed. The properties of solutions are investigated in detail. It is found that the blow up of solutions occurs at a finite time for both the nonperiodic and periodic cases due to the breakdown of certain analytic conditions imposed on the dependent variables.

I. INTRODUCTION

The study of soliton equations has made remarkable progress owing to the development of various exact methods for solving nonlinear evolution equations (NEEs). In particular, the bilinear transformation method (BTM)[1-3] has enabled us to analyze some classes of nonlinear *integrodifferential* evolution equations (NIDEEs). The essence of the BTM is to investigate bilinear equations that are obtained by means of appropriate dependent variable transformations of given NEEs. The first success of the method when applied to NIDEE is the case of the Benjamin–Ono (BO) equation.[3,4] The mathematical structure of the BTM has now been clarified considerably by the Sato theory.[5-7]

Recently, the BTM was applied to the sine-Hilbert (sH) equation[8-11]

$$H\theta_t = -\sin\theta, \quad \theta = \theta(x,t), \tag{1.1}$$

and the NIDEE of the form[12]

$$\theta_t = -\mu H\,\mathrm{Re}(e^{-i\theta}He^{i\theta}), \quad \theta = \theta(x,t), \tag{1.2}$$

where the operator H is the Hilbert transform defined by

$$H\theta(x,t) = \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{\theta(y,t)}{y-x}\,dy, \tag{1.3}$$

and Re denotes the real part. However, in a striking contrast to the BO equation, the above two equations are reducible to linear equations by introducing appropriate dependent variable transformations and hence the initial value problems for the equations can be solved exactly. A few classes of linealizable NIDEEs are also known at present.[13-16] In this paper, we propose new examples of NIDEEs that can be transformed into linear equations.

The basic equations that we present here are the following two nonlinear diffusion equations with the Hilbert kernel:

$$u_t - \nu u_{xx} + \nu(uHu)_x - \mu H(\sin\theta)_x = 0, \tag{1.4}$$

$$u_t - \nu u_{xx} + \nu(uHu)_x + \mu H[\mathrm{Re}(e^{-i\theta}He^{i\theta})]_x = 0, \tag{1.5}$$

where

$$\theta(x,t) = \int^x u(x',t)dx', \tag{1.6}$$

and μ and ν are positive constants. In the limit of $\nu\to 0$, Eqs. (1.4) and (1.5) are reduced, after integrating once with respect to x, to Eqs. (1.1) and (1.2), respectively. On the other hand, in the limit of $\mu\to 0$, these are reduced to a nonlinear diffusion equation that has already been studied by Satsuma.[13]

The outline of this paper is as follows. In Sec. II, the nonperiodic solutions of Eqs. (1.4) and (1.5) are constructed through the linearization and the properties of the solutions are investigated in detail. It is found that when $\nu\neq 0$, the solutions blow up at a finite time. The characteristics of the solutions are a striking contrast to those of the case, $\nu = 0$ where the blow up occurs only at an infinite time.[8-12] In Sec. III, the periodic case is considered in the same way. Section IV is devoted to concluding remarks.

II. THE NONPERIODIC CASE

A. Linearization

First, let us introduce the following dependent variable transformation

$$u = i\frac{\partial}{\partial x}\ln\frac{f^*}{f}, \tag{2.1a}$$

with

$$f = \prod_{j=1}^{N}[x - x_j(t)], \tag{2.1b}$$

where $x_j(t)$ $(j = 1,2,\ldots,N)$ are complex functions of t whose imaginary parts are all positive and $*$ denotes the complex conjugate. Substituting (2.1) into (1.4) and (1.5), integrating once with respect to x and taking an integration constant to be zero, both equations are reduced to the following *bilinear* equation for f and f^*:

$$i(f^*_t f - f_t f^*) - i\nu(f^*_{xx}f - f_{xx}f^*) - (\mu/2)(f^2 + f^{*2} - 2f^*f) = 0. \tag{2.2}$$

Here, we have used the formulas

$$H(f/f^* - 1) = i(f/f^* - 1), \tag{2.3a}$$

$$H(f^*/f - 1) = -i(f^*/f - 1), \tag{2.3b}$$

which stem from the fact that $f/f^* - 1$ $(f^*/f - 1)$ is an analytic function in the upper(lower)-half plane and vanishes at infinity. Furthermore, Eq. (2.2) is modified in the form

$$f^*[f_t - \nu f_{xx} - (\mu/2i)(f - f^*)] = f[f^*_t - \nu f^*_{xx} + (\mu/2i)(f^* - f)]. \tag{2.4}$$

Hence, one sees that Eq. (2.2) is satisfied identically provided that the following *linear* equation for f holds:

$$f_t - \nu f_{xx} - (\mu/2i)(f - f^*) = 0. \tag{2.5}$$

This expression is nothing but a linearization of Eqs. (1.4) and (1.5).

When $\mu = 0$, Eq. (2.5) is reduced to the well-known linear diffusion equation while when $\nu = 0$, it coincides with a linearized equation for Eqs. (1.1) and (1.2).[8-12] It should be noted that a term of the form, κf(κ:a real constant) may be added on the right-hand side of Eq. (2.5). However, one can easily show by investigating the asymptotic behavior of Eq. (2.5) for large x that the κ must be taken to be zero. The situation is different from the periodic case treated in Sec. III where the nonzero κ will be maintained.

Now, in order to derive the time evolution of $x_n(t)$ $(n = 1,2,...,N)$, we substitute (2.1b) into (2.5) and then take the coefficients of $(x - x_n)^{-1}$ to be zero. The resultant expression has the following form:

$$\dot{x}_n = \frac{\mu}{2i} \prod_{j=1}^{N} (x_n - x^*_j) \left(\prod_{\substack{j=1 \\ (j \neq n)}}^{N} (x_n - x_j) \right)^{-1} - 2\nu \sum_{\substack{j=1 \\ (j \neq n)}}^{N} \frac{1}{x_n - x_j} \quad (n = 1,2,...,N), \tag{2.6}$$

where a dot appended to x_n denotes the time differentiation. Thus the problem under consideration is seen to be equivalent to the finite-dimensional dynamical system whose motion is governed by Eq. (2.6). It follows from (2.6) that the time evolution of the imaginary part of x_n, namely, $\mathrm{Im}\, x_n$ is described by the equation

$$\mathrm{Im}\, \dot{x}_n = \left[\mu \left(\mathrm{Im} \prod_{\substack{j=1 \\ (j \neq n)}}^{N} \frac{x_n - x^*_j}{x_n - x_j} \right) + 2\nu \sum_{\substack{j=1 \\ (j \neq n)}}^{N} \frac{1}{|x_n - x_j|^2} \right] \mathrm{Im}\, x_n - 2\nu \sum_{\substack{j=1 \\ (j \neq n)}}^{N} \frac{\mathrm{Im}\, x_j}{|x_n - x_j|^2} \quad (n = 1,2,...,N). \tag{2.7}$$

Because of the presence of the second term on the right-hand side of Eq. (2.7), the condition, $\mathrm{Im}\, x_n(t) > 0$ that has been assumed in deriving Eq. (2.2) will be broken down at a finite time even if it holds at an initial time. This fact will result in the blow up of solutions of Eqs. (1.4) and (1.5) as shown explicitly in subsection C. In the case of $\nu = 0$, this condition is satisfied for all finite time if it holds at an initial time.[9]

B. Exact solutions

In this subsection, we shall solve Eq. (2.5). To do so, we expand f as

$$f = \sum_{j=0}^{N} (-1)^j s_j x^{N-j}, \tag{2.8a}$$

where s_j are elementary symmetric functions of $x_1, x_2, ..., x_N$:

$$s_0 = 1, \quad s_1 = \sum_{j=1}^{N} x_j,$$

$$s_2 = \sum_{\substack{j,k=1 \\ (j<k)}}^{N} x_j x_k, ..., s_N = \prod_{j=1}^{N} x_j, \tag{2.8b}$$

substitute (2.8) into (2.5) and then take the coefficients of x^{N-j} $(j = 1,2,...,N)$ to be zero. The time evolution of s_j thus obtained is now written in the form

$$s_0 = 1, \tag{2.9a}$$

$$s_1 = \mu a_1 t + b_1 + i a_1, \tag{2.9b}$$

$$\dot{s}_j = \mu\, \mathrm{Im}\, s_j + \nu(N + 2 - j)(N + 1 - j) s_{j-2} \quad (j = 2,3,...,N), \tag{2.9c}$$

where $a_1 > 0$ and b_1 is a real constant. The system of linear differential equations (2.9c) can be solved successively starting from (2.9a) and (2.9b).

Indeed, it follows from the real and imaginary parts of Eq. (2.9c) that

$$\mathrm{Re}\, \dot{s}_j = \mu\, \mathrm{Im}\, s_j + \nu(N + 2 - j)(N + 1 - j) \mathrm{Re}\, s_{j-2}, \tag{2.10a}$$

$$\mathrm{Im}\, \dot{s}_j = \nu(N + 2 - j)(N + 1 - j) \mathrm{Im}\, s_{j-2}. \tag{2.10b}$$

Expanding $\mathrm{Re}\, s_j$ and $\mathrm{Im}\, s_j$ in powers of νt as

$$\mathrm{Re}\, s_j = \sum_{s=0}^{[(j+1)/2]} c_s^{(j)} (\nu t)^{[(j+1)/2] - s}, \tag{2.11a}$$

$$\mathrm{Im}\, s_j = \sum_{s=0}^{[(j-1)/2]} d_j^{(s)} (\nu t)^{[(j-1)/2] - s}, \tag{2.11b}$$

and substituting these expressions into (2.10), we obtain the recursion relations for the coefficients $c_s^{(j)}$ and $d_s^{(j)}$ as follows:

$$c_s^{(j)} = \frac{(N + 2 - j)(N + 1 - j)}{[(j+1)/2 - s]} c_s^{(j-2)} + \frac{\mu\nu^{-1}}{[(j+1)/2 - s]} d_s^{(j)} \quad (s = 0,1,...,[(j-1)/2]), \tag{2.12a}$$

$$c^{(j)}_{[(j+1)/2]} = b_j, \tag{2.12b}$$

$$d_s^{(j)} = \frac{(N + 2 - j)(N + 1 - j)}{[(j-1)/2 - s]} d_s^{(j-2)} \quad (s = 0,1,...,[(j-3)/2]), \tag{2.12c}$$

$$d^{(j)}_{[(j-1)/2]} = a_j, \tag{2.12d}$$

where a_j and b_j are real constants and the notation $[(j+1)/2]$ means the maximum integer not exceeding $(j+1)/2$. The solutions for the recursion relations (2.12) are given explicitly in the forms:

For even N

$$\mathrm{Re}\, s_{2n} = \sum_{s=0}^{n} c_s^{(2n)} (\nu t)^{n-s} \quad (n = 1,2,...,N/2), \tag{2.13a}$$

$$\mathrm{Im}\, s_{2n} = \sum_{s=0}^{n-1} d_s^{(2n)} (\nu t)^{n-s-1} \quad (n = 1,2,...,N/2), \tag{2.13b}$$

and for odd N

$$\mathrm{Re}\, s_{2n+1} = \sum_{s=0}^{n+1} c_s^{(2n+1)} (\nu t)^{n-s+1} \quad [n = 1,2,...,(N-1)/2], \tag{2.13c}$$

$$\mathrm{Im}\, s_{2n+1} = \sum_{s=0}^{n} d_s^{(2n+1)} (\nu t)^{n-s} \quad [n = 1,2,...,(N-1)/2]. \tag{2.13d}$$

Here, various coefficients in (2.13) are defined by the following relations:

$$c_s^{(2n)} = \tilde{d}_{s-1}^{(2n)} b_{2s} + \mu\nu^{-1} a_{2(s+1)} \sum_{m=0}^{n-s-2} \frac{\tilde{d}_s^{(2n-2m-2)}}{n-m-s-1} \times \prod_{l=0}^{m} \frac{(N-2n+2l+2)(N-2n+2l+1)}{n-s-l} + \frac{\mu\nu^{-1}\tilde{d}_s^{(2n)}}{n-s} a_{2(s+1)} \quad (s = 0,1,...,n-2), \tag{2.14a}$$

$$c_{n-1}^{(2n)} = (N-2n+1)(N-2n+2) b_{2(n-1)} + \mu\nu^{-1} a_{2n}, \tag{2.14b}$$

$$c_s^{(2n+1)} = \tilde{d}_{s-1}^{(2n+1)} b_{2s+1} + \mu\nu^{-1} a_{2s+1} \sum_{m=0}^{n-s-1} \frac{\tilde{d}_s^{(2n-2m-1)}}{n-m-s} \times \prod_{l=0}^{m} \frac{(N-2n+2l+1)(N-2n+2l)}{n-s-l+1} + \frac{\mu\nu^{-1}\tilde{d}_s^{(2n+1)}}{n-s+1} a_{2s+1} \quad (s = 0,1,...,n-1), \tag{2.14c}$$

$$c_n^{(2n+1)} = (N-2n)(N-2n+1) b_{2n-1} + \mu\nu^{-1} a_{2n+1}, \tag{2.14d}$$

$$d_s^{(2n)} = \tilde{d}_s^{(2n)} a_{2(s+1)}, \quad (s = 0,1,...,n-2), \tag{2.14e}$$

$$d_{n-1}^{(2n)} = a_{2n}, \tag{2.14f}$$

$$d_s^{(2n+1)} = \tilde{d}_s^{(2n+1)} a_{2s+1}, \quad (s = 0,1,...,n-1), \tag{2.14g}$$

$$d_n^{(2n+1)} = a_{2n+1}, \tag{2.14h}$$

$$\tilde{d}_s^{(2n)} = \prod_{m=1}^{n-s-1} \frac{(N-2n+2m)(N-2n+2m+1)}{n-s-m} \quad (s = 0,1,...,n-2), \tag{2.14i}$$

$$\tilde{d}_s^{(2n+1)} = \prod_{m=0}^{n-s-1} \frac{(N-2n+2m+1)(N-2n+2m)}{n-s-m} \quad (s = 0,1,...,n-1), \tag{2.14j}$$

$$\tilde{d}_{-1}^{(2n)} = \tilde{d}_{-1}^{(2n+1)} = 0, \tag{2.14k}$$

$$b_0 = 1. \tag{2.14l}$$

Explicitly, the first few of s_j read in the forms:

$$s_2 = [\mu a_2 + \nu N(N-1)]t + b_2 + i a_2, \tag{2.15}$$

$$s_3 = \mu\nu(N-1)(N-2)a_1 t^2 + [\mu a_3 + \nu N(N-1)(N-2)b_1]t + b_3 + i[\nu(N-1)(N-2)a_1 t + a_3], \tag{2.16}$$

$$s_4 = [\mu\nu(N-2)(N-3)a_2 + \tfrac{1}{2}\nu^2 N(N-1)(N-2)(N-3)]t^2 + [\mu a_4 + \nu(N-2)(N-3)b_2]t + b_4 + i[\nu(N-2)(N-3)a_2 t + a_4]. \tag{2.17}$$

In the limit of $\nu \to 0$, s_j has a very simple expression as

$$s_j = \mu a_j t + b_j + i a_j \quad (j = 1,2,...,N). \tag{2.18}$$

The above solution may be derived directly from Eq. (2.9c) with $\nu = 0$. It should be remarked, however, that various restrictions must be imposed on real constants, a_j and $b_j (j = 1,2,...,N)$ to satisfy the conditions, $\mathrm{Im}\, x_j > 0 (j = 1,2,...,N)$.

C. Properties of solutions

In this subsection, we shall investigate the properties of solutions. Instead of entering into the discussion on solutions for general N, we consider the first two cases, namely, $N = 1$ and $N = 2$ in detail.

1. N=1

It follows from (2.8) and (2.9b) that

$$f = x - (\mu a_1 t + b_1 + i a_1) \quad (a_1 > 0), \tag{2.19}$$

and in this case f does not depend on ν. Then (2.1) yields a solution of the Lorentzian profile

$$u = \frac{2a_1}{(x - \mu a_1 t - b_1)^2 + a_1^2}, \tag{2.20}$$

which represents a pulse moving with a constant velocity. One can observe that the propagation velocity of the pulse is inversely proportional to the amplitude and hence a tall pulse propagates more slowly than a small pulse unlike the behavior of the usual soliton that propagates with a velocity proportional to its amplitude.

2. N=2

One finds from (2.8), (2.9a), (2.9b), and (2.15) that

$$f = x^2 - (\mu a_1 t + b_1 + i a_1)x + (\mu a_2 + 2\nu)t + b_2 + i a_2. \tag{2.21}$$

The time evolutions of x_1 and x_2 are obtained by solving the algebraic equation, $f = 0$ and they are expressed as follows:

$$x_{1,2} = \frac{1}{2}\left\{A \pm \mathrm{sgn}(D)\left(\frac{C + \sqrt{C^2 + D^2}}{2}\right)^{1/2} + i\left[B \pm \left(\frac{-C + \sqrt{C^2 + D^2}}{2}\right)^{1/2}\right]\right\}, \tag{2.22}$$

where the upper(lower) sign corresponds to x_1 (x_2) and A, B, C, and D are defined, respectively, by

$$A = \mu a_1 t + b_1, \tag{2.23a}$$

$$B = a_1, \tag{2.23b}$$

$$C = (\mu a_1 t)^2 + 2[\mu(a_1 b_1 - 2a_2) - 4\nu]t - a_1^2 - 4b_2 + b_1^2, \tag{2.23c}$$

$$D = 2\mu a_1^2 t + 2a_1 b_1 - 4a_2. \quad (2.23d)$$

It is easy to show that the conditions, $\mathrm{Im}\, x_j > 0 (j = 1,2)$ are equivalent to the following ones:

$$a_1 > 0, \quad (2.24a)$$

$$2\nu a_1^2 t < a_1 b_1 a_2 - a_2^2 - a_1^2 b_2. \quad (2.24b)$$

In the present case, one can observe that the condition, $\mathrm{Im}\, x_2 > 0$ breaks down at $t = t_c$ with

$$t_c = (a_1 b_1 a_2 - a_2^2 - a_1^2 b_2)/2\nu a_1^2, \quad (2.25)$$

where, of course, the inequality, $a_1 b_1 a_2 - a_2^2 - a_1^2 b_2 > 0$ must be assumed in order to yield a positive t_c. The breakdown of the conditions (2.24) also implies the blow up of the solution of Eqs. (1.4) and (1.5). To show this, we first examine the behavior of x_1 and x_2 near $t = t_c$. One finds from (2.22) that

$$x_1 \sim \mu a_1 t + b_1 - a_2/a_1 + i a_1, \quad (2.26a)$$

$$x_2 \sim a_2/a_1 + i(2\nu a_1/\sqrt{C^2 + D^2})|_{t=t_c} \quad (t_c - t), \quad (2.26b)$$

Then, the solution u given by (2.1) takes the form

$$u = \sum_{j=1}^{2} \frac{2\, \mathrm{Im}\, x_j}{(x - \mathrm{Re}\, x_j)^2 + (\mathrm{Im}\, x_j)^2}$$

$$\sim \frac{2a_1}{(x - \mu a_1 t - b_1 + a_2/a_1)^2 + a_1^2} + 2\pi\delta\left(x - \frac{a_2}{a_1}\right), \quad (2.27)$$

where we have used the formula

$$\lim_{\epsilon \to 0} \frac{\epsilon}{x^2 + \epsilon^2} = \pi\delta(x), [\delta(x)\text{:Dirac's delta function}]. \quad (2.28)$$

The first term on the right-hand side of (2.27) has the same form as (2.20) except for a "phase shift," $-a_2/a_1$ and the second term shows that the blow up of the solution takes place at $x = a_2/a_1$. The blow-up time and position are also obtained directly from (2.21) by solving the algebraic equation, $f = 0$. In this case, however, the variable x must be regarded as a real number. Indeed, rewriting (2.21) in the form

$$f = x^2 - (\mu a_1 t + b_1)x + (\mu a_2 + 2\nu)t + b_2 - i(a_1 x - a_2), \quad (2.29)$$

and taking both the real and imaginary parts of f to be zero, the results mentioned above are reproduced.

Finally, we consider an initial condition that is composed of a superposition of two pulses, each one is represented by (2.19). In this case, we have

$$f(x,0) = [x - (\tilde{b}_1 + i\tilde{a}_1)][x - (\tilde{b}_2 + i\tilde{a}_2)] \quad (\tilde{a}_1 > 0, \tilde{a}_2 > 0). \quad (2.30)$$

If we compare (2.30) with f given by (2.21), we find that the constants a_1, a_2, b_1, and b_2 must be taken as

$$a_1 = \tilde{a}_1 + \tilde{a}_2, \quad (2.31a)$$

$$a_2 = \tilde{a}_1 \tilde{b}_2 + \tilde{a}_2 \tilde{b}_1, \quad (2.31b)$$

$$b_1 = \tilde{b}_1 + \tilde{b}_2, \quad (2.31c)$$

$$b_2 = \tilde{b}_1 \tilde{b}_2 - \tilde{a}_1 \tilde{a}_2. \quad (2.31d)$$

The blow-up time of the solution is obtained by substituting (2.31) into (2.25) as

$$t_c = \tilde{a}_1 \tilde{a}_2 [(\tilde{a}_1 + \tilde{a}_2)^2 + (\tilde{b}_1 - \tilde{b}_2)^2]/[2\nu(\tilde{a}_1 + \tilde{a}_2)^2]. \quad (2.32)$$

The positiveness of t_c is obvious from the above expression.

D. Remark

In the case of $\nu = 0$, the blow up occurs only at an infinite time for solutions with general N.[8-12] Instead of writing down the condition[9] for $\mathrm{Im}\, x_j > 0 (j = 1,2,...,N)$, we shall present an explicit example of a set of constants, a_j and b_j that realize these conditions. They read as follows:

$$a_j = [(N-1)!/(N-j)!]^2/(j-1)! \quad (j = 1,2,...,N), \quad (2.33a)$$

$$b_j = [N!/(N-j)!]^2/j! \quad (j = 1,2,...,N). \quad (2.33b)$$

Asymptotic behavior of u for $t \to +\infty$ is then given by

$$u \sim \frac{2a_1}{(x - \mu a_1 t - b_1 + a_2/a_1)^2 + a_1^2} + 2\pi \sum_{j=1}^{N-1} \delta(x - \xi_j), \quad (2.34)$$

where $\xi_j (j = 1,2,...,N-1)$ are $N-1$ zeros of the Laguerre polynomial of order $N-1$, namely,

$$L_{N-1}(\xi_j) \equiv \sum_{s=0}^{N-1} \frac{(-1)^s (N-1)!}{(N-s-1)!(s!)^2} \xi_j^s = 0. \quad (2.35)$$

The expression (2.34) shows that the blow up takes place at the $N-1$ positions, $\xi_1, \xi_2, ..., \xi_{N-1}$.

III. THE PERIODIC CASE

A. Linearization

In the periodic case, we introduce the following dependent variable transformation

$$u = i \frac{\partial}{\partial x} \ln \frac{f^*}{f}, \quad (3.1a)$$

with

$$f = \prod_{j=1}^{N} \beta^{-1} \sin \beta [x - x_j(t)] (\mathrm{Im}\, x_j > 0), \quad (3.1b)$$

where β is a positive constant. In the limit of $\beta \to 0$, (3.1) is reduced to (2.1). We first consider Eq. (1.4). The bilinear equation corresponding to Eq. (2.2) now takes the form

$$i(f^*_t f - f_t f^*) - i\nu(f^*_{xx} f - f_{xx} f^*) - (\mu/2)(f^2 + f^{*2} - 2c f^* f) = -2\delta f^* f, \quad (3.2)$$

where δ is a real integration constant and c is a positive constant given by

$$c = \exp\left[-2\beta \sum_{j=1}^{N} \mathrm{Im}\, x_j\right]. \quad (3.3)$$

The constancy of c in time will be proved later [see (3.17)]. In deriving (3.2), use has been made of the formulas

$$H(f/f^* - c) = i(f/f^* - c), \quad (3.4a)$$

$$H(f^*/f - c) = -i(f^*/f - c), \quad (3.4b)$$

which are reduced, in the limit of $\beta\to 0$, to (2.3a) and (2.3b), respectively. Moreover, we modify Eq. (3.2) as

$$f^*\{f_t - \nu f_{xx} - (\mu/2i)(f-f^*) + i[\delta - (\mu/2)(1-c)]f\} = f\{f^*_t - \nu f^*_{xx} + (\mu/2i)(f^*-f) - i[\delta - (\mu/2)(1-c)]f^*\}. \quad (3.5)$$

Thus we may decouple Eq. (3.5) in the form

$$f_t - \nu f_{xx} - (\mu/2i)(f-f^*) = \mu(\eta - i\lambda)f, \quad (3.6)$$

and its complex conjugate expression. Here, we have chosen the integration constant δ as

$$\delta = (\mu/2)(1-c) + \mu\lambda, \quad (3.7)$$

where λ is a positive constant and η is a real constant to be determined later.

Next, we consider Eq. (1.5). The linear equation corresponding to Eq. (3.6) is given by

$$f_t - \nu f_{xx} - (\mu c/2i)(f-f^*) = \mu c(\eta - i\lambda)f, \quad (3.8)$$

which is transformed into Eq. (3.6) by rescaling the time and space variables as $t\to c^{-1}t$ and $x\to c^{-1/2}x$. Hence, we shall be concerned only with Eq. (1.4) in the following.

Now, the time evolution of x_n is obtained from (3.1) and (3.6) as

$$\dot{x}_n = \frac{\mu}{2i}\prod_{j=1}^{N}\beta^{-1}\sin\beta(x_n - x^*_j) \times\left(\prod_{\substack{j=1\\(j\neq n)}}^{N}\beta^{-1}\sin\beta(x_n - x_j)\right)^{-1} - 2\nu\beta\sum_{\substack{j=1\\(j\neq n)}}^{N}\cot\beta(x_n - x_j) \quad (n=1,2,\ldots,N). \quad (3.9)$$

The system of Eqs. (3.9) describes the dynamical motion of N variables, $x_1, x_2, \ldots, x_N$. It is reduced, in the limit of $\beta\to 0$, to Eq. (2.6). As in the nonperiodic case, the second term on the right-hand side of Eq. (3.9) will result in the blow up of solutions as shown in the subsection C.

B. Exact solutions

In order to solve Eq. (3.6), we expand f as

$$f(x,t) = \sum_{n=-N}^{N} c_n(t)e^{in\beta x}, \quad (3.10)$$

and substitute this expression into Eq. (3.6) to obtain the following linear equation for c_n:

$$\dot{c}_n + \nu(n\beta)^2 c_n - (\mu/2i)(c_n - c^*_{-n}) = \mu(\eta - i\lambda)c_n \quad (-N\leqslant n\leqslant N). \quad (3.11)$$

By comparing (3.1b) and (3.10), one finds

$$c_N(t) = (2i\beta)^{-N}\exp\left(-i\beta\sum_{j=1}^{N}x_j\right), \quad (3.12a)$$

$$c_{-N}(t) = (-2i\beta)^{-N}\exp\left(i\beta\sum_{j=1}^{N}x_j\right), \quad (3.12b)$$

and

$$c_{2m+1}(t) = c_{-(2m+1)}(t) = 0 \quad (m=0,1,\ldots,M-1), \quad (3.13)$$

for $N=2M$ and

$$c_{2m}(t) = c_{-2m}(t) = 0 \quad (m=0,1,\ldots,M), \quad (3.14)$$

for $N=2M+1$. Substitution of (3.12) into Eq. (3.11) with $n=N$ yields the time evolution of $\Sigma_{j=1}^{N}x_j$:

$$-i\beta\sum_{j=1}^{N}\dot{x}_j + \nu(\beta N)^2 = \frac{\mu}{2i}\left[1 - \exp\left(-2\beta\sum_{j=1}^{N}\mathrm{Im}\,x_j\right)\right] + \mu(\eta - i\lambda), \quad (3.15a)$$

$$i\beta\sum_{j=1}^{N}\dot{x}_j + \nu(\beta N)^2 = \frac{\mu}{2i}\left[1 - \exp\left(2\beta\sum_{j=1}^{N}\mathrm{Im}\,x_j\right)\right] + \mu(\eta - i\lambda). \quad (3.15b)$$

For both equations to be consistent, the unknown real constant η must be chosen as

$$\eta = \mu^{-1}\nu(\beta N)^2, \quad (3.16)$$

so that the integration of Eq. (3.15) is readily performed to yield the solution

$$\sum_{j=1}^{N}x_j = \frac{\mu\gamma}{\beta}t + b + i\frac{1}{2\beta}\ln(2\lambda + 2\gamma + 1), \quad (3.17a)$$

where γ is a positive constant defined by

$$\gamma = [\lambda(\lambda+1)]^{1/2}. \quad (3.17b)$$

A simple evolution of $\Sigma_{j=1}^{N}x_j$ should be remarked. In particular, the imaginary part of it does not depend on time, which proves the constancy of c defined by (3.3).

The general solution of Eq. (3.11) is given by

$$c_n(t) = e^{\nu\beta^2(N^2-n^2)t}\Big\{c_n(0)[\cos\gamma\mu t - (i/2)(1+2\lambda)(\sin\gamma\mu t/\gamma)] + \frac{i}{2}c^*_{-n}(0)\frac{\sin\gamma\mu t}{\gamma}\Big\} \quad (-N\leqslant n\leqslant N), \quad (3.18)$$

which, together with (3.10), constitutes the general solution for the initial value problem of Eq. (3.6).

C. Properties of solutions

Let us now investigate the properties of solutions given by (3.10) and (3.18). Since it is difficult to obtain a clear picture of the behavior of solutions for general N, we restrict ourselves to solutions for $N=1$ and $N=2$.

1. N=1

For $N=1$, it readily follows from (3.17) that

$$x_1(t) = (\mu\gamma_1/\beta)t + b_1 + i(1/2\beta)\ln(2\lambda_1 + 2\gamma_1 + 1), \quad (3.19a)$$

with

$$\gamma_1 = [\lambda_1(\lambda_1+1)]^{1/2}. \quad (3.19b)$$

The above expression may be rewritten in a more transparent form as

$$x_1(t) = \mu a_1 t + b_1 + i(\sinh^{-1}2\beta a_1/2\beta), \quad (3.20)$$

by introducing a parameter a_1 through the relation

$$\gamma_1 = a_1\beta, \quad (a_1 > 0). \tag{3.21}$$

Then, (3.1) yields a periodic solution of Eq. (1.4) of the form

$$u = \frac{4a_1\beta^2}{(1+4a_1^2\beta^2)^{1/2} - \cos 2\beta(x - \mu a_1 t - b_1)}. \tag{3.22}$$

The solution (3.22) is independent of ν and as is clear from (3.20), the condition, $\operatorname{Im} x_1 > 0$ holds for all time. Therefore, the blow up of the solution never occurs. The situation is the same as that for the nonperiodic case with $N = 1$ already presented in Sec. II. One can easily confirm that (3.22) is reduced to (2.20) in the limit of $\beta \to 0$.

2. $N=2$

The case $N = 2$ is somewhat complicated. To simplify the discussion, we consider the following initial condition for f:

$$f(x,0) = \beta^{-2} \sin\beta[x - x_1(0)] \sin\beta[x - x_2(0)], \tag{3.23a}$$

with

$$x_j(0) = \tilde{b}_j + i(1/2\beta)\ln\epsilon_j \quad (j = 1,2), \tag{3.23b}$$

where we have put

$$\epsilon_j = 2\lambda_j + 2\gamma_j + 1, \quad \lambda_j > 0 \quad (j = 1,2), \tag{3.23c}$$

$$\gamma_j = [\lambda_j(\lambda_j + 1)]^{1/2} \quad (j = 1,2). \tag{3.23d}$$

The expression (3.23) represents a superposition of two periodic pulses, each one is given by (3.19) with $t = 0$. Owing to the relation (3.17a), the parameters b and λ must be chosen as

$$b = \tilde{b}_1 + \tilde{b}_2, \tag{3.24a}$$

$$\lambda = \alpha^2/(1 + 2\alpha), \tag{3.24b}$$

where

$$\alpha = 2(\lambda_1 + \gamma_1)(\lambda_2 + \gamma_2) + \lambda_1 + \lambda_2 + \gamma_1 + \gamma_2. \tag{3.24c}$$

Then, we find from (3.10) and (3.23) that at the initial time, $t = 0$

$$c_0(0) = p_0 + iq_0, \tag{3.25a}$$

where

$$p_0 = \frac{1}{4\beta^2}\left[\left(\frac{\epsilon_2}{\epsilon_1}\right)^{1/2} + \left(\frac{\epsilon_1}{\epsilon_2}\right)^{1/2}\right]\cos\beta(\tilde{b}_1 - \tilde{b}_2), \tag{3.25b}$$

$$q_0 = \frac{1}{4\beta^2}\left[\left(\frac{\epsilon_2}{\epsilon_1}\right)^{1/2} - \left(\frac{\epsilon_1}{\epsilon_2}\right)^{1/2}\right]\sin\beta(\tilde{b}_1 - \tilde{b}_2). \tag{3.25c}$$

Now, the time evolution of c_0 follows from (3.18) and (3.25) as

$$c_0(t) = e^{4\nu\beta^2 t}\{p_0 \cos\gamma\mu t + q_0(1+\lambda)(\sin\gamma\mu t/\gamma) + i[q_0\cos\gamma\mu t - p_0\lambda(\sin\gamma\mu t/\gamma)]\} \equiv p + iq. \tag{3.26}$$

Here, p and q represent the real and imaginary parts of $c_0(t)$, respectively. Finally, the expression for $f(x,t)$ is given from (3.10), (3.12), (3.17), and (3.26) as follows:

$$f(x,t) = -(4\beta^2\epsilon^{-1/2})^{-1}z - (4\beta^2\epsilon^{1/2}z)^{-1} + p + iq, \tag{3.27a}$$

where we have put

$$z = \exp[2i\beta(x - (\gamma\mu/2\beta)t - b/2)], \tag{3.27b}$$

$$\epsilon = 2\lambda + 2\gamma + 1, \tag{3.27c}$$

for simplicity.

The time evolutions of x_1 and x_2 are obtained by solving the algebraic equation for $z, f = 0$ and these are expressed in the forms

$$x_{1,2}(t) = \frac{\gamma\mu}{2\beta}t + \frac{b}{2} + \frac{1}{2\beta}\tan^{-1}\left(\frac{Q \mp S}{P \mp R}\right) - \frac{i}{4\beta}\ln[(P \mp R)^2 + (Q \mp S)^2], \tag{3.28}$$

where the upper(lower) sign corresponds to x_1 (x_2). Here, P, Q, R, and S are defined, respectively, by

$$P = 2\beta^2 p/\epsilon^{1/2}, \tag{3.29a}$$

$$Q = 2\beta^2 q/\epsilon^{1/2}, \tag{3.29b}$$

$$R = (1/\sqrt{2})[P^2 - Q^2 - \epsilon^{-1} + \sqrt{(P^2 - Q^2 - \epsilon^{-1})^2 + 4(PQ)^2}]^{1/2}, \tag{3.29c}$$

$$S = [\operatorname{sgn}(PQ)/\sqrt{2}][-(P^2 - Q^2 - \epsilon^{-1}) + \sqrt{(P^2 - Q^2 - \epsilon^{-1})^2 + 4(PQ)^2}]^{1/2}. \tag{3.29d}$$

The x_1 (x_2) is shown to be a periodic analog of the x_1 (x_2) given by (2.22). One can also easily check by a direct calculation that the relation (3.17) is satisfied by (3.28). The expressions (3.28) are valid in so far as $\operatorname{Im} x_j > 0$ ($j = 1,2$). However, these conditions will be broken down at a finite time. As seen from (3.28), this happens when

$$(P \mp R)^2 + (Q \mp S)^2 = 1. \tag{3.30}$$

We shall now derive a breakdown time t_c on the basis of (3.30). First, if we use (3.29), the relation (3.30) is considerably simplified as

$$p^2/(\lambda + 1) + q^2/\lambda = (4\beta^2)^{-1}. \tag{3.31}$$

Furthermore, substituting (3.25) and (3.26) into (3.31), we find that the t_c is determined by the relation

$$\exp(-8\nu\beta^2 t_c) = [(\epsilon_1 + \epsilon_2)^2/(\epsilon_1\epsilon_2 + 1)^2]\cos^2\beta(\tilde{b}_1 - \tilde{b}_2) + [(\epsilon_1 - \epsilon_2)^2/(\epsilon_1\epsilon_2 - 1)^2]\sin^2\beta(\tilde{b}_1 - \tilde{b}_2). \tag{3.32}$$

It is worthwhile to note that the right-hand side of (3.32) does not depend on ν and that it is always less than unity due to the inequalities

$$0 < (\epsilon_1 + \epsilon_2)/(\epsilon_1\epsilon_2 + 1) < 1, \tag{3.33a}$$

$$-1 < (\epsilon_1 - \epsilon_2)/(\epsilon_1\epsilon_2 - 1) < 1, \tag{3.33b}$$

where $\epsilon_1 > 1$ and $\epsilon_2 > 1$. Hence, we see that (3.32) gives rise to a positive t_c. At the instant of $t = t_c$, the solution (3.1) blows up and which is the same situation as that has already been encountered in the nonperiodic case with $N = 2$. The position the blow up takes place is obtained from the real part of $x_j(t_c)$.

D. Remark

In the limit of $\beta \to 0$, various results presented here are shown to be reduced to those for the nonperiodic case. In concluding this section, we shall demonstrate this for the blow up time t_c. For the purpose, we put

$$\gamma_j = \tilde{a}_j \beta \quad (j=1,2), \tag{3.34}$$

and take the limit $\beta \to 0$. It then turns out that

$$\epsilon_j = 1 + 2\tilde{a}_j\beta + 2\tilde{a}_j^2\beta^2 + O(\beta^3) \quad (j=1,2), \tag{3.35a}$$

$$(\epsilon_1 + \epsilon_2)/(\epsilon_1\epsilon_2 + 1) = 1 - 2\tilde{a}_1\tilde{a}_2\beta^2 + O(\beta^3), \tag{3.35b}$$

$$(\epsilon_1 - \epsilon_2)/(\epsilon_1\epsilon_2 - 1) = (\tilde{a}_1 - \tilde{a}_2)/(\tilde{a}_1 + \tilde{a}_2) + O(\beta), \tag{3.35c}$$

$$\cos^2\beta(\tilde{b}_1 - \tilde{b}_2) = 1 - (\tilde{b}_1 - \tilde{b}_2)^2\beta^2 + O(\beta^4), \tag{3.35d}$$

$$\sin^2\beta(\tilde{b}_1 - \tilde{b}_2) = (\tilde{b}_1 - \tilde{b}_2)^2\beta^2 + O(\beta^4). \tag{3.35e}$$

Substituting (3.35) into (3.32) and retaining the terms of the order of β^2 on both sides, we can recover the blow-up time t_c for the nonperiodic case [see (2.32)].

IV. CONCLUDING REMARKS

In this paper, we have proposed two novel nonlinear diffusion equations with the Hilbert Kernel. By means of the method of linearization, the initial value problems of the proposed equations have been solved exactly for both nonperiodic and periodic cases. A common feature of solutions thus obtained is that the blow up occurs after a lapse of finite time even if we assume smooth initial conditions. In this respect, it should be remarked that a nonlinear diffusion equation of the form

$$u_t - \nu u_{xx} - uHu = 0, \tag{4.1}$$

has been proposed as a model vorticity equation describing a viscous incompressible fluid flow.[17] A particular solution of Eq. (4.1) has been shown to exhibit the blow up at a finite time.[17] However, in contrast to the equations proposed here, Eq. (4.1) cannot be linearized and hence it seems to be intractable in comparison with our equations. In particular, the initial value problem has not been solved as yet for Eq. (4.1).

In concluding this paper, we note that Eqs. (1.4) and (1.5) may be generalized by introducing the following singular integral kernels[14] in place of the Hilbert kernel

$$Tu(x,t) = \frac{1}{2\delta} P \int_{-\infty}^{\infty} \left[\coth \frac{\pi(y-x)}{2\delta}\right] u(y,t)dy, \tag{4.2}$$

$$T_p u(x,t) = \frac{1}{2L} P \int_{-L}^{L} \left[\cot \frac{\pi(y-x)}{2L}\right] u(y,t)dy, \tag{4.3}$$

In the limit of $\delta \to \infty$ ($L \to \infty$), the operator $T(T_p)$ is reduced to the Hilbert kernel defined by (1.3). To investigate these generalized equations will be left for the future work.

ACKNOWLEDGMENT

The author thanks Professor M. Nishioka for continual encouragement.

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