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Bäcklund transformation, conservation laws, and inverse scattering transform of a model integrodifferential equation for water waves

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The Bäcklund transformation (BT), an infinite number of conservation laws, and the inverse scattering transform (IST) of a model integrodifferential equation for water waves in fluids of finite depth [Y. Matsuno, J. Math. Phys. **29**, 49(1989)] are constructed by employing the bilinear transformation method. The model equation is also shown to pass the Painlevé test. These facts prove the complete integrability of the equation. Both the deep- and shallow-water limits of various results thus obtained are then investigated in detail. In addition, a new method to evaluate conserved quantities for pure N-soliton is developed by utilizing actively the time part of the BT. It is found that the structure of conservation laws exhibits peculiar characteristics in comparison with those of usual water wave equations such as the Benjamin–Ono and the Korteweg–de Vries equations. The most important problem left open in this paper is to solve various IST equations.

I. INTRODUCTION

In this paper, we consider the following integrodifferential evolution equation:

$$u_t + u_x - 2uu_t + 2u_x \int_x^{\infty} u_t\, dx - Tu_{tx} = 0, \quad u = u(x,t), \tag{1.1a}$$

where the operator T is defined by

$$Tu(x,t) = \frac{1}{2\delta} P \int_{-\infty}^{\infty} \left\{ \coth\left[\frac{\pi(y-x)}{2\delta}\right] - \operatorname{sgn}(y-x) \right\} u(y,t)dy, \tag{1.1b}$$

and the subscripts t and x appended to u denote partial differentiation.

Equation (1.1) has already been proposed as a model equation describing wave phenomena in fluids of finite depth.[1] The parameter δ in (1.1b) represents the depth of fluids. In the deep-water limit $\delta \to \infty$, it reduces to the equation

$$u_t + u_x - 2uu_t + 2u_x \int_x^{\infty} u_t\, dx - Hu_{tx} = 0, \tag{1.2a}$$

where H is the Hilbert transform given by

$$Hu(x,t) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{u(y,t)}{y-x} dy. \tag{1.2b}$$

On the other hand, in the shallow-water limit $\delta \to 0$, if we introduce the new variables τ and ξ by the relations

$$\tau = \sqrt{3/\delta}\, t, \tag{1.3a}$$

$$\xi = \sqrt{3/\delta}\, x, \tag{1.3b}$$

and use the expansion formula (B8) in Appendix B, Eq. (1.1) reduces to the shallow-water wave equation that has already been proposed by Hirota and Satsuma[2]

$$u_\tau + u_\xi - 2uu_\tau + 2u_\xi \int_\xi^{\infty} u_\tau\, d\xi - u_{\tau\xi\xi} = 0. \tag{1.4}$$

Therefore, Eq. (1.1) is an intermediate version between Eqs. (1.2) and (1.4). In this respect, it should be remarked that an evolution equation of the form[3,4]

$$u_t + u_x + 2uu_x + Tu_{xx} = 0 \tag{1.5}$$

is known as an intermediate equation between the Benjamin–Ono (BO)[5,6] and the Korteweg–de Vries (KdV) equations. The mathematical structure of Eq. (1.5) is now completely understood.[7–9] However, the characteristics of Eq. (1.1) are quite different from those of Eq. (1.5) as shown in this paper.

The multisoliton solutions[1,2] of Eqs. (1.1), (1,2), and (1.4) have already been obtained by employing the bilinear transformation method.[10–12] In particular, the properties of solutions of Eq. (1.2) have been studied in great detail[13] and it was found that it exhibits an algebraic N-soliton solution expressed in terms of Pfaffians.[14]

The purpose of the present paper is to construct the Bäcklund transformation (BT), an infinite number of conservation laws and the inverse scattering transform (IST) of Eq. (1.1) and to prove its complete integrability. Both the deep- and shallow-water limits are then taken for various results thus obtained and which give rise to the corresponding ones for Eqs. (1.2) and (1.4), respectively. Throughout this paper, we mainly use the bilinear transformation method[10–12] as a mathematical tool.

In Sec. II, the BT, an infinite number of conservation laws, and the IST are constructed for Eq. (1.1). It is also demonstrated that Eq. (1.1) passes the so-called Painlevé test.[15–18] In Sec. III, the deep-water limit of the results obtained in Sec. II is investigated in detail. In addition to this, the conserved quantities are evaluated explicitly for pure N-soliton solution and an initial condition evolving into pure N solitons is briefly discussed. In Sec. IV, the shallow-water limit is considered in the same way. Section V is devoted to concluding remarks where a few comments are made concerning problems left open in this paper. In Appendix A, the formulas of bilinear operators are noted for the convenience of the reader unfamiliar with the bilinear formalism. Finally, in Appendix B, various properties of the singular integral operators T and H are described.

II. STUDY OF EQ. (1.1)

First of all, let us bilinearize Eq. (1.1). For the purpose, we introduce the following dependent variable transformation:

$$u = \frac{i}{2}\frac{\partial}{\partial x}\ln\frac{f_+}{f_-}, \tag{2.1a}$$

with

$$f_+(x,t) = f(x - i\delta,t), \tag{2.1b}$$

$$f_-(x,t) = f(x + i\delta,t), \tag{2.1c}$$

where $f(z,t)$ is an analytic function of z and it is assumed that $f(z - i\delta,t)$ has no zero in the complex region $0 \leqslant \mathrm{Im}\, z \leqslant 2\delta$. It then follows by using the Cauchy residue theorem that

$$Tu_x = -\frac{1}{2}\frac{\partial^2}{\partial x^2}(f_+ f_-) + \delta^{-1}u. \tag{2.2}$$

Substituting (2.1a) and (2.2) into Eq. (1.1) and integrating once with respect to x, we obtain the following bilinear equation for f_+ and f_-:[1]

$$[i(1-\delta^{-1})D_t + iD_x + D_tD_x]f_+\cdot f_- = 0. \tag{2.3}$$

Here, the bilinear operators D_t and D_x are defined by the relation

$$D_t^m D_x^n f_+\cdot f_- = \left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^m\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^n f_+(x,t) \times f_-(x',t')\Big|_{t'=t,\ x'=x},$$

$$(m, n = 0, 1, \ldots). \tag{2.4}$$

Equation (2.3) is the basic equation and plays a fundamental role in this section.

A. Bäcklund transformation

In this subsection, starting from the bilinear equation (2.3) we construct the BT in bilinear form and then transform it into ordinary form.

1. BT in bilinear form

We now construct the BT of Eq. (1.1) in bilinear form. Let another solution of Eq. (2.3) be $g_\pm$, namely,

$$[i(1-\delta^{-1})D_t + iD_x + D_tD_x]g_+\cdot g_- = 0, \tag{2.5}$$

and consider the following equation:

$$\begin{aligned}P \equiv{}& g_+g_-\,[i(1-\delta^{-1})D_t + iD_x + D_tD_x]f_+\cdot f_- \\ &- \{[i(1-\delta^{-1})D_t + iD_x + D_tD_x] \\ &\times g_+\cdot g_-\}f_+f_- = 0.\end{aligned} \tag{2.6}$$

Obviously, if $f_\pm$ is a solution of Eq. (2.3), then $g_\pm$ gives an another solution and vice versa. We prove that Eq. (2.6) is satisfied identically by the following system of bilinear equations that relate $f_\pm$ and $g_\pm$ and hence these equations constitute a BT of Eq. (2.3):

$$D_tg_+\cdot f_- - D_tg_-\cdot f_+ = -i(g_+f_- + g_-f_+) + \nu(g_+f_- - g_-f_+), \tag{2.7a}$$

$$D_xg_+\cdot f_- + D_xg_-\cdot f_+ = -i(1-\delta^{-1})\times(g_+f_- - g_-f_+) + \mu(g_+f_- + g_-f_+), \tag{2.7b}$$

$$D_tD_xg_+\cdot f_+ = \mu D_tg_+\cdot f_+ + \nu D_xg_+\cdot f_+ + \kappa f_+g_+, \tag{2.7c}$$

$$D_tD_xg_-\cdot f_- = \mu D_tg_-\cdot f_- + \nu D_xg_-\cdot f_- + \kappa g_-f_-. \tag{2.7d}$$

Here, μ, ν, and κ are arbitrary parameters that may be related to each other by specifying boundary conditions. To show the above BT, we first use (A3) and (A4) to modify (2.6) into the form

$$\begin{aligned}P ={}& D_t\{g_-f_+\cdot[D_xg_+\cdot f_- + i(1-\delta^{-1})g_+f_-]\} \\ &+ D_x[g_-f_+\cdot(D_tg_+\cdot f_- + ig_+f_-)] \\ &+ (D_tD_xg_+\cdot f_+)g_-f_- - g_+f_+D_tD_xg_-\cdot f_-.\end{aligned} \tag{2.8a}$$

Substituting (2.7c) and (2.7d) into (2.8a) and using (A3), P is transformed into

$$\begin{aligned}P ={}& D_t\{g_-f_+\cdot[D_xg_+\cdot f_- \\ &+ (i(1-\delta^{-1}) - \mu)g_+f_-]\} \\ &+ D_x\{g_-f_+\cdot[D_tg_+\cdot f_- + (i-\nu)g_+f_-]\}.\end{aligned} \tag{2.8b}$$

Finally, introducing (2.7a) and (2.7b) into (2.8b) and using (A1) and (A2), P becomes

$$P = -D_t(g_-f_+\cdot D_xg_-\cdot f_+) + D_x(g_-f_+\cdot D_tg_-\cdot f_+) = 0. \tag{2.8c}$$

This completes the proof of Eq. (2.6).

2. BT in ordinary form

In order to rewrite the BT in ordinary form, it is convenient to introduce the potentials $\bar u$ and $\bar v$ by

$$\bar u = (i/2)\ln(f_+/f_-), \tag{2.9a}$$

$$\bar v = (i/2)\ln(g_+/g_-). \tag{2.9b}$$

It then follows from (2.2) and (2.9) that

$$f_\pm = f_{\pm,0}\left[\mp 2i\hat P_\mp(\bar u - \bar u_0) + \delta^{-1}\int_{-\infty}^x(\bar u - \bar u_0)dx\right], \tag{2.10a}$$

$$g_\pm = g_{\pm,0}\left[\mp 2i\hat P_\mp(\bar v - \bar v_0) + \delta^{-1}\int_{-\infty}^x(\bar v - \bar v_0)dx\right]. \tag{2.10b}$$

Here, $P_\pm$ are integral operators defined by

$$\hat P_\pm = \tfrac{1}{2}(1 \pm iT), \tag{2.11}$$

and $\bar u_0 = \bar u(-\infty,t)$, $\bar v_0 = \bar v(-\infty,t)$, $f_{\pm,0} = f_\pm(-\infty,t)$, $g_{\pm,0} = g_\pm(-\infty,t)$. If we subtract (2.7d) from (2.7c) after substituting (2.10) into these equations and define the function w by

$$w = \bar v - \bar u, \tag{2.12}$$

with the boundary conditions

$$w(+\infty,t)=w(-\infty,t)=w_0, \tag{2.13}$$

then, we obtain the time part of the BT as follows:

$$w_{tx}-w_tTw_x-w_xTw_t+\delta^{-1}w_x\int_{-\infty}^{x}w_t\,dx$$
$$+\delta^{-1}(w-w_0)w_t$$
$$=\mu w_t+\nu w_x-2\bar{u}_{tx}. \tag{2.14}$$

Integrating (2.14) and using (B5), we obtain an important relation

$$\int_{-\infty}^{\infty}w_t\,dx=0. \tag{2.15}$$

This fact will be used in Sec. II B to derive conservation laws.

The space part of the BT stems by substituting (2.10) into (2.7b) and the result is expressed in the form

$$e^{2iw}=\frac{2i\hat{P}_-w_x-\delta^{-1}(w-w_0)+2iu-i(1-\delta^{-1})+\mu}{2i\hat{P}_+w_x+\delta^{-1}(w-w_0)+2iu-i(1-\delta^{-1})-\mu}. \tag{2.16}$$

Similarly, Eq. (2.7a) can be rewritten as

$$e^{2iw}=-\frac{2i\hat{P}_-w_t-\delta^{-1}\int_{-\infty}^{x}w_t\,dx+2i\bar{u}_t-i+\nu}{2i\hat{P}_+w_t+\delta^{-1}\int_{-\infty}^{x}w_t\,dx+2i\bar{u}_t-i-\nu}. \tag{2.17}$$

Equation (2.17) may be regarded as a time part of the BT in place of Eq. (2.14). Now, by taking $|x|\to\infty$ in (2.17) and using (2.13) with the boundary conditions $u(\pm\infty,t)=w_x(\pm\infty,t)=0$, one sees that w_0 is expressed in terms of ν as

$$w_0=(1/2i)\ln[(\nu-i)/(\nu-i)]. \tag{2.18}$$

Furthermore, if we equate (2.16) and (2.17) and then take $|x|\to\infty$, we find that μ is related to ν as

$$\mu=-(1-\delta^{-1})/\nu. \tag{2.19}$$

It should be remarked here that w also satisfies the following equation:

$$(2\bar{u}_x-1)w_t+(2\bar{u}_t-1)w_x+Tw_{tx}+2w_tw_x=0. \tag{2.20}$$

To show this, we first introduce (2.9) into (1.1) and then integrate once with respect to x to obtain

$$\bar{u}_t+\bar{u}_x-T\bar{u}_{tx}-2\bar{u}_t\bar{u}_x=0, \tag{2.21a}$$

$$\bar{v}_t+\bar{v}_x-T\bar{v}_{tx}-2\bar{v}_t\bar{v}_x=0. \tag{2.21b}$$

Subtracting (2.21a) from (2.21b) and noting (2.12) leads to Eq. (2.20). It can be verified by direct calculation with the aid of (B10) that Eqs. (2.14), (2.16), (2.17), and (2.21) are all compatible.

Remark 1: One can use the BT presented here to generate multisoliton solutions for Eq. (1.1). However, since this problem has already been solved by a more direct method on the basis of Eq. (2.5),[1] we restrict ourselves only on the one-soliton solution. For this case, we start with a vacuum solution of Eq. (2.3), namely, $f_+=f_-=1$. Then, Eqs. (2.7) reduce to the following system of linear differential equations for $g_\pm$:

$$g_{+,t}-g_{-,t}=-i(g_++g_-)+\nu(g_+-g_-), \tag{2.22a}$$

$$g_{+,x}+g_{-,x}=-i(1-\delta^{-1})(g_+-g_-)$$
$$+\mu(g_++g_-), \tag{2.22b}$$

$$g_{+,tx}=\mu g_{+,t}+\nu g_{+,x}+\kappa g_+, \tag{2.22c}$$

$$g_{-,tx}=\mu g_{-,t}+\nu g_{-,x}+\kappa g_-, \tag{2.22d}$$

It is easy to see that Eqs. (2.22) exhibit the solutions of the forms

$$g_+=\alpha\cosh(\gamma/2\delta)(x-ct-x_{01}-i\delta), \tag{2.23a}$$

$$g_-=\beta\cosh(\gamma/2\delta)(x-ct-x_{01}+i\delta), \tag{2.23b}$$

with the choice of the parameters

$$\kappa=-(\mu\nu+1-\delta^{-1}), \tag{2.23c}$$

$$\mu=i\left[\left(1-\delta^{-1}+\frac{\gamma}{2\delta}\cot\frac{\gamma}{2}\right)\times\left(1-\delta^{-1}-\frac{\gamma}{2\delta}\tan\frac{\gamma}{2}\right)\right]^{1/2}, \tag{2.23d}$$

$$\nu=i\left[\left(1-\frac{c\gamma}{2\delta}\cot\frac{\gamma}{2}\right)\left(1+\frac{c\gamma}{2\delta}\tan\frac{\gamma}{2}\right)\right]^{1/2}. \tag{2.23e}$$

Here, c is a propagation velocity of the soliton given by

$$c=(1-\delta^{-1}+\delta^{-1}\gamma\cot\gamma)^{-1}, \tag{2.23f}$$

α and β are constants with the ratio

$$\frac{\alpha}{\beta}=-\left(1-\delta^{-1}-i\mu+\frac{\gamma}{2\delta}\cot\frac{\gamma}{2}\right)\times\left(1-\delta^{-1}+i\mu+\frac{\gamma}{2\delta}\cot\frac{\gamma}{2}\right)^{-1}, \tag{2.23g}$$

and γ is an arbitrary constant within the range $0<\gamma<\pi$. It then follows from (2.1a) and (2.23) that

$$u_1=\frac{(\gamma/2\delta)\sin\gamma}{\cosh(\gamma/\delta)(x-ct-x_{01})+\cos\gamma}, \tag{2.24}$$

which is nothing but a one-soliton solution of Eq. (1.1).[1]

B. Conservation laws

In this subsection, we derive an infinite number of conservation laws of Eq. (1.1). For the purpose, it is most straightforward to employ the results obtained in subsection B 2. We first put

$$w=w_0+\sum_{j=1}^{\infty}w_j\epsilon^j\quad(w_j(\pm\infty,t)=0,\ j=1,2,\ldots), \tag{2.25a}$$

$$\epsilon=-2\nu/(1-\delta^{-1}). \tag{2.25b}$$

Then, Eq. (2.15) implies that

$$I_j\equiv\int_{-\infty}^{\infty}w_j\,dx\quad(j=1,2,\ldots), \tag{2.26}$$

are conserved quantities. Substitution of (2.19) and (2.25b) into (2.16) yields

$$2i(w-w_0) = \ln\{1+(\alpha\epsilon)^2/4 + i\epsilon(1+i\alpha\epsilon/2) \times [u + (i/2\delta)(w-w_0) + \hat{P}_- w_x]\} - \ln\{1+(\alpha\epsilon)^2/4 - i\epsilon(1-i\alpha\epsilon/2) \times [u - (i/2\delta)(w-w_0) + \hat{P}_+ w_x]\}, \quad (2.27)$$

where $\alpha \equiv 1-\delta^{-1}$. Finally, substituting (2.25a) into (2.27) and comparing the coefficients of ϵ^j on both sides of (2.27), we can derive w_j successively by means of purely algebraic procedure. It is easily seen that only I_{2j+1} ($j=0, 1, ...$) survive and I_{2j} ($j=1, 2, ...$) vanish identically. Indeed, the first three of I_{2j+1} read in the forms:

$$I_1 = \int_{-\infty}^{\infty} u\,dx, \quad (2.28a)$$

$$I_3 = -\int_{-\infty}^{\infty} \{\tfrac{1}{3}u^3 - \tfrac{1}{2}u^2 + \tfrac{1}{2}uTu_x + \tfrac{1}{4}(1-\delta^{-1})u\}dx, \quad (2.28b)$$

$$I_5 = \int_{-\infty}^{\infty} \{\tfrac{1}{5}u^5 - \tfrac{1}{6}(3+\delta^{-1})u^4 + \tfrac{1}{8}[\tfrac{16}{3}u^3 - 8u^2 + (4-3\delta^{-1}-\delta^{-2})u - u_{xx}]Tu_x + \tfrac{1}{12}(6+\delta^{-1}-\delta^{-2})u^3 + \tfrac{1}{4}(2u-1+\delta^{-1})(Tu_x)^2 - \tfrac{1}{8}(2-\delta^{-1}-\delta^{-2})u^2 + \tfrac{1}{8}(2u-1){u_x}^2 + \tfrac{1}{16}(1-\delta^{-1})^2u\}dx. \quad (2.28c)$$

It is observed that I_{2j+1} always includes a term u^{2j+1}. The lack of I_{2j} in (2.28) is a remarkable feature of conservation laws when compared with those of Eq. (1.5). In fact, Eq. (1.5) has a conserved density that includes a term u^j in I_j for all j (Ref. 8). Finally, it is worthwhile to remark that Eq. (1.1) also has an independent conserved quantity of the form

$$J_1 = \int_{-\infty}^{\infty} dx \int_x^{\infty} u_t\,dy, \quad (2.29)$$

in addition to (2.28). The constancy of (2.29) in time is verified by using Eq. (2.35) below and (B5).

Remark 2: The evaluation of the jth conservation law for pure N-soliton solution of Eq. (1.1) is an interesting problem since this may provide an approximate method to obtain amplitudes of solitons evolving from arbitrary initial conditions. However, we have not as yet suitable procedure for the purpose. The main difficulty is found to arise due to the right-hand side of the formula (B7). Indeed, if this term vanishes, integration of Eq. (2.14) twice with respect to x would yield a desired result. In order to overcome the difficulty, we must probably rely on the IST method. Nevertheless, for the special cases for both deep- and shallow-water limits, the evaluation of I_j can be performed completely as will be demonstrated in Sec. III B 2 and Sec. IV B 2, respectively.

C. Inverse scattering transform

The IST of Eq. (1.1) is easily derived by employing a standard procedure in the bilinear formalism.[11,12] First, we define the wave functions $\psi_\pm$ by the relations

$$\psi_+ = g_+/f_+, \quad (2.30a)$$

$$\psi_- = g_-/f_-. \quad (2.30b)$$

Substituting (2.30) into (2.7) and using (2.9) and (2.10), we find the following system of linear differential equations:

$$(\psi_+ + \psi_-)_x = i(2\bar{u}_x - 1 + \delta^{-1})(\psi_+ - \psi_-) + \mu(\psi_+ + \psi_-), \quad (2.31a)$$

$$(\psi_+ - \psi_-)_t = i(2\bar{u}_t - 1)(\psi_+ + \psi_-) + \nu(\psi_+ - \psi_-), \quad (2.31b)$$

$$\psi_{+,tx} = \mu\psi_{+,t} + \nu\psi_{+,x} - (2\delta^{-1}\bar{u}_t - 4i\hat{P}_-\bar{u}_{tx} - \kappa)\psi_+, \quad (2.31c)$$

$$\psi_{-,tx} = \mu\psi_{-,t} + \nu\psi_{-,x} - (2\delta^{-1}\bar{u}_t + 4i\hat{P}_+\bar{u}_{tx} - \kappa)\psi_-. \quad (2.31d)$$

It is easily verified by a cross differentiation, namely, $\psi_{+,tx} = \psi_{+,xt}$ that Eqs. (2.31) yield Eq. (1.1) as a compatibility condition. Notice also that Eqs. (2.31) must be complimented by the relation

$$(1+iT)\ln\frac{\psi_+}{\psi_{+,0}} - i\delta^{-1}\int_{-\infty}^{x} \ln\frac{\psi_+}{\psi_{+,0}}\,dx = (-1+iT)\ln\frac{\psi_-}{\psi_{-,0}} - i\delta^{-1}\int_{-\infty}^{x} \ln\frac{\psi_-}{\psi_{-,0}}\,dx, \quad (2.32)$$

which stems from (2.1), (2.2), and (2.30), where

$$\psi_{+,0} = g_{+,0}/f_{+,0} = g_+(-\infty,t)/f_+(-\infty,t), \quad (2.33a)$$

$$\psi_{-,0} = g_{-,0}/f_{-,0} = g_-(-\infty,t)/f_-(-\infty,t). \quad (2.33b)$$

A system of equations (2.31) and (2.32) constitute a complete set of the IST for Eq. (1.1).

Remark 3: One final comment to be noted here is concerned with the initial value of u_t. It can be obtained from $u(x,0)$ as follows. Let us introduce a function v through the relation

$$v = \int_x^{\infty} u_t\,dx, \quad (2.34)$$

and introduce (2.34) into Eq. (1.1). Integrating the resultant equation, one finds

$$-v + u + 2uv + Tv_x = 0. \quad (2.35)$$

If we solve Eq. (2.35) for a given initial condition $u(x,0)$, we can get $v(x,0)$, from which $u_t(x,0)$ follows immediately by differentiating (2.34).

D. Painlevé test

The Painlevé test provides a useful information about integrability of given system of equations. Ablowitz, Ramani, and Segur[15] conjectured that every ordinary differential equation (ODE) obtained by an exact reduction of a partial differential equation (PDE) solvable by the IST method possesses the Painlevé property, namely, solutions for ODEs have only poles as movable singularities. Thus the Painlevé property is seen to be closely related to integrability of PDEs. A drawback of the above-mentioned procedure is that one must always reduce the PDE to the ODE. In order to overcome this point, Weiss, Tabor, and Carnevale[16] have proposed a direct method that is applicable to PDEs themselves and showed that almost all soliton equations have the Painlevé property.[16,17]

Recently, Grammaticos, Dorizzi, and Ramani[19] showed that the conjecture by Ablowitz, Ramani, and Segur can also be applied to nonlinear integrodifferential evolution equations such as the BO equation and Eq. (1.5). In this subsection, following the idea due to Grammaticos, Dorizzi, and Ramani,[19] we show that Eq. (1.1) has the Painlevé property.

In the beginning, it should be observed that the problem under consideration is essentially a two-space dimensional problem. Indeed, Eq. (1.1) may be interpreted as an equation that describes internal waves propagating in the x direction in two-layered fluids, the depth of the bottom layer being δ while that of the upper layer being very shallow compared with the former one. Under this situation, the following Laplace equation for the velocity potential V must be satisfied in fluids:

$$V_{xx} + V_{yy} = 0,$$
$$V = V(x,y,t), \quad (-\infty < x < \infty, \ -\delta \leqslant y \leqslant 0), \tag{2.36a}$$

together with the boundary condition

$$V = V_0(x,t), \quad \text{at } y = 0. \tag{2.36b}$$

Although Grammaticos *et al.*[19] imposed another boundary condition $V_y = 0$ at $y = -\delta$, we found it inappropriate for Eq. (1.1). Indeed, this condition is shown to be incompatible with condition (2.38) given below. Now, Eq. (1.1) is equivalent to the following system of first-order PDEs:

$$U_t + V_x = 0, \quad \text{at } y = 0, \tag{2.37a}$$

$$U - V - V_y + 2UV = 0, \quad \text{at } y = 0, \tag{2.37b}$$

with a subsidiary condition

$$V_y = -TV_x, \quad \text{at } y = 0, \tag{2.38}$$

where $U = U(x,y,t)$. Indeed, if we put

$$U(x,0,t) = u(x,t), \tag{2.39a}$$

$$V(x,0,t) = v(x,t), \tag{2.39b}$$

then, the above fact readily follows from (2.34) and (2.35). Before performing the Painlevé test, it should be remarked that a solution of Eq. (2.36) satisfying the boundary conditions (2.36b) and (2.38) exists and it reads explicitly in the form

$$V(x,y,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dz\, V_0(z,t) \times \int_{-\infty}^{\infty} dk\, [\cosh ky + c(k)\sinh ky]\, e^{ik(x-z)}, \tag{2.40a}$$

where

$$c(k) = \coth k\delta - (k\delta)^{-1}. \tag{2.40b}$$

One can verify (2.40) by using the Fourier transform method and the formula (B4).

Now, let us begin the Painlevé test. First, note that the general solution of Eq. (2.36a) is readily written as

$$V = F(x+iy,t) + G(x-iy,t), \tag{2.41}$$

where F and G are arbitrary functions. The above expression implies that the singularities of Eq. (2.36a) propagate along the characteristics $x - iy = \chi(t)$ and $x + iy = \phi(t)$. Next, consider the singularity manifold $\psi = x + iy - \phi(t)$ where $\phi(t)$ is an arbitrary function of t and expand U and V around the singularity $\psi = 0$ as

$$U = \sum_{j=0}^{\infty} U_j \psi^{j-m} \quad (U_j = U_j(t), j = 0,1, \ldots), \tag{2.42a}$$

$$V = \sum_{j=0}^{\infty} V_j \psi^{j-n} + G(x-iy,t) \quad (V_j = V_j(t), j = 0,1, \ldots), \tag{2.42b}$$

where m and n are positive integers. These are determined by an analysis of the leading-order singularities in Eq. (2.37). The result is

$$m = n = 1 \tag{2.43}$$

Finally, substituting (2.42) with (2.43) into (2.37), expanding G around $\psi = 0$ as

$$G = \sum_{j=0}^{\infty} \frac{G^{(j)}}{j!} \psi^j \quad \left(G^{(j)} \equiv \frac{\partial^j G(x,t)}{\partial x^j}\bigg|_{x=\phi - iy} \right), \tag{2.44}$$

and then taking the limit $y \to 0$, we can determine the coefficients U_j and V_j successively by balancing various powers in ψ^j. We quote only the final results as follows:

$$U_0 = -i/2, \tag{2.45a}$$

$$V_0 = -i\phi_t/2, \tag{2.45b}$$

$$V_1 = -\phi_t U_1 - G^{(0)} + \phi_t/2 - \tfrac{1}{2}, \tag{2.45c}$$

$$(j+1)\phi_t U_{j+2} - (j+1)V_{j+2} = U_{j+1,t} + G^{(j+1)}/j! \quad (j \geqslant 0), \tag{2.45d}$$

$$\phi_t U_{j+2} + (j+2)V_{j+2} = -i(U_{j+1} - V_{j+1}) + iG^{(j)}/j! - jG^{(j+1)}/(j+1)! - 2i\sum_{k=0}^{j} U_{j-k+1}(V_{k+1} + G^{(k)}/k!) \quad (j \geqslant 0). \tag{2.45e}$$

In these expressions, U_1 is taken to be an arbitrary function, which means that a resonance condition for U_1 is satisfied automatically. From (2.45d) and (2.45e), we see that U_j and V_j ($j \geqslant 2$) are uniquely expressed in terms of the two arbitrary functions ϕ and U_1. Hence, the expansions (2.42) are of Painlevé type. This completes the proof that Eq. (1.1) has a Painlevé property.

Before concluding this subsection, we briefly discuss both the deep- and shallow-water limits of the results presented here. For the deep-water limit $\delta \to \infty$, it follows from (2.40b) that

$$c(k) = \text{sgn}(k) + O(\delta^{-1}). \tag{2.46}$$

Substituting (2.46) into (2.40a), we obtain

$$V_y\big|_{y=0} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dz\, V_0(z,t) \int_{-\infty}^{\infty} dk\, |k| e^{ik(x-z)} = -HV_{0,x}(x,t) = -H\,V_x\big|_{y=0}, \tag{2.47}$$

where the formula (B11) has been used. Then, Eq. (1.2) follows by combining Eqs. (2.37) and (2.47).

For the shallow-water limit $\delta \to 0$, since

$$c(k) = k\delta/3 + O(\delta^3), \tag{2.48}$$

the relation corresponding to (2.47) becomes

$$V_y\big|_{y=0} = \frac{\delta}{6\pi}\int_{-\infty}^{\infty} dz\, V_0(z,t) \int_{-\infty}^{\infty} dk\, k^2 e^{ik(x-z)} = -(\delta/3) V_{xx}\big|_{y=0}. \tag{2.49}$$

Thus we find that Eqs. (2.37) and (2.49) together with the scalings (1.3) reproduce Eq. (1.4). The above two limits imply that Eqs. (1.2) and (1.4) also possess the Painlevé property.

III. STUDY OF EQ. (1.2): DEEP-WATER LIMIT

In this section, we consider the deep-water limit $\delta \to \infty$ of various results presented in Sec. II. Since the limiting procedure can be performed quite simply, we shall be concerned only with conservation laws and related topics.

A. Conservation laws

In order to obtain conservation laws, we use the BT. The time part and the space part of the BT are derived from (2.14) and (2.16), respectively, by taking the limit $\delta \to \infty$. The results are expressed as follows:

$$w_{tx} - w_t H w_x - w_x H w_t = \mu w_t + \nu w_x - 2\bar{u}_{tx}, \tag{3.1}$$

$$e^{2iw} = -\frac{2iP_- w_x + 2iu - i + \mu}{2iP_+ w_x + 2iu - i - \mu}, \tag{3.2a}$$

where $P_\pm$ are projection operators given by

$$P_\pm = \tfrac{1}{2}(1 \pm iH). \tag{3.2b}$$

If we impose the boundary conditions

$$w(+\infty,t) = w(-\infty,t) = w_0, \tag{3.3}$$

(2.18) and (2.19) reduce, respectively, to

$$w_0 = (1/2i)\ln[(\nu - i)/(\nu + i)], \tag{3.4}$$

$$\mu = -1/\nu. \tag{3.5}$$

It is obvious from (3.1), (3.3), and (B12) that the quantity

$$\int_{-\infty}^{\infty} (w - w_0)\, dx$$

is conserved.

Let us now derive an infinite number of conservation laws of Eq. (1.2). We first expand w as

$$w = w_0 + \sum_{j=1}^{\infty} w_j \epsilon^j, \tag{3.6a}$$

where

$$\epsilon = -2\nu, \tag{3.6b}$$

and substitute (3.6) into (3.2). It turns out by using (3.4), (3.5), and (3.6b) that

$$2i(w - w_0) = \ln[1 + \epsilon^2/4 + i\epsilon(1 + i\epsilon/2)(u + P_- w_x)] - \ln[1 + \epsilon^2/4 - i\epsilon(1 - i\epsilon/2) \times (u + P_+ w_x)]. \tag{3.7}$$

One can also derive (3.7) directly from (2.27) by noting $\hat{P}_\pm \to P_\pm$ and $\alpha \to 1$ in the limit of $\delta \to \infty$. Now, $w_j\,(j = 1,2,\ldots)$ are determined successively by comparing the same powers of ϵ^j on both sides of (3.7). The first three of the nontrivial conservation laws are written explicitly as follows:

$$I_1 = \int_{-\infty}^{\infty} u\, dx, \tag{3.8a}$$

$$I_3 = -\int_{-\infty}^{\infty} \left[\tfrac{1}{3}(u - \tfrac{1}{2})^3 + \tfrac{1}{2}uHu_x + \tfrac{1}{24}\right] dx, \tag{3.8b}$$

$$I_5 = \int_{-\infty}^{\infty} \left[\tfrac{1}{5}(u - \tfrac{1}{2})^5 + \tfrac{1}{8}(\tfrac{16}{3}u^3 - 8u^2 + 4u - u_{xx})Hu_x + \tfrac{1}{2}u(Hu_x)^2 + \tfrac{1}{8}(2u - 3)u_x^2 + \tfrac{1}{160}\right] dx. \tag{3.8c}$$

The above expressions are also obtained directly by taking the limit $\delta \to \infty$ in (2.28).

B. Evaluation of conservation laws

In this subsection, we evaluate conservation laws for N-soliton solution of Eq. (1.2). A new method developed here is independent of the IST, and is based on a previous work by the author.[20] First, we derive the corresponding result for the one-soliton solution, namely,[1]

$$u = u_1 = a/[a^2(x - ct - x_{01})^2 + 1], \tag{3.9a}$$

where a is an amplitude such as $0 < a < 1$, x_{01} is a phase constant and c is a propagation velocity given by

$$c = (1 - a)^{-1}. \tag{3.9b}$$

For the above one-soliton solution, it is obvious that w must have the following simple functional dependence on t and x

$$w = w(x - ct - x_{01}), \tag{3.10}$$

so that

$$w_t = -c w_x. \tag{3.11}$$

Substitution of (3.11) into (3.1) leads, after integrating once with respect to x, to

$$w_x - 2\int_{-\infty}^{x} w_y H w_y\, dy = \left(\mu - \frac{\nu}{c}\right)(w - w_0) - 2u_1. \tag{3.12}$$

Integrating once again and using (3.5), (3.6b), and (B16), one arrives at the formula

$$\int_{-\infty}^{\infty} (w - w_0)\, dx = \frac{\pi\epsilon}{1 - (a - 1)(\epsilon/4)^2}. \tag{3.13}$$

This expression is interpreted as a generating function for conservation laws. The final step for obtaining I_j consists of substitution of (3.6) into (3.13) to yield

$$\sum_{j=1}^{\infty} I_j \epsilon^j = \pi \sum_{j=1}^{\infty} 2^{-2(j-1)}(a-1)^{j-1}\epsilon^{2j-1}, \qquad (3.14)$$

and comparison of the coefficients of ϵ^j on both sides of (3.14). The result is very simple and it reads as follows:

$$I_{2j+1} = \pi(a-1)^j/2^{2j} \quad (j=0,1,\ldots), \qquad (3.15a)$$

$$I_{2j} = 0 \quad (j=1,2,\ldots). \qquad (3.15b)$$

The generalization of (3.15) to N-soliton solution u_N is done quite easily since after lapse of large time, u_N is represented by a superposition of one-soliton solutions:[1]

$$u_N \sim \sum_{n=1}^{N} \frac{a_n}{a_n^2(x-c_nt-x_{0n})^2+1}, \qquad (3.16a)$$

with

$$c_n = (1-a_n)^{-1}, \quad 0<a_n<1 \quad (n=1,2,\ldots). \qquad (3.16b)$$

Thus the relations corresponding to (3.15) become

$$I_{2j+1} = \pi \sum_{n=1}^{N} \frac{(a_n-1)^j}{2^{2j}} \quad (j=0,1,\ldots), \qquad (3.17a)$$

$$I_{2j} = 0 \quad (j=1,2,\ldots). \qquad (3.17b)$$

C. Initial condition evolving into pure solitons

In order to solve the initial value problem of Eq. (1.2) for arbitrary initial conditions, one must solve the IST equations. However, if we use (3.17) and an explicit form of N-soliton solution, we can find an initial condition evolving into pure solitons after lapse of large time. For the BO equation, a useful method has already been developed for the purpose.[21] In the present case, it has been conjectured that the initial condition of the form

$$u(x,0) = N\lambda/(x^2+\lambda^2), \qquad (3.18)$$

would evolve into pure N solitons, where λ is a positive constant.[13] We now proceed to verify the conjecture at least up to $N=4$. Let an N-soliton solution of Eq. (1.2) expressed in terms of bilinear variable be $f_N(x,t)$. Then, it is obvious that the initial condition (3.18) evolves into pure N solitons if the equation

$$f_N(x,0) = (x-i\lambda)^N, \qquad (3.19)$$

holds identically for arbitrary values of x. For the first few N, the explicit forms of $f_N(x,0)$ are given as follows:[1]

$$f_2 = \xi_1\xi_2 - B_{12}, \qquad (3.20a)$$

$$f_3 = \xi_1\xi_2\xi_3 - B_{23}\xi_1 - B_{13}\xi_2 - B_{12}\xi_3, \qquad (3.20b)$$

$$f_4 = \xi_1\xi_2\xi_3\xi_4 - B_{34}\xi_1\xi_2 - B_{24}\xi_1\xi_3 - B_{23}\xi_1\xi_4 - B_{14}\xi_2\xi_3 - B_{13}\xi_2\xi_4 - B_{12}\xi_3\xi_4 + B_{12}B_{34} + B_{13}B_{24} + B_{14}B_{23}, \qquad (3.20c)$$

where

$$\xi_j = x - ia_j^{-1} \quad (j=1,2,\ldots,N), \qquad (3.20d)$$

$$B_{jk} = 2(2-a_j-a_k)/(a_j-a_k)^2 \quad (j\neq k, j,k=1,2,\ldots,N), \qquad (3.20e)$$

$$0<a_j<1 \, (j=1,2,\ldots,N). \qquad (3.20f)$$

In general, it is shown that f_N can be expressed in terms of Pfaffians.[14] One can observe by comparing the coefficients of x^j ($j=0,1,\ldots,N-1$) on both sides of Eq. (3.19) that it yields a system of N algebraic equations for N unknowns a_1, $a_2,\ldots,a_N$ and hence there exist solutions.

A general method to obtain a_j is as follows. First, introduce the following fundamental symmetric functions of a_j ($j=1,2,\ldots,N$):

$$s_1 = \sum_{j=1}^{N} a_j, \qquad (3.21a)$$

$$s_2 = \sum_{\substack{j,k=1\\(j<k)}}^{N} a_j a_k, \qquad (3.21b)$$

$$\vdots$$

$$s_N = \prod_{j=1}^{N} a_j, \qquad (3.21c)$$

and then rewrite the algebraic equations in terms of these new variables. This is always possible because of the symmetry property of N-soliton solution with respect to a_j ($j=1,2,\ldots,N$). By solving these equations, s_j ($j=1,2,\ldots,N$) are expressed as functions of λ. Finally, a_j ($j=1,2,\ldots,N$) are determined from the algebraic equation of order N

$$\sum_{j=0}^{N} (-1)^j s_j A^{N-j} = 0 \quad (s_0=1). \qquad (3.22)$$

It should be remarked, however, that certain restriction must be imposed on λ to satisfy conditions (3.20f). It is also interesting to observe that if (3.18) evolves into pure N solitons, s_j ($j=1,2,\ldots,N$) are calculated directly from the relations

$$\pi \sum_{n=1}^{N} (a_n-1)^j/2^{2j} = I_{2j+1}(0) \quad (j=1,2,\ldots), \qquad (3.23)$$

which stem from (3.17). Here, $I_{2j+1}(0)$ are conserved quantities evaluated at an initial time $t=0$ by using (3.18). One may also employ (3.23) to obtain approximate values of a_n for general initial conditions which would evolve into a train of solitons and tail parts.

Now, we write down the results for $N=2,3,4$.

(a) $N=2$

In this case, (3.22) becomes

$$A^2 - 2\lambda^{-2}(2\lambda-1)A + \lambda^{-3}(2\lambda-1) = 0, \qquad (3.24)$$

and a_1 and a_2 are given explicitly by

$$a_1 = \{2\lambda - 1 - [(\lambda-1)(2\lambda-1)]^{1/2}\}/\lambda^2, \qquad (3.25a)$$

$$a_2 = \{2\lambda - 1 + [(\lambda-1)(2\lambda-1)]^{1/2}\}/\lambda^2. \qquad (3.25b)$$

The condition for λ is now expressed as

$$\lambda > \lambda_m^{(2)} \cong 2.62, \qquad (3.26)$$

where $\lambda_m^{(2)}$ is the largest root of the algebraic equation

$$\lambda^2 - 3\lambda + 1 = 0. \qquad (3.27)$$

(b) $N=3$

$$A^3 - 9\lambda^{-2}(\lambda-1)A^2 + \tfrac{9}{4}\lambda^{-4}(8\lambda^2 - 12\lambda + 3)A - \tfrac{3}{4}\lambda^{-5}(8\lambda^2 - 12\lambda + 3) = 0, \qquad (3.28)$$

$$\lambda > \lambda_m^{(3)} \cong 4.36. \quad (3.29)$$

Here, $\lambda_m^{(3)}$ is the largest root of the algebraic equation

$$\lambda^3 - 6\lambda^2 + \tfrac{15}{2}\lambda - \tfrac{3}{2} = 0. \quad (3.30)$$

(c) $N = 4$

$$A^4 - 8\lambda^{-2}(2\lambda - 3)A^3 + 8\lambda^{-4}(4\lambda^2 - 10\lambda + 5)A^2 - 12\lambda^{-6}(8\lambda^3 - 24\lambda^2 + 18\lambda - 3)A + 3\lambda^{-7}(8\lambda^3 - 24\lambda^2 + 18\lambda - 3) = 0, \quad (3.31)$$

$$\lambda > \lambda_m^{(4)} \cong 6.15. \quad (3.32)$$

Here, $\lambda_m^{(4)}$ is the largest root of the algebraic equation

$$\lambda^4 - 10\lambda^3 + 27\lambda^2 - 21\lambda + 3 = 0. \quad (3.33)$$

For $N = 3,4$, the amplitudes may be written explicitly by using the famous formulas of Cardano and Ferrari. Instead of doing this, we have presented numerical values of a_j for some λ in Table I($N = 3$) and Table II($N = 4$), where the amplitudes are ordered such that $a_j < a_k$ for $j < k$.

Although the above procedure becomes complicated for large N, asymptotic expressions of the amplitudes $a_j\,(j = 1,2,\dots,N)$ for large values of λ are derived easily. To show this, by observing (3.25) we put

$$x = i\lambda y, \quad (3.34a)$$

$$a_j = \tilde{a}_j/\lambda \quad (j = 1,2,\dots,N), \quad (3.34b)$$

in (3.19) and take the limit $\lambda \to \infty$. Then, (3.19) becomes

$$\tilde{f}_N(y,0) = (y - i)^N, \quad (3.35)$$

where $\tilde{f}_N$ is an N-soliton solution of the BO equation[12,22] given by

$$\tilde{f}_N(y,0) = \det M, \quad (3.36a)$$

with the $N \times N$ matrix M:

$$M = (m_{jk}) = \begin{cases} y - i\tilde{a}_j^{-1} & (j = k), \\ 2/(\tilde{a}_j - \tilde{a}_k) & (j \neq k). \end{cases} \quad (3.36b)$$

Thus, we see from Ref. 21 that $\tilde{a}_j\,(j = 1,2,\dots,N)$ coincide with the N roots of the Laguerre polynomial of order N:

$$L_N(\tilde{A}) \equiv \sum_{j=0}^{N} (-1)^j \binom{N}{j} \frac{\tilde{A}^j}{j!} = 0, \quad (3.37a)$$

where $\binom{N}{j}$ is the binomial coefficient

$$\binom{N}{j} = \frac{N!}{(N-j)!\,j!}. \quad (3.37b)$$

For the special case of $N = 2$, the above results are easily confirmed with the aid of (3.25), (3.34b), and (3.37).

One final remark to be noted here is that the equation corresponding to Eq. (2.35), namely,

$$-v + u + 2uv + Hv_x = 0, \quad (3.38)$$

can be solved explicitly for the initial condition (3.18). We quote only the results. The solution of Eq. (3.38) is expressed in the form

$$v(x,0) = \sum_{j=1}^{N} \frac{\alpha_j}{(x^2 + \lambda^2)^j}. \quad (3.39)$$

Here, $\alpha_j\,(j = 1,2,\dots,N)$ are determined uniquely from the following system of linear algebraic equations

$$(1 - \lambda^{-1})\alpha_1 - \sum_{r=1}^{N-1} \frac{(2r)!}{(r!)^2 2^{2r} \lambda^{2r+1}} \alpha_{r+1} = N\lambda, \quad (3.40a)$$

$$2\lambda(N-j)\alpha_j - [1 - (j+1)/\lambda]\alpha_{j+1} + \sum_{r=j+2}^{N} \frac{[2(r-j-1)]!(r-j)r}{[(r-j)!]^2 2^{2(r-j-1)} \lambda^{2(r-j)-1}} \times \alpha_r = 0 \quad (j = 1,2,\dots,N-1). \quad (3.40b)$$

If we write these equations in a matrix form as

$$\hat{A}\boldsymbol{\alpha} = \boldsymbol{\beta}, \quad (3.41)$$

with the column vectors $\boldsymbol{\alpha} = (\alpha_1,\alpha_2,\dots,\alpha_N)^t$ and $\boldsymbol{\beta} = (N\lambda,0,\dots,0)^t$ and an $N \times N$ matrix $\hat{A}$, then the equation $\det \hat{A} = 0$ yields an algebraic equation of order N for λ. Let the largest root of the equation be $\lambda_m^{(N)}$. It is now conjectured that the restriction for λ would be $\lambda > \lambda_m^{(N)}$. The conjecture is found to be true for $N = 2,3,4$ for which the explicit forms of the algebraic equations are already given, respectively, by (3.27), (3.30), and (3.33). It is an interesting open problem to verify the conjecture for general N.

IV. STUDY OF EQ. (1.4): SHALLOW-WATER LIMIT

In this section, the shallow-water limit $\delta \to 0$ of the results presented in Sec. II is considered. Although the limiting procedure is somewhat complicated in comparison with the deep-water limit, it can be done straightforwardly. Hence, we shall not enter into the details of the derivations of the BT and the IST and describe only the final results. However, the properties of conservation laws are investigated in detail.

A. Conservation laws

We first derive the BT as a first step to obtain conservation laws. In the limit $\delta \to 0$, the function defined by (2.12) is expressed with the aid of (1.3), (2.1b), (2.1c), and (2.9) as

TABLE I. Amplitudes of solitons for $N = 3$.

λ	a_1	a_2	a_3
5.0	0.086	0.432	0.922
7.0	0.061	0.313	0.728
9.0	0.047	0.246	0.596
11.0	0.038	0.203	0.503
13.0	0.032	0.172	0.434

TABLE II. Amplitudes of solitons for $N = 4$.

λ	a_1	a_2	a_3	a_4
7.0	0.047	0.244	0.568	0.936
9.0	0.037	0.190	0.455	0.799
11.0	0.030	0.156	0.380	0.690
13.0	0.025	0.133	0.325	0.606
15.0	0.022	0.115	0.285	0.538

$$w = (3\delta)^{1/2} \frac{\partial}{\partial \xi} \ln \frac{g}{f} + O(\delta). \tag{4.1}$$

Suggested by this expression, we introduce the new variable W through the relation

$$W = \left(\frac{\delta}{3}\right)^{-1/2} w = 3 \frac{\partial}{\partial \xi} \ln \frac{g}{f}, \tag{4.2}$$

together with the scaled parameters $\tilde{\mu}$ and $\tilde{\nu}$

$$\tilde{\mu} = -(3\delta)^{-1/2}\mu, \tag{4.3a}$$

$$\tilde{\nu} = (3\delta)^{1/2}\nu. \tag{4.3b}$$

Substituting (1.3), (4.2), (4.3), and (B8) into (2.17) and expanding with δ, one finds after neglecting higher-order terms in δ:

$$W_\tau + \frac{1}{3} W \int_{-\infty}^{\infty} W_\tau \, d\xi = -2\tilde{u}_\tau + \frac{1}{3}\tilde{\nu} W + 1. \tag{4.4}$$

The boundary value $W_0 \equiv W_0(\pm\infty, \tau)$ is obtained by using (2.18), (4.2), and (4.3b) as

$$W_0 = -3/\tilde{\nu}. \tag{4.5}$$

Equation (4.4) with (4.5) represents the time part of the BT. If we take $\xi \to \infty$ in (4.4) and use (4.5), we obtain

$$\int_{-\infty}^{\infty} W_\tau \, d\xi = 0, \tag{4.6}$$

and this relation implies that the quantity

$$\tilde{I} \equiv \int_{-\infty}^{\infty} (W - W_0) d\xi \tag{4.7}$$

is conserved.

The space part of the BT follows similarly by introducing (2.18), (2.19), (4.2), (4.3), (4.5), and (B5) into (2.16) and performing the limiting procedure as follows:

$$W_{\xi\xi} + (W_\xi + 2u - 1) W + \tfrac{1}{9} W^3 = 3(\tilde{\nu}^{-1} - \tilde{\nu}^{-3}). \tag{4.8}$$

It is useful to note that Eq. (4.4) and Eq. (4.8) can also be derived from a pair of the BT of Eq. (1.4) expressed in terms of bilinear variables[11] with the aid of (4.2) and the formulas (A6)–(A8).

In order to derive conservation laws, we first introduce the following quantities

$$W_j = (\delta/3)^{(j-1)/2} w_j \quad (j = 1,2,...), \tag{4.9a}$$

$$\tilde{\epsilon} = (\delta/3)^{-1/2}\epsilon. \tag{4.9b}$$

It then follows from (2.25), (4.2), and (4.9) that

$$W = W_0 + \sum_{j=1}^{\infty} W_j \tilde{\epsilon}^j. \tag{4.10}$$

Notice that (2.25b), (4.3b), and (4.9b) yield a relation

$$\tilde{\epsilon} = 2\tilde{\nu}. \tag{4.11}$$

Expanding $\tilde{I}$ as

$$\tilde{I} = \sum_{j=1}^{\infty} \tilde{I}_j \tilde{\epsilon}^j, \tag{4.12}$$

and substituting (4.10) and (4.12) into (4.7), one finds that

$$\tilde{I}_j = \int_{-\infty}^{\infty} W_j \, d\xi \quad (j = 1,2,...), \tag{4.13}$$

are conserved quantities. If we use (4.9) and

$$W_0 = -3/\tilde{\nu} = -6/\tilde{\epsilon}, \tag{4.14}$$

Eq. (4.8) is rewritten in the form

$$W_{\xi\xi} + (W_\xi + 2u) W + \tfrac{1}{9}(W - W_0)^3 - (2/\tilde{\epsilon})(W - W_0)^2 - \left(1 - \frac{12}{\tilde{\epsilon}^2}\right)(W - W_0) = 0. \tag{4.15}$$

Finally, substituting (4.10) into (4.15) and taking the coefficients of $\tilde{\epsilon}^j (j = 1,2, ...)$ zero, we obtain the recursion relations that generate W_j. One then finds that only $\tilde{I}_j$ with odd j survive. Explicitly up to $j = 11$, they read as follows:

$$\tilde{I}_1 = \int_{-\infty}^{\infty} u \, d\xi, \tag{4.16a}$$

$$\tilde{I}_3 = \frac{1}{12} \int_{-\infty}^{\infty} u \, d\xi, \tag{4.16b}$$

$$\tilde{I}_5 = -\frac{1}{108} \int_{-\infty}^{\infty} \left(u^3 - \frac{3}{2} u^2 - \frac{3}{2} u_\xi^2 - \frac{3}{4} u\right) d\xi, \tag{4.16c}$$

$$\tilde{I}_7 = -\frac{1}{648} \int_{-\infty}^{\infty} \left(u^4 + \frac{1}{2} u^3 - \frac{9}{2} u u_\xi^2 + \frac{3}{4} u_{\xi\xi}^2 - \frac{9}{4} u^2 - \frac{3}{2} u_\xi^2 - \frac{3}{8} u\right) d\xi, \tag{4.16d}$$

$$\tilde{I}_9 = -\frac{7}{7776} \int_{-\infty}^{\infty} \left(u^4 - \frac{4}{7} u^3 - \frac{9}{2} u u_\xi^2 + \frac{3}{4} u_{\xi\xi}^2 - \frac{9}{14} u^2 + \frac{3}{28} u_\xi^2 - \frac{3}{56} u\right) d\xi, \tag{4.16e}$$

$$\begin{aligned} \tilde{I}_{11} = \frac{1}{157464} \int_{-\infty}^{\infty} \Big(& u^6 - 3u^5 - \frac{75}{4} u^3 u_\xi^2 - \frac{45}{16} u^4 \\ & + \frac{225}{8} u^2 u_\xi^2 + 9u^2 u_{\xi\xi}^2 - \frac{51}{16} u_\xi^4 + \frac{15}{4} u^3 + \frac{495}{32} u u_\xi^2 \\ & - 9u u_{\xi\xi}^2 - \frac{63}{32} u u_{\xi\xi\xi}^2 + \frac{3}{2} u_{\xi\xi}^2 + \frac{63}{64} u_{\xi\xi\xi}^2 + \frac{45}{32} u^2 \\ & - \frac{135}{64} u_\xi^2 - \frac{171}{64} u_{\xi\xi}^2 + \frac{9}{64} u_{\xi\xi\xi\xi}^2 + \frac{9}{128} u \Big) d\xi. \end{aligned} \tag{4.16f}$$

These quantities can also be derived from (2.28) by introducing the scalings $\tilde{I}_j = (\delta/3)^{-j/2} I_j (j = 1,2,...)$ in addition to (1.3). It is quite interesting to observe that in (4.16), only $\tilde{I}_1, \tilde{I}_5, \tilde{I}_7$, and $\tilde{I}_{11}$ are independent. Indeed, $\tilde{I}_3$ and $\tilde{I}_9$ are represented in terms of these quantities as

$$\tilde{I}_3 = \frac{1}{12} \tilde{I}_1, \tag{4.17a}$$

$$\tilde{I}_9 = \frac{1}{2304} \tilde{I}_1 - \frac{5}{48} \tilde{I}_5 + \frac{7}{12} \tilde{I}_7. \tag{4.17b}$$

The peculiar structure of these conserved quantities will be clarified in the following (see C).

B. Evaluation of conservation laws

We now develop a new method to evaluate conservation laws for N-soliton solution. For the purpose, we use actively the time part of the BT. In the beginning, we do this for a one-soliton solution of Eq. (1.4):

$$u = u_1 = 3a^2 \operatorname{sech}^2 a(\xi - c\tau - \xi_{01}) \quad [c = (1-4a^2)^{-1}]. \tag{4.18}$$

It then turns out that W has the following functional form

$$W = W(\xi - c\tau - \xi_{01}). \tag{4.19}$$

Substituting (4.19) into (4.4) and integrating once with respect to ξ, we obtain after some modifications

$$3W_\xi + (W - W_0)^2 + \left(-\frac{3}{\tilde{\nu}} + \frac{\tilde{\nu}}{c}\right)(W - W_0) + 6u_1 = 0. \tag{4.20}$$

This is an equation of the Riccati type. Owing to the second term on the left-hand side of Eq. (4.20), one cannot rely on the method developed in Sec. III B. Indeed, for Eq. (3.12) corresponding to Eq. (4.20), the second term on the left-hand side has been shown to vanish identically by integrating with respect to x from $-\infty$ to ∞. However, luckly in the present case, the difficulty is overcome as follows. We start by introducing new variables Y and η through the relations

$$W - W_0 = 3Y_\xi / Y, \tag{4.21a}$$

$$\eta = a(\xi - c\tau - \xi_{01}), \tag{4.21b}$$

and setting

$$\frac{-3}{\tilde{\nu}} + \frac{\tilde{\nu}}{c} = \frac{-6}{\tilde{\epsilon}} + \frac{(1-4a^2)\tilde{\epsilon}}{2} \equiv -6a\sigma. \tag{4.21c}$$

Then, Eq. (4.20) becomes

$$Y_{\eta\eta} - 2\sigma Y_\eta + 2 \operatorname{sech}^2 \eta Y = 0. \tag{4.22}$$

Furthermore, if we make a change of the independent variable η as

$$\zeta = (1 - \tanh \eta)/2, \tag{4.23}$$

Eq. (4.22) is transformed into the form

$$\zeta(1-\zeta) Y_{\zeta\zeta} + (1 + \sigma - 2\zeta) Y_\zeta + 2Y = 0. \tag{4.24}$$

Equation (4.24) is a special case of the following hypergeometric differential equation of Gauss[23]

$$\zeta(1-\zeta) Y_{\zeta\zeta} + [\gamma - (\alpha + \beta + 1)\zeta] Y_\zeta - \alpha\beta Y = 0, \tag{4.25}$$

with $\alpha = 2, \beta = -1$, and $\gamma = \sigma + 1$. As is well known,[23] a solution of Eq. (4.25) regular at $\zeta = 0$ is represented by an infinite series

$$Y = F(\alpha, \beta, \gamma; \zeta) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \frac{\zeta^n}{n!}, \tag{4.26a}$$

with

$$(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1). \tag{4.26b}$$

If we note the boundary condition $\lim_{|\xi|\to\infty} Y = \text{const}$ which is a consequence of (4.21a), we find that an appropriate solution of Eq. (4.24) satisfying the boundary condition is written in a finite series of the form

$$Y = F(2, -1, \sigma+1; \zeta) = 1 - [2/(\sigma+1)]\zeta. \tag{4.27}$$

Using the above expression, one can easily evaluate the quantities given by (4.13). Indeed, it follows from (4.7) and (4.21a) that

$$\begin{aligned} \tilde{I} &= [3 \ln Y]_{\eta=-\infty}^{\eta=+\infty} \\ &= [3 \ln Y]_{\zeta=1}^{\zeta=0} \\ &= 3 \ln[(\sigma+1)/(\sigma-1)]. \end{aligned} \tag{4.28}$$

Substitution of (4.12) and (4.21c) into (4.28) leads to

$$\sum_{j=1}^{\infty} \tilde{I}_j \tilde{\epsilon}^j = 3 \ln\left[\frac{1 + a\tilde{\epsilon} - (1-4a^2)\tilde{\epsilon}^2/12}{1 - a\tilde{\epsilon} - (1-4a^2)\tilde{\epsilon}^2/12}\right]. \tag{4.29}$$

Expanding the right-hand side of (4.29) with $\tilde{\epsilon}$ and then comparing the coefficients of $\tilde{\epsilon}^j$ on both sides, one arrives at the final result as follows:

$$\tilde{I}_{2j+1} = 6 \sum_{r=0}^{j} \frac{\binom{2j-r+1}{r}}{2j-r+1} \times \frac{a^{2j-2r+1}(1-4a^2)^r}{12^r} \quad (j = 0,1,\ldots), \tag{4.30a}$$

$$\tilde{I}_{2j} = 0 \quad (j = 1,2,\ldots). \tag{4.30b}$$

Explicitly, the nonzero $\tilde{I}_j$ up to $j = 11$ read in the forms

$$\tilde{I}_1 = 6a, \tag{4.31a}$$

$$\tilde{I}_3 = \frac{a}{2}, \tag{4.31b}$$

$$\tilde{I}_5 = -\frac{5}{12}\left(a^5 - \frac{5}{4}a^3 - \frac{5}{16}a\right), \tag{4.31c}$$

$$\tilde{I}_7 = -\frac{2}{63}\left(a^7 - \frac{21}{16}a^3 - \frac{7}{64}a\right), \tag{4.31d}$$

$$\tilde{I}_9 = -\frac{1}{54}\left(a^7 - \frac{3}{4}a^5 - \frac{3}{8}a^3 - \frac{1}{64}a\right), \tag{4.31e}$$

$$\tilde{I}_{11} = \frac{2}{891}\left(a^{11} - \frac{11}{4}a^9 - \frac{11}{16}a^7 + \frac{143}{64}a^5 + \frac{55}{128}a^3 + \frac{11}{1024}a\right). \tag{4.31f}$$

The generalization of the above results to those for an N-soliton solution is obvious. One may simply replace (4.30a) by the expressions

$$\tilde{I}_{2j+1} = 6 \sum_{r=0}^{j} \frac{\binom{2j-r+1}{r}}{2j-r+1} \times \sum_{n=1}^{N} \frac{a_n^{2j-2r+1}(1-4a_n^2)^r}{12^r} \quad (j = 0,1,\ldots), \tag{4.32}$$

where $3a_n^2$ represents the amplitude of the nth soliton.

C. Structure of conservation laws

As easily observed from the explicit expressions (4.31) for $\tilde{I}_{2j+1}$, all these quantities are not independent. One can confirm by direct calculation that the relations (4.17) hold for (4.31). This fact provides a useful information about the structure of conservation laws. In this subsection, we show that $\tilde{I}_{2j+1}$ $(j \neq 3n+1, n = 0,1,\ldots)$ are only independent conserved quantities. In other words, if $2j+1$ is equal to an odd integer times 3, namely, $2j+1 = 3(2n+1)$ $(n = 0,1,\ldots)$, then $\tilde{I}_{2j+1}$ are expressed as a linear combination of independent conserved quantities. It is sufficient to prove this statement only for (4.31) since the corresponding result for an N-soliton solution readily follows from (4.32).

First, we note from (4.29) that $\tilde{I}_{2j+1}$ is also written in the form

$$\tilde{I}_{2j+1} = \frac{6\cdot 12^{-(2j+1)/2}}{2j+1}[(\sqrt{3}a+\sqrt{1-a^2})^{2j+1} - (\sqrt{3}a-\sqrt{1-a^2})^{2j+1}]. \quad (4.33)$$

If we put

$$a = \cos\theta, \quad (4.34)$$

then, (4.33) becomes

$$\tilde{I}_{2j+1} = \frac{2\cdot 3^{-(2j-1)/2}}{2j+1} \times\left[\cos^{2j+1}\left(\theta-\frac{\pi}{6}\right)+\cos^{2j+1}\left(\theta+\frac{\pi}{6}\right)\right]. \quad (4.35)$$

Using the well-known formulas

$$\cos^{2j+1}\theta = 2^{-2j}\sum_{r=0}^{j}\binom{2j+1}{r}\cos(2j-2r+1)\theta, \quad (4.36a)$$

$$\cos\theta_1+\cos\theta_2 = 2\cos\tfrac{1}{2}(\theta_1+\theta_2)\cos\tfrac{1}{2}(\theta_1-\theta_2), \quad (4.36b)$$

(4.35) is modified as

$$\tilde{I}_{2j+1} = \frac{2^{-2(j-1)}3^{-(2j-1)/2}}{2j+1} \times\sum_{r=0}^{j}\binom{2j+1}{r}\cos\frac{\pi}{6}(2j-2r+1) \times\cos(2j-2r+1)\theta. \quad (4.37)$$

This is a desired form. However, it is sometimes more convenient to express (4.37) in terms of a. To do so, we use the formula

$$\cos(2j+1)\theta = \sum_{r=0}^{j}\frac{(-1)^r(2j+1)}{2(2j+1-r)}\binom{2j+1-r}{r} \times(2\cos\theta)^{2j-2r+1}. \quad (4.38)$$

Substitution of (4.34) and (4.38) into (4.37) yields, after some manipulations, the following alternative expression for $\tilde{I}_{2j+1}$:

$$\tilde{I}_{2j+1} = \frac{12^{-(j-1)}}{2\sqrt{3}(2j+1)}\sum_{r=0}^{j}(2a)^{2r+1} \times\sum_{s=0}^{j-r}\frac{(-1)^s(2r+2s+1)}{2r+s+1} \times\binom{2(j-r-s)+1}{j-r-s}\binom{2r+s+1}{s} \times\cos\frac{\pi}{6}[2(r+s)+1]. \quad (4.39)$$

It is seen from (4.39) that $\tilde{I}_{2j+1}$ with $2j+1\neq 3(2n+1)$ includes a term a^{2j+1} as a maximum power of a, but for $2j+1=3(2n+1)$ it lacks the term and instead has a^{2j-1} as a corresponding term.

Under these preparations, let us now prove the dependence of conserved quantities mentioned at the beginning of this section. For the purpose, consider the following quantity:

$$\tilde{J}_{6n+3} \equiv \sum_{j=0}^{3n}\alpha_{n,j}\tilde{I}_{2j+1} = \sum_{j=0}^{3n}\alpha_{n,j}\sum_{r=0}^{j}\binom{2j+1}{j-r}c_jF_r, \quad (4.40a)$$

where

$$F_j = \cos\frac{\pi}{6}(2j+1)\cos(2j+1)\theta, \quad (4.40b)$$

$$c_j = \frac{2^{-2}(j-1)3^{-(2j-1)/2}}{2j+1}, \quad (4.40c)$$

and $\alpha_{n,j}$ are unknown constants determined later. One then modifies (4.40) in the form

$$\tilde{J}_{6n+3} = \sum_{j=0}^{3n}\sum_{r=0}^{3n-j}\alpha_{n,j+r}\binom{2(j+r)+1}{r}c_{j+r}F_j. \quad (4.41)$$

Furthermore, if $\alpha_{n,j}$ $(j=0,1,\ldots,3n)$ satisfy the following system of linear algebraic equations:

$$\sum_{r=0}^{3n-j}\alpha_{n,j+r}\binom{2(j+r)+1}{r}c_{j+r} = \binom{6n+3}{3n+1-j}c_{3n+1} \quad (j=0,1,\ldots,3n), \quad (4.42)$$

then, (4.40) becomes

$$\tilde{J}_{6n+3} = \sum_{j=0}^{3n}\binom{6n+3}{3n+1-j}c_{3n+1}F_j. \quad (4.43)$$

However, one easily sees by noting $F_{3n+1}=0$ that $\tilde{J}_{6n+3}$ coincides with $\tilde{I}_{6n+3}$ and this means that $\tilde{I}_{6n+3}$ is expressed by a linear combination of $\tilde{I}_{2j+1}$ $(j=0,1,\ldots,3n)$. The existance of $\alpha_{n,j}$ is obvious since the determinant constructed from the coefficients of $\alpha_{n,j}$ in (4.42) is

$$\prod_{r=0}^{3m}c_r$$

and hence never vanishes due to (4.40c). Thus we have finished the proof.

Remark 1: The similar structure of conservation laws exists[24-26] for the Sawada–Kotera(SK) equation:[27]

$$u_\tau + 180u^2u_\xi + 30(uu_{\xi\xi\xi}+u_\xi u_{\xi\xi}) + u_{\xi\xi\xi\xi\xi} = 0, \quad (4.44)$$

for which the IST problem has been partially solved.[26,28]

Remark 2: The discussion on the dependence of conservation laws developed here is based on an N-soliton solution. However, since the conservation laws hold for arbitrary initial conditions, the conclusion obtained in this section gives a necessary condition for the dependence. Nevertheless, the investigation of lower conservation laws [see (4.17), for instance] strongly suggests that it is also sufficient. This is an interesting problem to be pursued further.

Remark 3: The IST equations arise naturally from (2.31) by noting the relations

$$\psi_\pm = \frac{g_\pm}{f_\pm} = \exp\left(\mp i\delta\frac{\partial}{\partial x}\right)\frac{g}{f}. \quad (4.45)$$

If we define the wave function ψ by

$$\psi = g/f, \quad (4.46)$$

substitute (4.46), (1.3), and (4.3) into (2.31) and take the limit $\delta\to 0$, we obtain the following system of linear differential equations

$$3\psi_{\tau\xi} = \tilde{v}\psi_\xi - (2\bar{u}_\tau - 1)\psi, \tag{4.47a}$$

$$\psi_{\xi\xi\xi} + (2u-1)\psi_\xi = \tilde{\mu}\psi, \tag{4.47b}$$

which constitute the IST of Eq. (1.4). It is important to remark that the space part of the IST (4.47b) is essentially the same form as that corresponding to the SK equation (4.44).[25,26]

V. CONCLUDING REMARKS

In this paper, starting from the bilinear form of a model equation for water waves in fluids of finite depth we have constructed the BT, an infinite number of conservation laws and the IST for the equation. Then, both the deep- and shallow-water limits of various results thus obtained have been investigated in detail. In particular, the structure of conservation laws has been clarified and it was found that it exhibits quite peculiar characteristics in comparison with those of usual water wave equations such as the BO and the KdV equations.

The most important open problem left in this paper would be to solve various IST equations. In this respect, it should be remarked that the space part of the IST equation for the shallow-water wave equation considered in Sec. IV has the same form as that for the SK equation. However, one must keep in mind that the IST problem for the SK equation has been solved only partially.[26,28] For instance, the explicit form of N-soliton solution has not been derived as yet within the framework of the IST formalism. Concerning this point, it may be instructive to note that Hirota recently found Pfaffian expressions of N-soliton solutions for the SK and related equations[29] and derived a new type of linear integral equation[30] that corresponds to the well-known Gel'fand–Levitan equation. The above-mentioned problems must be studied further in future works.

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APPENDIX A: FORMULAS OF BILINEAR OPERATORS

The following formulas are easily verified by direct calculations using the definition of bilinear operators (2.4), where f, f', g, and g' are functions of t and x and $\phi = \ln(f/g), \rho = \ln(fg)$:

$$D_x f\cdot f = 0, \tag{A1}$$

$$D_t(fg\cdot D_x f\cdot g) = D_x(fg\cdot D_t f\cdot g), \tag{A2}$$

$$f'g'D_x f\cdot g - (D_x f'\cdot g')fg = D_x g'f\cdot f'g, \tag{A3}$$

$$\begin{aligned} &f'g'D_t D_x f\cdot g - (D_t D_x f'\cdot g')fg \\ &\quad = (D_t D_x f'\cdot f)g'g - f'fD_t D_x g'\cdot g \\ &\qquad + D_t[g'f\cdot(D_x f'\cdot g)] + D_x[g'f\cdot(D_t f'\cdot g)], \end{aligned} \tag{A4}$$

$$\exp(i\delta D_x)f\cdot g = f(x+i\delta)g(x-i\delta), \tag{A5}$$

$$D_x f\cdot g/fg = \phi_x, \tag{A6}$$

$$D_t D_x f\cdot g/fg = \rho_{tx} + \phi_t\phi_x, \tag{A7}$$

$$D_x^3 f\cdot g/fg = \phi_{xxx} + 3\phi_x\rho_{xx} + \phi_x^3. \tag{A8}$$

APPENDIX B: PROPERTIES OF OPERATORS *T* AND *H*

The following formulas are obtained by employing the Fourier transform method together with the Chauchy residue theorem and the formulas

$$\begin{aligned} &[\coth k_1\delta + \coth k_2\delta]\coth(k_1+k_2)\delta \\ &\quad = \coth k_1\delta\coth k_2\delta + 1, \end{aligned} \tag{B1}$$

$$\lim_{\delta\to\infty}\coth k\delta = \mathrm{sgn}(k). \tag{B2}$$

In the following, $\tilde{f}(k)$ denotes the Fourier transform of $f(x)$:

$$\tilde{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx, \tag{B3}$$

and the existence of $\tilde{f}(0)$ is assumed.

1. *T* operator

$$Te^{ikx} = i[\coth k\delta - (1/k\delta)]e^{ikx}, \tag{B4}$$

$$\int_{-\infty}^{\infty}(fTg + gTf)dx = 0, \tag{B5}$$

$$\int_{-\infty}^{\infty} Tf\,dx = 0, \tag{B6}$$

$$\begin{aligned} &\int_{-\infty}^{\infty}dx\int_{-\infty}^{x} f_y Tf_y\,dy - \frac{1}{2\delta}\int_{-\infty}^{\infty} f^2\,dx \\ &\quad = -\pi\delta\int_{-\infty}^{\infty}\tilde{f}(k)\tilde{f}(-k)(k/\sinh k\delta)^2 dk, \end{aligned} \tag{B7}$$

$$Tf = \frac{\delta}{3}f_x + \frac{\delta^3}{45}f_{xxx} + \frac{2}{945}\delta^5 f_{xxxxx} + O(\delta^7). \tag{B8}$$

If we define the operator $\hat{T}$ by

$$\hat{T}f(x) = P\int_{-\infty}^{\infty}\frac{1}{2\delta}\coth\frac{\pi(y-x)}{2\delta}f(y)\,dy, \tag{B9}$$

then, for the functions such that $\lim_{k\to 0}\tilde{f}(k) = O(k)$ and $\lim_{k\to 0}\tilde{g}(k) = O(k)$

$$\hat{T}(f\hat{T}g + g\hat{T}f) = (\hat{T}f)(\hat{T}g) - fg. \tag{B10}$$

2. *H* operator

$$He^{ikx} = i\,\mathrm{sgn}(k)e^{ikx}, \tag{B11}$$

$$\int_{-\infty}^{\infty}(fHg + gHf)dx = 0, \tag{B12}$$

$$\int_{-\infty}^{\infty} Hf\,dx = 0, \tag{B13}$$

$$H^2 f = -f, \tag{B14}$$

$$H(fg) = H[(Hf)(Hg)] + fHg + gHf, \tag{B15}$$

$$\int_{-\infty}^{\infty}dx\int_{-\infty}^{x} f_y Hf_y\,dy = 0. \tag{B16}$$

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