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A class of exact solutions of Yang's K gauge equation for SU(2) gauge fields

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Using a simple ansatz, Yang's K gauge equation for SU(2) gauge fields is reduced to a system of nonlinear ordinary differential equations. Exact solutions for the equations are obtained together with corresponding gauge potentials.

I. INTRODUCTION

In search of the SU(2) gauge fields in four-dimensional Euclidean space, Yang derived conditions of self-duality and obtained nonlinear partial differential equations that describe gauge potentials.[1] His equations are divided into the two types according to the choice of the gauge. One is an equation with a Hermitian gauge or the K gauge, and the other is an equation with the R gauge. The latter equation has been studied extensively and various classes of exact solutions have been published.[2–5] The investigation of the former one, on the other hand, has scarcely been done.

The purpose of this paper is to construct exact solutions of Yang's self-dual equation with the K gauge. Since the equation itself is a quite complicated nonlinear partial differential equation for a vector field and hence it seems to be intractable, we shall introduce a simple ansatz that the fields depend only on one variable. Under the assumption, Yang's equation is reduced to a system of nonlinear ordinary differential equations. Exact solutions for the equations are constructed and corresponding gauge potentials are calculated explicitly.

II. EXACT SOLUTIONS

Yang's K gauge equation for a real vector field $\mathbf{v}$ is written in the form[1]

$$\tfrac{1}{2}(1-v^2)\mathbf{v}_{\mu\mu}+2(\mathbf{v}\cdot\mathbf{v}_\mu)\mathbf{v}_\mu-(\mathbf{v}_\mu\cdot\mathbf{v}_\mu)\mathbf{v} - 2(\mathbf{v}_1\times\mathbf{v}_2-\mathbf{v}_3\times\mathbf{v}_4)=0, \quad (v\equiv|\mathbf{v}|), \tag{2.1}$$

where the subscript μ indicates the differentiation with respect to the Euclidean coordinate x_μ and the repeated greek index μ runs from 1 to 4, i.e., $\mathbf{v}_{\mu\mu}=\Sigma_{\mu=1}^4\partial^2\mathbf{v}/\partial x_\mu^2$, for example.

We shall seek solutions of Eq. (2.1) that depend only on one variable ϕ, where ϕ is a function of x_μ $(\mu=1\sim4)$. Under this situation, the fourth term on the left-hand side of Eq. (2.1) vanishes identically and Eq. (2.1) is reduced to the equation

$$\tfrac{1}{2}(1-v^2)\mathbf{v}'\Delta\phi+\{\tfrac{1}{2}(1-\mathbf{v}^2)\mathbf{v}''+2(\mathbf{v}\cdot\mathbf{v}')\mathbf{v}'-(\mathbf{v}'\cdot\mathbf{v}')\mathbf{v}\} \times(\nabla\phi)^2=0. \tag{2.2}$$

Here, the prime appended to $\mathbf{v}$ denotes the differentiation with respect to ϕ, and Δ and ∇ are the Laplace and the gradient operators in four-dimensional Euclidean space, respectively. Furthermore, we may decouple Eq. (2.2) into the following two equations:

$$\tfrac{1}{2}(1-v^2)\mathbf{v}''+2(\mathbf{v}\cdot\mathbf{v}')\mathbf{v}'-(\mathbf{v}'\cdot\mathbf{v}')\mathbf{v}=0, \tag{2.3}$$

$$\Delta\phi=0. \tag{2.4}$$

These are the basic equations that we consider in this paper.

We shall now integrate Eq. (2.3). First, it follows from the vector product of $\mathbf{v}$ and Eq. (2.3) that

$$\tfrac{1}{2}(1-v^2)(\mathbf{v}\times\mathbf{v}')'+2vv'\mathbf{v}\times\mathbf{v}'=0, \tag{2.5}$$

which is readily integrated as

$$\mathbf{v}\times\mathbf{v}'=\mathbf{c}(1-v^2)^2, \tag{2.6}$$

where

$$\mathbf{c}=(c_1,c_2,c_3), \quad c=|\mathbf{c}| \tag{2.7}$$

is a real constant vector. Denoting the components of $\mathbf{v}$ by

$$\mathbf{v}=(F,G,H), \tag{2.8}$$

and introducing the functions f, g, and h through the relations

$$F=(1-v^2)f, \tag{2.9a}$$

$$G=(1-v^2)g, \tag{2.9b}$$

$$H=(1-v^2)h, \tag{2.9c}$$

the vector equation (2.6) is equivalent to the following equations:

$$fg'-f'g=c_1, \tag{2.10a}$$

$$gh'-g'h=c_2, \tag{2.10b}$$

$$hf'-h'f=c_3. \tag{2.10c}$$

From (2.10), one finds that g and h are expressed in terms of f as

$$g=-c_1 f\int^\phi\frac{d\phi}{f^2}, \tag{2.11}$$

$$h=-\frac{1}{c_1}\left(c_2-c_1c_3\int^\phi\frac{d\phi}{f^2}\right)f. \tag{2.12}$$

Next, taking the scalar product of Eq. (2.3) and $\mathbf{v}$ gives

$$\tfrac{1}{2}(1-v^2)(vv')'+2(vv')^2-\tfrac{1}{2}(1+v^2)\mathbf{v}'\cdot\mathbf{v}'=0. \tag{2.13}$$

Substituting the relation

$$\mathbf{v}'\cdot\mathbf{v}'=v'^2+c^2(1-v^2)^4/v^2, \tag{2.14}$$

which stems from the square of Eq. (2.6), into (2.13), one obtains the following ordinary differential equation for v:

$$\tfrac{1}{2}(1-v^2)(vv')'+2(vv')^2-\tfrac{1}{2}(1+v^2) \times\{v'^2+c^2(1-v^2)^4/v^2\}=0. \tag{2.15}$$

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If we introduce the variable P by the relation

$$v^2 = 1 - (2/(P+1)), \tag{2.16}$$

then Eq. (2.15) is considerably simplified and it reads in the form

$$(P^2 - 1)P'' - PP'^2 - 16c^2P = 0. \tag{2.17}$$

This equation is readily integrated to yield the solution

$$P = d\cosh(a\phi + b), \tag{2.18a}$$

with

$$d = \sqrt{1 + 16c^2/a^2}, \tag{2.18b}$$

where a and b are real integration constants. Therefore, we have

$$v^2 = 1 - 2/[d\cosh(a\phi + b) + 1]. \tag{2.19}$$

At this stage, the procedure to obtain $\mathbf{v}$ is straightforward. First, substitution of Eq. (2.11) and Eq. (2.12) into the relation $v^2 = F^2 + G^2 + H^2 = (1-v^2)^2 \times (f^2 + g^2 + h^2)$ yields

$$\int^Q \left\{ (c_1^2 + c_3^2)Q^2 - \frac{2c_2c_3}{c_1}Q + 1 + \frac{c_2^2}{c_1^2} \right\}^{-1} dQ = \int^\phi \frac{(1-v^2)^2}{v^2} d\phi, \tag{2.20}$$

where we have put

$$Q = \int^\phi \frac{d\phi}{f^2} \tag{2.21}$$

for simplicity. Integration of Eq. (2.20) is easily performed by noting (2.19). The result is

$$(c_1^2 + c_3^2)Q = (c_2c_3/c_1) + c\tan[\tan^{-1}\{(1/\sqrt{d^2-1}) \times \tanh(a\phi + b)\} + \theta], \tag{2.22}$$

where θ is a real constant.

Finally, it follows from (2.9), (2.11), (2.12), (2.21), and (2.22) that one obtains, after some tedious calculations, the explicit expressions for the vector $\mathbf{v} = (F,G,H)$ as follows:

$$F = \pm\alpha R\cosh(a\phi + b - \delta), \tag{2.23a}$$

$$G = \pm\beta R\cosh(a\phi + b + \epsilon), \tag{2.23b}$$

$$H = \pm\gamma R\cosh(a\phi + b - \eta), \tag{2.23c}$$

where

$$R = \{d\cosh(a\phi + b) + 1\}^{-1}, \tag{2.24a}$$

$$\alpha = c^{-1}\sqrt{(c_1^2 + c_3^2)(d^2\cos^2\theta - 1)}, \tag{2.24b}$$

$$\beta = c^{-1}\sqrt{\{d^2(c_2c_3\cos\theta + cc_1\sin\theta)^2 - (cc_1)^2 - (c_2c_3)^2\}/(c_1^2 + c_3^2)}, \tag{2.24c}$$

$$\gamma = c^{-1}\sqrt{\{d^2(c_1c_2\cos\theta - cc_3\sin\theta)^2 - (c_1c_2)^2 - (cc_3)^2\}/(c_1^2 + c_3^2)}, \tag{2.24d}$$

$$\delta = \tanh^{-1}(\tan\theta/\sqrt{d^2-1}), \tag{2.24e}$$

$$\epsilon = \tanh^{-1}\left\{\frac{1}{\sqrt{d^2-1}}\,\frac{cc_1 - c_2c_3\tan\theta}{c_2c_3 + cc_1\tan\theta}\right\}, \tag{2.24f}$$

$$\eta = \tanh^{-1}\left\{\frac{1}{\sqrt{d^2-1}}\,\frac{cc_3 + c_1c_2\tan\theta}{c_1c_2 - cc_3\tan\theta}\right\}. \tag{2.24g}$$

It should be remarked that the arbitrary constants included in (2.23) are c_1, c_2, c_3, a, b, and θ, while c and d are expressed by these constants [see (2.7) and (2.18b)] and hence (2.23) represents a general solution of Eq. (2.3). The solutions of Eq. (2.1) are then determined perfectly by (2.23) and solutions of Eq. (2.4). Although various classes of exact solutions exist for Eq. (2.4), we shall not discuss them here.

III. GAUGE POTENTIALS

The gauge potentials $\mathbf{b}_\mu$ ($\mu = 1\sim4$) in the K gauge are expressed in terms of $\mathbf{v}$ as follows[1]:

$$\mathbf{b}_1 = 2(\mathbf{v}\times\mathbf{v}_1 + \mathbf{v}_2)(1-v^2)^{-1}, \tag{3.1a}$$

$$\mathbf{b}_2 = 2(\mathbf{v}\times\mathbf{v}_2 - \mathbf{v}_1)(1-v^2)^{-1}, \tag{3.1b}$$

$$\mathbf{b}_3 = 2(\mathbf{v}\times\mathbf{v}_3 - \mathbf{v}_4)(1-v^2)^{-1}, \tag{3.1c}$$

$$\mathbf{b}_4 = 2(\mathbf{v}\times\mathbf{v}_4 + \mathbf{v}_3)(1-v^2)^{-1}. \tag{3.1d}$$

These quantities are easily evaluated by using (2.6) and (2.23). The results are expressed in the form

$$\mathbf{b}_1 = R(4\phi_1\mathbf{c} + \phi_2\mathbf{B}), \tag{3.2a}$$

$$\mathbf{b}_2 = R(4\phi_2\mathbf{c} - \phi_1\mathbf{B}), \tag{3.2b}$$

$$\mathbf{b}_3 = R(4\phi_3\mathbf{c} - \phi_4\mathbf{B}), \tag{3.2c}$$

$$\mathbf{b}_4 = R(4\phi_4\mathbf{c} + \phi_3\mathbf{B}), \tag{3.2d}$$

where the components of the vector $\mathbf{B} = (B_1,B_2,B_3)$ are given by

$$B_1 = \pm a\alpha\{\sinh(a\phi + b - \delta) - d\sinh\delta\}, \tag{3.3a}$$

$$B_2 = \pm a\beta\{\sinh(a\phi + b + \epsilon) + d\sinh\epsilon\}, \tag{3.3b}$$

$$B_3 = \pm a\gamma\{\sinh(a\phi + b - \eta) - d\sinh\eta\}, \tag{3.3c}$$

and $\phi_\mu = \partial\phi/\partial x_\mu$. The field strengths $\mathbf{f}_{\mu\nu}$, defined by

$$\mathbf{f}_{\mu\nu} = \frac{\partial\mathbf{b}_\mu}{\partial x_\nu} - \frac{\partial\mathbf{b}_\nu}{\partial x_\mu} - \mathbf{b}_\mu\times\mathbf{b}_\nu, \tag{3.4}$$

are then derived from (3.2), the explicit expressions of which are not written down here. One can observe from (3.2) and (3.3) that the gauge potentials take finite values provided that ϕ_μ ($\mu = 1\sim4$) are finite.

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