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New integrable nonlinear integrodifferential equations and related solvable finite dimensional dynamical systems

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A new integrable nonlinear integrodifferential equation (NIDE) is proposed. This equation may be interpreted as a model equation for deep-water waves. The N-periodic and N-soliton solutions for the equation are constructed by means of the bilinear transformation method. These solutions have the same structure as that for the Benjamin–Ono equation which describes internal waves in stratified fluids of great depth. Furthermore, it is shown that the motion of the positions of the poles of solutions is related to certain solvable finite-dimensional dynamical systems described by first-order nonlinear ordinary differential equations. The discussion is also made on a more general NIDE that may be interpreted as a model equation describing nonlinear waves in fluids of finite depth.

I. INTRODUCTION

Recently, much attention has been paid to integrable nonlinear integrodifferential equations (NIDE's) of both physical and mathematical interests such as the Benjamin–Ono (BO) equation,[1–5] the intermediate long wave (ILW) equation,[6,8] the sine–Hilbert equation,[9–11] and some other related NIDE's.[9,12,13] In this paper, we shall propose a new NIDE that exhibits exact N-periodic wave and N-soliton solutions. The equation that we consider here reads

$$u_t - Hu_{tx} - uu_t + u_x \int_x^\infty u_t\, dx + u_x = 0, \quad u = u(x,t), \tag{1.1a}$$

with

$$Hu(x,t) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{u(y,t)}{y-x}\, dy, \tag{1.1b}$$

where the operator H is the Hilbert transform [the symbol P in (1.1b) stands for the Cauchy principal value] and the abbreviations $u_t = \partial u/\partial t$, $u_x = \partial u/\partial x$, and $u_{tx} = \partial^2 u/\partial t\, \partial x$ have been used. Equation (1.1) includes both the definite and indefinite integrals and in this respect Eq. (1.1) is quite different from the known NIDE's mentioned above. We note that Eq. (1.1) is reduced to the following model equation for shallow water waves introduced by Hirota and Satsuma[14]:

$$u_t - u_{txx} - uu_t + u_x \int_x^\infty u_t\, dx + u_x = 0, \tag{1.2}$$

provided that the H operator is replaced formally by an x derivative. Mathematically, this formal derivation is entirely analogous to that of the Korteweg–de Vries (KdV) equation from the BO equation. Physically, a new NIDE (1.1) may be interpreted as a model equation that describes nonlinear waves in fluids of great depth.

In Sec. II, we analyze Eq. (1.1) by means of the well-known bilinear transformation method[15,16] and construct the N-periodic wave and N-soliton solutions. The latter solutions stem quite naturally from the long-wave limit of the former solutions. The initial value problem for a linearized version of Eq. (1.1) is also solved exactly in the last part of this section. In Sec. III, it is shown that the motion of the poles of solutions presented in Sec. II is closely related to certain solvable finite-dimensional dynamical systems described by first-order ordinary differential equations. It then follows from the integrability of Eq. (1.1) that solutions for the dynamical systems are determined by solving *algebraic equations* of order N. This remarkable fact implies an aspect of the integrability of the dynamical systems related to Eq. (1.1). In Sec. IV, we generalize Eq. (1.1) to a more general NIDE that is reduced to Eq. (1.1) and Eq. (1.2) in the deep-water and shallow-water limits, respectively. This equation may describe relevantly nonlinear waves in fluids of finite depth. The N-soliton and some rational solutions for the generalized NIDE are presented and subsequently we show that the NIDE is related to a solvable finite-dimensional dynamical system. In addition, the two limiting procedures, namely the deep-water and shallow-water limits, are taken for both solutions and a dynamical system obtained here. The results are consistent with corresponding solutions and a related dynamical system for Eq. (1.1), in the deep-water limit and those for Eq. (1.2), in the shallow-water limit, respectively. Section V is devoted to the conclusion.

II. EXACT SOLUTIONS

A. *N*-periodic wave solution

First, we focus our attention on a real and finite periodic-wave solution of Eq. (1.1) and seek it in the form

$$u = i\frac{\partial}{\partial x} \ln\left(\frac{f_+}{f_-}\right), \quad f_\pm = f_\pm(x,t), \tag{2.1}$$

where $f_+(f_-)$ is a complex analytic function with zeros lying only in the lower (upper) half complex x plane. The dependent variable transformation (2.1) is the same as that used for the BO[1,2] and the ILW[6,7] equations. It then follows by using the property of the H operator that

$$Hu = -\frac{\partial}{\partial x} \ln(f_+ f_-). \tag{2.2}$$

Now, substituting (2.1) and (2.2) into Eq. (1.1) and integrating once with respect to x, Eq. (1.1) is transformed into the following bilinear equation for f_+ and f_-:

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$$(iD_t + D_t D_x + iD_x) f_+ \cdot f_- = 0, \tag{2.3}$$

where the integration constant has been taken to be zero and the bilinear operators D_t and D_x are defined by the relation

$$D_t^n D_x^m f_+ \cdot f_- = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \times f_+(x,t) f_-(x',t')\Big|_{\substack{x'=x\\ t'=t}} \quad (n,m = 0,1,...). \tag{2.4}$$

Applying the standard procedure in the bilinear transformation method[15,16] to Eq. (2.3), we have found the following N-periodic wave solution of Eq. (1.1):

$$u = -\sum_{j=1}^{N} k_j + i \frac{\partial}{\partial x} \ln\left(\frac{f_+}{f_-}\right), \tag{2.5a}$$

$$f_- = \sum_{\mu = 0,1} \exp\left[\sum_{j=1}^{N} \mu_j (i\xi_j + \phi_j) + \sum_{j<l}^{(N)} \mu_j \mu_l A_{jl}\right], \tag{2.5b}$$

$$f_+ = f_-^* \quad (*\text{: complex conjugate}), \tag{2.5c}$$

$$\xi_j = k_j (x - a_j t - x_{0j}) + \xi_j^{(0)} \quad (j = 1,2,...,N), \tag{2.5d}$$

$$a_j = (1 - k_j \coth \phi_j)^{-1}, \quad \phi_j / k_j > 0 \quad (j = 1,2,...,N), \tag{2.5e}$$

$$\exp A_{jl} = \frac{(a_j - a_l)^2 - a_j a_l (a_j k_j - a_l k_l)(k_j - k_l)}{(a_j - a_l)^2 - a_j a_l (a_j k_j + a_l k_l)(k_j + k_l)}. \tag{2.5f}$$

Here, $\Sigma_{\mu=0,1}$ denotes the summation over all possible combinations of $\mu_1 = 0,1, \mu_2 = 0,1,..., \mu_N = 0,1$, $\Sigma_{j<l}^{(N)}$ means the summation under the condition $j<l$ and k_j, ϕ_j, x_{0j}, and $\xi_j^{(0)}$ $(j = 1,2,...,N)$ are real constants.

For $N = 1$, the solution (2.5) is written explicitly in the form

$$u = \frac{k_1 \tanh \phi_1}{1 + \operatorname{sech} \phi_1 \cos \xi_1} \quad \left(\frac{\phi_1}{k_1} > 0\right), \tag{2.6}$$

which represents a real and finite one-periodic wave solution of Eq. (1.1). Except for the phase velocity a_1, the functional form of (2.6) coincides with the periodic solution of the BO equation presented by Benjamin[17] and Ono.[18] Note that in the BO case, $a_1 = k_1 \coth \phi_1$.

B. *N*-soliton solution

The N-soliton solution is easily constructed by taking the long-wave limit of the N-periodic wave solution (2.5). To show this, we set in (2.5d),

$$\xi_j^{(0)} = \pi \quad (j = 1,2,...,N), \tag{2.7}$$

and take the long-wave limit $k_j \to 0$ $(j = 1,2,...,N)$ with the phase velocities a_j $(j = 1,2,...,N)$ keeping finite values. It then turns out that

$$A_{jl} = \frac{2(a_j + a_l) a_j a_l}{(a_j - a_l)^2} k_j k_l + O(k_j^4). \tag{2.8}$$

Introducing the expansion (2.8) into (2.5b), one finds, in the long-wave limit, the explicit expression of the N-soliton solution of Eq. (1.1) as follows:

$$u = i \frac{\partial}{\partial x} \ln\left(\frac{f^*}{f}\right), \tag{2.9a}$$

$$f = \det M. \tag{2.9b}$$

Here, M is an $N \times N$ matrix whose elements are given by

$$M_{jk} = \begin{cases} i(x - a_j t - x_{0j}) + a_j/(a_j - 1), & \text{for } j = k, \\ [2(a_j + a_k) a_j a_k]^{1/2}/(a_j - a_k), & \text{for } j \neq k, \end{cases} \tag{2.9c}$$

and the phase velocities are restricted by the conditions $a_j > 1$ and $a_j \neq a_k$ for $j \neq k$ $(j, k = 1,2,...,N)$. It is interesting to note that the N-soliton solution of the BO equation

$$u_t + 2uu_x + Hu_{xx} = 0, \tag{2.10}$$

is expressed in the form[1,16]

$$u = i \frac{\partial}{\partial x} \ln\left(\frac{\tilde{f}^*}{\tilde{f}}\right), \tag{2.11a}$$

$$\tilde{f} = \det \tilde{M}, \tag{2.11b}$$

with an $N \times N$ matrix $\tilde{M}$ given by

$$\tilde{M}_{jk} = \begin{cases} i(x - \tilde{a}_j t - \tilde{x}_{0j}) + 1/\tilde{a}_j, & \text{for } j = k, \\ 2/(\tilde{a}_j - \tilde{a}_k), & \text{for } j \neq k, \end{cases} \tag{2.11c}$$

where $\tilde{a}_j$ $(j = 1,2,...,N)$ are positive constants such that $\tilde{a}_j \neq \tilde{a}_k$ for $j \neq k$ and $\tilde{x}_{0j}$ $(j = 1,2,...,N)$ are arbitrary phase constants. Therefore we see that the N-soliton solution of Eq. (1.1) has the same structure as that of the BO equation.

The one-soliton solution is readily derived from (2.9) with $N = 1$. It takes a Lorentzian profile as

$$u = \frac{2b_1}{(x - a_1 t - x_{01})^2 + b_1^2} \quad \left(b_1 = \frac{a_1}{a_1 - 1}, \ a_1 > 1\right). \tag{2.12}$$

The amplitude and the velocity of the soliton (2.12) are given, respectively, by $2(a_1 - 1)/a_1$ and a_1. Hence one can observe that the amplitude approaches a constant value 2 indefinitely when the velocity becomes large while it approaches zero in the limit of $a_1 \to 1$. Asymptotic behavior of the N-soliton solution (2.9) for large time is easily obtained following the same argument as that for the BO case.[1,16] The result is expressed simply as a superposition of N independent algebraic solitons as follows:

$$u \underset{t \to \pm\infty}{\sim} \sum_{j=1}^{N} \frac{2b_j}{(x - a_j t - x_{0j})^2 + b_j^2} \quad [b_j = a_j/(a_j - 1), \quad a_j > 1]. \tag{2.13}$$

This asymptotic expression shows that no phase shift appears as the result of collisions of solitons in contrast to collisions that take place between the KdV solitons. Thus we have presented the second example of the one space-dimensional algebraic N-soliton solution that is real and finite over all x and t. The first example is, of course, that of the BO equation.[1]

C. Solution for a linearized equation

Here, we consider the initial value problem for a linearized version of Eq. (1.1), namely

$$u_t - Hu_{tx} + u_x = 0, \tag{2.14}$$

with the boundary condition $u(x,t) \to 0$ as $|x| \to \infty$. If $u(x,t)$ is represented in the form of the Fourier transform

$$u(x,t) = \int_{-\infty}^{\infty} v(k) \exp[i(kx - \omega t)] dk, \tag{2.15}$$

we obtain the dispersion relation

$$\omega = k/(1+|k|), \tag{2.16}$$

with the aid of the formula

$$\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{e^{ikx}}{x} dx = i \frac{k}{|k|}. \tag{2.17}$$

It is interesting to note that for small k, (2.16) behaves like

$$\omega = k - k|k| + O(k^3). \tag{2.18}$$

If we retain only up to the second term in the expansion, the expression (2.18) coincides perfectly with the dispersion relation of the following linearized BO equation

$$u_t + u_x + Hu_{xx} = 0. \tag{2.19}$$

This fact may suggest the suitability for interpreting Eq. (1.1) as a model equation which describes nonlinear wave propagations in fluids of great depth. In comparison with the dispersion relation of Eq. (2.19), Eq. (2.16) is well behaved for a wide range of the values of k, in particular for large k and hence Eq. (1.1) may be more relevant than the BO equation itself as a model equation for deep-water waves.

Now, the unknown function $v(k)$ appeared in (2.15) is determined from the initial value $u(x,0)$ as

$$v(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(x,0) e^{-ikx} dx. \tag{2.20}$$

Substituting (2.16) and (2.20) into (2.15), we obtain a general solution of Eq. (2.14) as follows:

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(y,0) \times \exp\{i[k(x-y) - (1+|k|)^{-1}kt]\} dk\, dy. \tag{2.21}$$

To investigate asymptotic behaviors of (2.21) for large time is an interesting problem. But we shall not be concerned with this problem here and the details will be reported elsewhere.

III. DYNAMICAL SYSTEMS RELATED TO EQ. (1.1)

In this section, we consider the dynamical systems related to Eq. (1.1). The relationships between integrable nonlinear evolution equations and solvable finite-dimensional dynamical systems have been studied extensively by many authors.[19–23] The basic idea due to Kruskal[19] is to investigate the time evolution of the positions of the poles of solutions of nonlinear evolution equations. In the following, we show that Eq. (1.1) is related to certain solvable finite-dimensional dynamical systems. The periodic and nonperiodic dynamical systems are treated separately.

A. Periodic dynamical system

We first consider the periodic dynamical system. As easily seen from (2.5a)–(2.5c), it is possible to express the periodic wave solution (2.5) with $k_j = k$ $(j = 1,2,...,N)$ in the form

$$u = i \frac{\partial}{\partial x} \ln\left(\frac{\tilde{f}_+}{\tilde{f}_-}\right), \tag{3.1a}$$

$$\tilde{f}_- = \prod_{j=1}^{N} \frac{2}{k} \sin\left[\frac{k}{2}(x - x_j)\right], \quad x_j = x_j(t), \tag{3.1b}$$

$$\tilde{f}_+ = \tilde{f}_-^*, \tag{3.1c}$$

where x_j $(j = 1,2,...,N)$ are complex functions of t whose imaginary parts are all positive, i.e., $\mathrm{Im}\, x_j > 0$ and $x_j \neq x_k$ for $j \neq k$ $(j,k = 1,2,...,N)$. The functions x_j represent positions of the complex poles of the periodic-wave solution (2.5) with $k_j = k$ $(j = 1,2,...,N)$. In order to find the time evolution of x_j, we substitute (3.1) into Eq. (2.3), use the trigonometric identity

$$\cot A \cot B = -1 - (\cot A - \cot B)\cot(A - B), \tag{3.2}$$

and then equate the coefficient of $\cot[k(x - x_j)/2]$ zero. The resultant equations for x_j are written in the form

$$\dot{x}_j = 1 - i\frac{k}{2} \sum_{\substack{l=1 \\ (l \neq j)}}^{N} (\dot{x}_j + \dot{x}_l) \cot\left[\frac{k(x_j - x_l)}{2}\right] + i\frac{k}{2} \sum_{l=1}^{N} (\dot{x}_j + \dot{x}_l^*) \cot\left[\frac{k(x_j - x_l^*)}{2}\right] \quad (j = 1,2,...,N), \tag{3.3}$$

where the dot appended to x_j and x_l means the time differentiation. One can also obtain from the coefficient of $\cot[k(x - x_j^*)/2]$ the complex conjugate expressions of Eqs. (3.3). For $N = 1$, Eq. (3.3) reads

$$\dot{x}_1 = 1 + k(\mathrm{Re}\, \dot{x}_1)\coth(k\, \mathrm{Im}\, x_1), \tag{3.4}$$

and this equation is readily integrated to yield the solution

$$x_1 = (1 - k \coth \phi_1)^{-1} t + x_{01} + i\phi_1/k. \tag{3.5}$$

Substituting (3.5) into (3.1), we recover the one-periodic wave solution (2.6). For general N, on the other hand, Eqs. (3.3) and their complex conjugate expressions constitute the system of $2N$ algebraic equations for $\dot{x}_j$ and $\dot{x}_j^*$ $(j = 1,2,...,N)$ and hence it is possible by using Cramer's formula to express these variables in terms of x_n and x_n^* $(n = 1,2,...,N)$ in the form

$$\dot{x}_j = F_j \quad (j = 1,2,...,N), \tag{3.6}$$

together with their complex conjugate expressions, where the F_j are uniquely determined functions of x_n and x_n^* $(n = 1,2,...,N)$. The explicit functional forms of F_j will not be written here. The system of equations (3.6) consists of quite complicated first-order nonlinear ordinary differential equations and hence it could not be solved analytically in general. Nevertheless, in the present situation, one can obtain explicit periodic-wave solutions of Eqs. (3.6). In order to clarify this statement, we compare the expressions (2.5a)–(2.5c) with $k_j = k$ $(j = 1,2,...,N)$ and the expressions (3.1a)–(3.1c). It then follows from (2.5b) with $k_j = k$ that

$$f_- = c_0 e^{ikNx} + c_1 e^{ik(N-1)x} + \cdots + 1 = c_0[z^N + (c_1/c_0)z^{N-1} + \cdots + c_0^{-1}] \quad (z = e^{ikx}), \tag{3.7}$$

where c_j $(j = 1,2,...,N)$ are known functions expressible in terms of t and various constant parameters. Consequently, the $\exp(ikx_j)$ are determined by solving the *algebraic equation* of order N, $f_- = 0$, with f_- being given by (3.7). In other words, this result reveals an aspect of the complete integrability of the system of equations (3.6).

B. Nonperiodic dynamical system

Next, we investigate the nonperiodic dynamical system. The time evolutions for x_j $(j=1,2,...,N)$ are derived quite naturally by taking the long-wave limit, $k\to 0$ in Eqs. (3.3). The equations corresponding to Eqs. (3.3) are written in the form

$$\dot{x}_j = 1 - i\sum_{\substack{l=1\\(l\neq j)}}^{N}\frac{\dot{x}_j+\dot{x}_l}{x_j-x_l} + i\sum_{l=1}^{N}\frac{\dot{x}_j+\dot{x}_l^*}{x_j-x_l^*} \quad (j=1,2,...,N). \tag{3.8}$$

The solutions for this system of equations are readily found by solving the *algebraic equation* of order N, $f=0$, where f is given by (2.9b) with (2.9c). Finally, it should be remarked that the dynamical system related to the BO equation (2.10) is completely integrable and it is expressed in the form[3,4]

$$\dot{x}_j = -2i\sum_{\substack{l=1\\(l\neq j)}}^{N}\frac{1}{x_j-x_l} + 2i\sum_{l=1}^{N}\frac{1}{x_j-x_l^*} \quad (j=1,2,...,N). \tag{3.9}$$

IV. GENERALIZATION TO MORE GENERAL NIDE

In this section, we generalize Eq. (1.1) to a more general NIDE that is reduced to Eq. (1.1) in the deep-water limit and to Eq. (1.2) in the shallow-water limit, respectively, and construct the N-soliton solution together with some rational solutions for the NIDE. We also investigate the motion of the poles of the N-soliton solution to show that the generalized NIDE is related to certain solvable finite-dimensional dynamical systems. Since the discussion is almost the same as that for Eq. (1.1), we shall not enter into detail but present only the main results.

A generalized version of Eq. (1.1) which we propose here reads

$$u_t - Tu_{tx} - uu_t + u_x\int_x^{\infty}u_t\,dx + u_x = 0, \quad u=u(x,t), \tag{4.1a}$$

with the operator T defined by

$$Tu(x,t) = \frac{1}{2\delta}P\int_{-\infty}^{\infty}\left\{\coth\left[\frac{\pi(y-x)}{2\delta}\right] - \mathrm{sgn}(y-x)\right\}u(y,t)dy, \tag{4.1b}$$

where δ is a positive parameter that may be interpreted as a depth of fluids. The T operator has been first introduced by Joseph[24–26] in the context of his NIDE which describes nonlinear waves in stratified fluids of finite depth. Presently, his equation is known as the ILW equation.[6–8] In the deep-water limit $\delta\to\infty$ the T operator is reduced to the H operator defined by (1.1b) while in the shallow-water limit $\delta\to 0$ it takes the form

$$Tu = \delta u_x/3 + \delta^3 u_{xxx}/45 + O(\delta^5). \tag{4.2}$$

Therefore Eq. (4.1) is an intermediate version between Eq. (1.1) and Eq. (1.2).

A. N-soliton solution

First, introduce the following dependent variable transformation:

$$u = i\frac{\partial}{\partial x}\ln\left(\frac{f_+}{f_-}\right), \tag{4.3a}$$

with

$$f_+(x,t) = f(x-i\delta,t), \tag{4.3b}$$

$$f_-(x,t) = f(x+i\delta,t), \tag{4.3c}$$

where the complex function $f(z,t)$ is such that $f(z-i\delta,t)$ has no zero in the region $0\leqslant \mathrm{Im}\, z\leqslant 2\delta$. Then, it is straightforward by using the Cauchy residue theorem to show that

$$Tu_x = -\frac{\partial^2}{\partial x^2}\ln(f_+f_-) + \delta^{-1}u. \tag{4.4}$$

Substituting (4.3a) and (4.4) into Eq. (4.1) and integrating once with respect to x, we obtain the bilinear form of Eq. (4.1) as follows:

$$[i(1-\delta^{-1})D_t + D_tD_x + iD_x]f_+\cdot f_- = 0. \tag{4.5}$$

In comparison with Eq. (2.3), Eq. (4.5) differs only from a numerical factor in front of the operator D_t and therefore it may share many of the integrability properties of Eq. (2.3).

The procedure for constructing the N-soliton solution of Eq. (4.5) is a routine work in the context of the bilinear formalism.[15,16] The result is written compactly as follows:

$$f_- = f_+^* = \sum_{\mu=0,1}\exp\left[\sum_{n=1}^{N}\mu_n(\delta^{-1}\gamma_n\theta_n + i\gamma_n) + \sum_{j<m}^{(N)}\mu_j\mu_m B_{jm}\right], \tag{4.6a}$$

with

$$\theta_n = x - a_nt - x_{0n} \quad (n=1,2,...,N), \tag{4.6b}$$

$$a_n = (1-\delta^{-1}+\delta^{-1}\gamma_n\cot\gamma_n)^{-1}, \quad 0<\gamma_n<\pi \quad (n=1,2,...,N), \tag{4.6c}$$

$$\exp B_{jm} = \frac{\delta^2(1-\delta^{-1})(a_j-a_m)^2 + a_ja_m(a_j\gamma_j - a_m\gamma_m)(\gamma_j-\gamma_m)}{\delta^2(1-\delta^{-1})(a_j-a_m)^2 + a_ja_m(a_j\gamma_j + a_m\gamma_m)(\gamma_j+\gamma_m)}, \tag{4.6d}$$

where γ_n and x_{0n} $(n=1,2,...,N)$ are constants. The explicit one-soliton solution follows from (4.3a) and (4.6) with $N=1$ as

$$u = \frac{\delta^{-1}\gamma_1\sin\gamma_1}{\cosh[\delta^{-1}\gamma_1(x-a_1t-x_{01})] + \cos\gamma_1}, \tag{4.7}$$

which has the same functional form as that of the one-soliton solution of the ILW equation.[6–8,24] The interaction between solitons is easily investigated by using the explicit formula (4.6) for the N-soliton solution. The asymptotic form of the N-soliton solution is simply expressed as a superposition of N independent one-soliton solutions (4.7). In this case, however, the phase shift appears as the result of collisions between solitons. Since the explicit formula for the phase shift is easily derived following a procedure similar to that for the ILW equation,[6] the result will not be presented here.

We can also obtain more general periodic solutions of Eq. (4.1) which are expressed in terms of Riemann's theta function on the basis of the bilinear equation (4.5). The procedure for constructing solutions is almost the same as that for the ILW equation.[16,27] Details will not be discussed here.

Now, we consider the deep- and shallow-water limits, respectively, of the bilinear equation (4.5) and the N-soliton solution (4.6).

1. Deep-water limit: $\delta \to \infty$

In this limit, Eq. (4.5) is reduced to Eq. (2.3) as expected. For the purpose of a limiting procedure for the N-soliton solution (4.6), it is appropriate to introduce the positive constants b_n through the relations

$$\gamma_n = \pi(1 - b_n/\delta) \quad (n = 1,2,\dots,N). \tag{4.8}$$

Then, it follows in the deep-water limit $\delta \to \infty$ that

$$\cot \gamma_n = -\delta/\pi b_n + \pi b_n/3\delta + O(\delta^{-3}), \tag{4.9a}$$

$$a_n = b_n/(b_n - 1) \quad (b_n > 1), \tag{4.9b}$$

$$B_{jm} = -\frac{2(a_j + a_m)a_j a_m}{(a_j - a_m)^2}\left(\frac{\pi}{\delta}\right)^2 + O(\delta^{-4}). \tag{4.9c}$$

Substituting (4.8) and (4.9) into (4.3) and (4.6), one finds that the N-soliton solution coincides perfectly with that of Eq. (1.1), namely the expression (2.9). The one-soliton solution (4.7) is of course reduced to (2.12), the one-soliton solution of Eq. (1.1).

2. Shallow-water limit: $\delta \to 0$

In this limit, it is appropriate to introduce the variables $\tilde{t}$ and $\tilde{x}$ by

$$t = (\delta/3)^{1/2}\tilde{t}, \tag{4.10a}$$

$$x = (\delta/3)^{1/2}\tilde{x}. \tag{4.10b}$$

Then, it is easy to show by using the properties of the bilinear operator,[16]

$$\exp[-i\delta D_x] f(x)\cdot f(x) = f(x - i\delta) f(x + i\delta), \tag{4.11a}$$

$$D_t D_x^{2m} f\cdot f = 0 \quad (m = 0,1,2,\dots), \tag{4.11b}$$

that

$$\begin{aligned} D_t\, f_+\cdot f_- &= D_t \exp(-i\delta D_x) f\cdot f \\ &= -3i D_{\tilde{t}} D_{\tilde{x}} f\cdot f + \tfrac{3}{2} i\delta D_{\tilde{t}} D_{\tilde{x}}^3 f\cdot f + O(\delta^2), \end{aligned} \tag{4.12}$$

$$D_x D_t\, f_+\cdot f_- = 3\delta^{-1} D_{\tilde{t}} D_{\tilde{x}} f\cdot f - \tfrac{3}{2} D_{\tilde{t}} D_{\tilde{x}}^3 f\cdot f + O(\delta), \tag{4.13}$$

$$D_x\, f_+\cdot f_- = -3i D_{\tilde{x}}^2 f\cdot f + O(\delta). \tag{4.14}$$

We then have, by substituting (4.10) and (4.12)–(4.14) into Eq. (4.5), the following bilinear equation for f:

$$D_{\tilde{x}}(D_{\tilde{t}} - D_{\tilde{t}} D_{\tilde{x}}^2 + D_{\tilde{x}}) f\cdot f = 0. \tag{4.15}$$

The dependent variable transformation for u follows from (4.3) and (4.10) with the aid of the expressions for small δ,

$$f_+ = f - i\delta f_x + O(\delta^2), \tag{4.16a}$$

$$f_- = f + i\delta f_x + O(\delta^2), \tag{4.16b}$$

as

$$u = 6\frac{\partial^2}{\partial \tilde{x}^2}\ln f. \tag{4.17}$$

Equation (4.15) with (4.17) is nothing but the bilinear form of Eq. (1.2) with the variables $\tilde{t}$ and $\tilde{x}$ instead of t and x, respectively.[14]

In order to derive the explicit functional form for f, we introduce the positive constants p_n by the relations

$$\gamma_n = (3\delta)^{1/2} p_n \quad (n = 1,2,\dots,N). \tag{4.18}$$

It then turns out that

$$\cos\gamma_n = [(3\delta)^{1/2} p_n]^{-1} - (3\delta)^{1/2} p_n/3 + O(\delta^{3/2}), \tag{4.19a}$$

$$a_n = 1/(1 - p_n^2), \tag{4.19b}$$

$$B_{jm} \equiv \widetilde{B}_{jm} = \frac{(p_j - p_m)^2(-3 + p_j^2 - p_j p_m + p_m^2)}{(p_j + p_m)^2(-3 + p_j^2 + p_j p_m + p_m^2)}, \tag{4.19c}$$

and

$$f = \sum_{\mu=0,1}\exp\left[\sum_{n=1}^{N}\mu_n p_n(\tilde{x} - a_n\tilde{t} - \tilde{x}_{0n}) + \sum_{j<m}^{(N)}\mu_j\mu_m\widetilde{B}_{jm}\right],$$
$$[\tilde{x}_{0n} = (3/\delta)^{1/2}x_{0n}]. \tag{4.19d}$$

The expression (4.19d) coincides perfectly with that given by Hirota and Satsuma.[14]

B. Rational solutions

Rational solutions of certain nonlinear evolution equations may be constructed by taking an appropriate limit on soliton solutions.[28,29] Owing to the freedom to choose an arbitrary constant, x_{0n} in the present case, it is possible to reduce soliton solutions to corresponding rational ones. In this subsection, we shall briefly discuss some rational solutions that are reduced from the one-soliton solution of Eq. (4.1), namely the expression (4.6) with $N = 1$. The rational solutions reduced from the general N-soliton solution will be presented elsewhere.

Now, it follows from (4.6) with $N = 1$ that the one-soliton solution of Eq. (4.1) is written in terms of the bilinear variables as

$$f = 1 + \exp[\delta^{-1}\gamma_1(x - a_1 t - x_{01})], \tag{4.20a}$$

with

$$a_1 = (1 - \delta^{-1} + \delta^{-1}\gamma_1\cot\gamma_1), \quad 0 < \gamma_1 < \pi. \tag{4.20b}$$

The deduction of the rational solution from the one-soliton solution (4.20) is possible if we choose the phase constant x_{01} as

$$x_{01} = -\pi\delta\gamma^{-1} - i\alpha, \quad |\alpha| > \delta, \tag{4.21}$$

with α being a real constant, and then take a limit $\gamma_1 \to 0$. It should be noted that the condition $|\alpha| > \delta$ is necessary because of the assumption in deriving (4.4). Indeed, a_1 becomes in this limit

$$a_1 = 1 + (3\delta)^{-1}\gamma_1^2 + O(\gamma_1^4), \tag{4.22}$$

and hence f has an expansion for small γ_1,

$$f = -\delta^{-1}\gamma_1(x - t + i\alpha) + O(\gamma_1^2). \tag{4.23}$$

Substituting (4.23) into (4.3) and taking a limit $\gamma_1 \to 0$, we have a rational solution of Eq. (1.1) as follows:

$$u = -2\delta/[(x - t + i\alpha)^2 + \delta^2]. \tag{4.24}$$

The solution (4.24) is regular but complex for real t and x. We now consider the two limiting cases of $\delta \to \infty$ and $\delta = 0$.

1. Deep-water limit: $\delta \to \infty$

In this limit, it is convenient to start from the expression (4.20). Various limiting procedures are possible, which we shall treat separately below.

(a) $\gamma_1 = \pi(1 - b_1/\delta), \quad b_1 > 1.$

In this case, it follows that

$$a_1 = b_1/(b_1 - 1), \tag{4.25a}$$

$$f_- = 1 + \exp[\delta^{-1}\gamma_1(x - a_1 t + i\delta - x_{01})]$$
$$= -\delta^{-1}\pi(x - a_1 t - x_{01} - ib_1) + O(\delta^{-2}), \tag{4.25b}$$

$$f_+ = f_-^*, \tag{4.25c}$$

so that

$$u = i\frac{\partial}{\partial x}\ln\left(\frac{f_+}{f_-}\right)$$
$$= \frac{2b_1}{(x - a_1 t - x_{01})^2 + b_1^2} \quad \left(b_1 = \frac{a_1}{a_1 - 1}\right), \tag{4.25d}$$

which is nothing but the rational one-soliton solution of Eq. (1.1) already given by (2.12).

(b) $\gamma_1 = \pi(1 - c_1/\delta)/2 \ (c_1 > 0), \quad x_{01} = \pi i\delta\gamma_1^{-1}.$

In this case, one finds that

$$a_1 = 1 + O(\delta^{-1}), \tag{4.26a}$$

$$f_- = \pi(x - t - ic_1)/2\delta + O(\delta^{-2}), \tag{4.26b}$$

$$f_+ = 2 + O(\delta^{-1}), \tag{4.26c}$$

so that

$$u = -i/(x - t - ic_1), \tag{4.26d}$$

which is a single pole solution of Eq. (1.1).

(c) $\gamma_1 = \beta(1 - c_1/\delta), \quad \beta \neq \pi/2, \pi, \quad x_{01} = \pi i\delta\gamma_1^{-1}.$

In this case, following the same procedure as case (b), we find a single pole solution (4.26d).

2. Shallow-water limit: $\delta \to 0$

In this limit, we also take $\alpha \to 0$. Then, introduction of the new variables $\tilde{t}$ and $\tilde{x}$ defined by (4.10a) and (4.10b), respectively, into (4.24) yields

$$u = -6/(\tilde{x} - \tilde{t})^2, \tag{4.27}$$

which is a rational solution of Eq. (1.2) with the variables $\tilde{t}$ and $\tilde{x}$ instead of t and x, respectively. This fact can also easily be confirmed by direct substitution of (4.27) into Eq. (1.2).

C. Solution for a linearized equation

An appropriate linearized version of Eq. (4.1) may be written as

$$u_t - Tu_{tx} + u_x = 0. \tag{4.28}$$

The solution of the initial value problem for Eq. (4.28) with the boundary condition $u \to 0$ as $|x| \to \infty$ can easily be constructed by employing the Fourier transform. The result is expressed in the form

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(y,0)$$
$$\times \exp\{i[k(x - y) - \omega(k)t]\}dk\,dy. \tag{4.29a}$$

Here, $\omega(k)$ is a dispersion relation for Eq. (4.28) given by

$$\omega(k) = k/[1 - \delta^{-1} + k\coth(k\delta)]. \tag{4.29b}$$

In the deep-water limit $\delta \to \infty$, (4.29) is reduced to

$$\omega(k) = k/(1 + |k|), \tag{4.30}$$

which is in accordance with (2.16), the dispersion relation for Eq. (2.14). In the shallow-water limit $\delta \to 0$, on the other hand, it becomes

$$\tilde{\omega}(\tilde{k}) = 1/(1 + \tilde{k}^2), \tag{4.31a}$$

with the new variables $\tilde{\omega}$ and $\tilde{k}$ defined by

$$\tilde{\omega} = (\delta/3)^{1/2}\omega, \tag{4.31b}$$

$$\tilde{k} = (\delta/3)^{1/2}k. \tag{4.31c}$$

The expression (4.31) is just the dispersion relation for the linearized version in Eq. (1.2) with the variables $\tilde{t}$ and $\tilde{x}$ instead of t and x, namely,

$$u_{\tilde{t}} - u_{\tilde{t}\tilde{x}\tilde{x}} + u_{\tilde{x}} = 0. \tag{4.32}$$

The problem for investigating asymptotic behaviors of (4.29) for large time will be left for future work.

D. Dynamical systems related to Eq. (4.1)

In this subsection, we derive the dynamical systems related to Eq. (4.1). The discussion almost parallels that for Eq. (1.1) or that for the ILW equation.[29,30] In so doing, we assume the bilinear variable f defined in (4.3) in the form

$$f = \prod_{j=1}^{M}(x - x_j), \quad |\mathrm{Im}\,x_j| > \delta, \tag{4.33}$$

where the x_j are complex functions of t and M is a positive integer. The conditions $|\mathrm{Im}\,x_j| > \delta$ ($j = 1,2,\ldots,M$) are required because of the assumption for the analytical property of f [see (4.3) and (4.4)]. Substituting (4.33) into (4.5) and using partial fraction decomposition, we obtain the equation

$$\sum_{j=1}^{M}\left(\frac{1}{x - x_j + i\delta} - \frac{1}{x - x_j - i\delta}\right)(\dot{x}_j - 1) + 2\delta$$
$$\times \sum_{j=1}^{M}\sum_{\substack{l=1 \\ (l\neq j)}}^{M}\left[\frac{1}{x - x_j + i\delta}\,\frac{1}{(x_j - x_l)(x_j - x_l - 2i\delta)}\right.$$
$$\left. - \frac{1}{x - x_j - i\delta}\,\frac{1}{(x_j - x_l)(x_j - x_l + 2i\delta)}\right]$$
$$\times(\dot{x}_j + \dot{x}_l) = 0. \tag{4.34}$$

Then we have, by taking the coefficients of $(x-x_j+i\delta)^{-1}$ and $(x-x_j-i\delta)^{-1}$ zero, respectively, the following system of equations for x_j:

$$\dot{x}_j = 1 - 2\delta \sum_{\substack{l=1\\(l\neq j)}}^{M} \frac{\dot{x}_j+\dot{x}_l}{(x_j-x_l)(x_j-x_l-2i\delta)}, \tag{4.35a}$$

$$\dot{x}_j = 1 - 2\delta \sum_{\substack{l=1\\(l\neq j)}}^{M} \frac{\dot{x}_j+\dot{x}_l}{(x_j-x_l)(x_j-x_l+2i\delta)} \quad (j=1,2,...,M). \tag{4.35b}$$

Adding (4.5a) and (4.5b) yields

$$\dot{x}_j = 1 - 4\delta \sum_{\substack{l=1\\(l\neq j)}}^{M} \frac{\dot{x}_j+\dot{x}_l}{(x_j-x_l)^2+4\delta^2}, \tag{4.36a}$$

while subtracting (4.35a) from (4.35b) yields

$$\sum_{\substack{l=1\\(l\neq j)}}^{M} \frac{\dot{x}_j+\dot{x}_l}{(x_j-x_l)\{(x_j-x_l)^2+4\delta^2\}} = 0. \tag{4.36b}$$

Equation (4.36a) is a finite-dimensional dynamical system with a constraint (4.36b) and it is closely related to the motion of the positions of the poles of solutions of Eq. (4.1). The detailed analysis of Eq. (4.36) will not be done here. Instead, we consider two limiting cases of $\delta\to\infty$ and $\delta\to 0$.

1. Deep-water limit: $\delta\to\infty$

In this limit, it is appropriate to introduce the new variables $\tilde{x}_j$ $(j=1,2,...,M)$ by the relations[29,30]

$$\tilde{x}_j = x_j - i\delta, \quad \operatorname{Im} x_j > \delta \quad (j=1,2,...,N), \tag{4.37a}$$

$$\tilde{x}_j = x_j + i\delta, \quad \operatorname{Im} x_j < -\delta \quad (j=N+1,N+2,...,M). \tag{4.37b}$$

Hence $\tilde{x}_j$, for $j=1,2,...,N$, lie in the upper half plane ($\operatorname{Im}\tilde{x}_j>0$) and $\tilde{x}_j$ for $j=N+1,N+2,...,M$ lie in the lower half plane ($\operatorname{Im}\tilde{x}_j<0$). Substituting (4.37) into Eq. (4.34), we find, in the limit $\delta\to\infty$,

$$\sum_{j=1}^{N}\frac{1}{x-\tilde{x}_j}\left[\dot{\tilde{x}}_j - 1 + i\sum_{\substack{l=1\\(l\neq j)}}^{N}\frac{\dot{\tilde{x}}_j+\dot{\tilde{x}}_l}{\tilde{x}_j-\tilde{x}_l} - i\sum_{l=N+1}^{M}\frac{\dot{\tilde{x}}_j+\dot{\tilde{x}}_l}{\tilde{x}_j-\tilde{x}_l}\right] - \sum_{j=N+1}^{M}\frac{1}{x-\tilde{x}_j} \times\left[\dot{\tilde{x}}_j - 1 + i\sum_{l=1}^{N}\frac{\dot{\tilde{x}}_j+\dot{\tilde{x}}_l}{\tilde{x}_j-\tilde{x}_l} - i\sum_{\substack{l=N+1\\(l\neq j)}}^{M}\frac{\dot{\tilde{x}}_j+\dot{\tilde{x}}_l}{\tilde{x}_j-\tilde{x}_l}\right] = 0, \tag{4.38}$$

whereupon we readily obtain by taking the coefficient of $(x-\tilde{x}_j)^{-1}$ for $j=1,2,...,N$ and that for $j=N+1$, $N+2,...,M$ zero, respectively,

$$\dot{\tilde{x}}_j = 1 - i\sum_{\substack{l=1\\(l\neq j)}}^{N}\frac{\dot{\tilde{x}}_j+\dot{\tilde{x}}_l}{\tilde{x}_j-\tilde{x}_l} + i\sum_{l=N+1}^{M}\frac{\dot{\tilde{x}}_j+\dot{\tilde{x}}_l}{\tilde{x}_j-\tilde{x}_l} \quad (j=1,2,...,N), \tag{4.39a}$$

$$\dot{\tilde{x}}_j = 1 - i\sum_{l=1}^{N}\frac{\dot{\tilde{x}}_j+\dot{\tilde{x}}_l}{\tilde{x}_j-\tilde{x}_l} + i\sum_{\substack{l=N+1\\(l\neq j)}}^{M}\frac{\dot{\tilde{x}}_j+\dot{\tilde{x}}_l}{\tilde{x}_j-\tilde{x}_l} \quad (j=N+1,N+2,...,M). \tag{4.39b}$$

This system of equations is a dynamical system without any constraint. If we take $M=2N$ and $\tilde{x}_{N+j}=\tilde{x}_j^*$ $(j=1,2,...,N)$, Eqs. (4.39a) become

$$\dot{\tilde{x}}_j = 1 - i\sum_{\substack{l=1\\(l\neq j)}}^{N}\frac{\dot{\tilde{x}}_j+\dot{\tilde{x}}_l}{\tilde{x}_j-\tilde{x}_l} + i\sum_{l=1}^{N}\frac{\dot{\tilde{x}}_j+\dot{\tilde{x}}_l^*}{\tilde{x}_j-\tilde{x}_l^*} \quad (j=1,2,...,N), \tag{4.40}$$

and Eqs. (4.39b) become the complex conjugate expressions of (4.40). Furthermore, in the limit $\delta\to\infty$, we have from (4.3) and (4.37)

$$u = i\frac{\partial}{\partial x}\ln\left(\frac{\tilde{f}^*}{\tilde{f}}\right), \tag{4.41a}$$

with

$$\tilde{f} = \sum_{j=1}^{N}(x-\tilde{x}_j), \quad \operatorname{Im}\tilde{x}_j>0 \quad (j=1,2,...,N). \tag{4.41b}$$

The system of equations (4.40) is identical to Eqs. (3.8) with the variables $\tilde{x}_j$ in place of x_j which have already been reduced from the periodic dynamical system related to Eq. (1.1).

2. Shallow-water limit: $\delta\to 0$

In this limit, it is convenient to introduce the variables $\tilde{t}$ and $\tilde{x}$ defined in (4.10) and the new variables $\tilde{x}_j=(3/\delta)^{1/2}x_j$ $(j=1,2,...,M)$. We then immediately find from (4.36)

$$\dot{\tilde{x}}_j = 1 - 12\sum_{\substack{l=1\\(l\neq j)}}^{M}\frac{\dot{\tilde{x}}_j+\dot{\tilde{x}}_l}{(\tilde{x}_j-\tilde{x}_l)^2}, \tag{4.42a}$$

$$\sum_{\substack{l=1\\(l\neq j)}}^{M}\frac{\dot{\tilde{x}}_j+\dot{\tilde{x}}_l}{(\tilde{x}_j-\tilde{x}_l)^3} = 0, \tag{4.42b}$$

which is a dynamical system with a constraint. The dependent variable u, (4.3) now takes the form

$$u = 6\frac{\partial^2}{\partial\tilde{x}^2}\ln\tilde{f}, \tag{4.43a}$$

with

$$\tilde{f} = \sum_{j=1}^{M}(\tilde{x}-\tilde{x}_j). \tag{4.43b}$$

The system of equations (4.42) represents a dynamical system related to Eq. (1.2) and the solutions may be constructed from the N-soliton solution (4.19) of Eq. (1.2) by taking an appropriate limiting procedure. The solutions $\tilde{x}_j$ thus obtained are expected to have a form of algebraic functions of $\tilde{t}$ and $\tilde{x}$. However, solutions u themselves constructed from $\tilde{x}_j$ would become singular as seen from a rational solution (4.27), for example, which has been reduced from a one-soliton solution of Eq. (4.1). The situation mentioned here would be the same as that for rational solutions of the KdV equation.[20] The detailed analysis of the system of equations (4.42) will be dealt with elsewhere.

V. CONCLUSION

In this paper, we have proposed a new integrable NIDE that exhibits the N-periodic and N-soliton solutions and showed that it is closely related to certain solvable dynamical systems. The NIDE may be relevant as a model equation that describes wave propagations in fluids of great depth. Moreover, we have generalized our NIDE to a more general one that is an intermediate version between our equation and that of Hirota and Satsuma.[14] The generalized equation may also describe suitably wave phenomena in fluids of finite depth.

In the context of the soliton theory, it is quite interesting to derive the inverse scattering transforms (IST's), the Bäcklund transformations, and an infinite number of conservation laws, etc., for these NIDE's. In this respect, the bilinear equations (2.3) and (4.5) may offer a proper starting point for analyzing these problems since the systematic method for constructing IST's, etc., on the basis of the bilinear equations has already been established considerably within the framework of the bilinear formalism.[16]

Finally, it is useful to comment on another type of integrable model equations for shallow water waves proposed by Ablowitz *et al.*[31] It may read in the form

$$u_t - u_{txx} - 2uu_t + u_x \int_x^\infty u_t\, dx + u_x = 0. \tag{5.1}$$

Although only the coefficient of the nonlinear term uu_t of Eq. (5.1) differs in comparison with Eq. (1.2), Eq. (5.1) is never reducible to Eq. (1.2) by means of any scale transformations. If we replace the dispersive term u_{txx} by Hu_{tx}, new NIDE's will arise. The method for exact solution developed in this paper may be applied to these NIDE's in order to obtain various results corresponding to those presented here. A number of problems proposed in this paper will be dealt with in the near future.

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