On second order weakly hyperbolic equations and the ultradifferentiable classes

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Abstract

We consider second order weakly hyperbolic equations with time dependent coefficients in the ultradifferentiable classes. Our main purpose of the present paper is an investigation the relation between the classes of the functions to be well-posed and the following properties of the coefficients: the order of degeneration, stabilization to a monotonic function and their smoothness in the ultradifferentiable classes.

Dedicated to the memory of Professor Rentaro Agemi

1 Introduction

We study the Cauchy problem for second order weakly hyperbolic equations with time dependent coefficients

$$\begin{cases} \left(\partial_t^2 - a(t)^2\Delta\right) u = 0, & (t,x) \in (0,T] \times \mathbf{R}^n, \\ (u(0,x), u_t(0,x)) = (u_0(x), u_1(x)), & x \in \mathbf{R}^n, \end{cases} \tag{1.1}$$

where $\Delta = \sum_{k=1}^n \partial_{x_k}^2$, $T > 0$, $\sup_{t\in(0,T)}\{a(t)\} < \infty$ and $\min_{t\in[0,T]}\{a(t)\} =: a_0 \ge 0$. It is well-known that the energy of the wave equation with constant coefficients is conserved, but it is not so for general equations with variable coefficients. Actually, the energy may be unbounded due to the loss of regularity of the solution which is brought by some influence of variable coefficients. Our main purpose of the present paper is to describe the order of regularity loss by using several properties of variable coefficients.

Let us define the energy of the solution to (1.1) in the phase space $[0,T] \times \mathbb{R}^n_\xi$ by

$$\mathcal{E}(t,\xi) = \begin{cases} |\xi|^2 \left|\hat{u}(t,\xi)\right|^2 + \left|\hat{u}_t(t,\xi)\right|^2 & (a_0 > 0), \\ \left|\hat{u}(t,\xi)\right|^2 + \left|\hat{u}_t(t,\xi)\right|^2 & (a_0 = 0), \end{cases}$$

where $\hat{f}(t,\xi)$ denotes the partial Fourier transform of $f(t,x)$ with respect to the space variables x. Then the order of regularity loss is represented by the following estimate

$$\mathcal{E}(t,\xi) \le \exp\left(C\mu(\langle\xi\rangle)\right)\mathcal{E}(0,\xi), \tag{1.2}$$

where $\mu(r)$ is a positive and monotonically increasing function on $[1,\infty)$, $\langle\xi\rangle = \sqrt{1+|\xi|^2}$ and C is some positive constant. In the other words, the estimate (1.2) can conclude that (1.1) is well-posed in the space of μ-ultradifferentiable functions of Beurling-Roumieu type (see [1, 3]). If $a_0 > 0$, then we assume that

$$\frac{1}{\mu(r)} = O(1) \quad (r \to \infty). \tag{1.3}$$

In particular, if the estimate (1.2) is valid for $\mu(r) = 1$, then (1.1) is well-posed in L^2, which means no loss of regularity occurs. On the other hand, if $a_0 = 0$, then we cannot expect the estimate (1.2) with $\mu(r) = 1$ in general, so that it is reasonable to restrict ourselves to

$$\frac{\log r}{\mu(r)} = O(1) \quad (r \to \infty) \tag{1.4}$$

instead of (1.3). Here the estimate (1.2) with $\mu(r) = O(\log r)$ $(r \to \infty)$ implies that (1.1) is well-posed in C^∞ class. If the estimate (1.2) holds for $\mu(r) = r^{1/s}$ with $s > 1$, then (1.1) is well-posed in the Gevrey class of order $s > 1$, and the limiting case $s = 1$ leads to the well-posedness in real analytic class. Thus it is also reasonable to restrict ourselves to

$$\frac{\mu(r)}{r} = O(1) \quad (r \to \infty). \tag{1.5}$$

If $a(t)$ is Lipschitz continuous on $[0, T]$ and $a_0 > 0$, then (1.1) is well-posed in L^2, that is, (1.2) holds for $\mu(r) = 1$. On the other hand, if $a(t)$ is not Lipschitz continuous or $a_0 = 0$, we shall call such a coefficient *singular*, then the estimate (1.2) holds only if $\lim_{r\to\infty} \mu(r) = \infty$ in general; refer to [4] in case of $a_0 > 0$, and [5, 8, 9, 10, 11] in case of $a_0 = 0$ for instance. In particular, it is examined in [2, 5, 6, 7, 13, 19] that $a(t)$ is singular only at $t = T$, and our main theorem is based on their researches. Here we note that the linear wave equations with singular coefficients are studied by motivated to apply the time global solvability of Kirchhoff equation, which is a sort of non-linear wave equations with non-local nonlinearity; for the details refer to [12, 15, 17, 18].

Let us review some previous works lead to the main theorem in the present paper. If $a_0 > 0$, $a(t) \in C^1([0, T))$ and if

$$|a'(t)| \le M_1 (T - t)^{-\beta},$$

where M_1 is a positive constant and $\beta \in [0, 1) \cup (1, \infty)$, then (1.2) is valid with $\mu(r) = 1$ for $\beta < 1$, and $\mu(r) = r^{1-1/\beta} =: \mu_\beta(r)$ for $\beta > 1$ respectively. Moreover, if $a(t) \in C^m([0, T))$ with $m \ge 2$ satisfies that

$$\left| a^{(k)}(t) \right| \le M_k (T - t)^{-k\beta} \quad (k = 1, \ldots, m), \tag{1.6}$$

where $M_1, \ldots, M_m$ are positive constants, $\beta \in [0, 1]$, and there exist constants a_T and $\alpha \ge 1$ such that

$$\int_t^T |a(s) - a_T| \, ds = O((T - \alpha)^\alpha) \quad (t \to T), \tag{1.7}$$

then (1.2) is valid with

$$\mu(r) = r^{\kappa(\alpha,\beta,m)}, \quad \kappa(\alpha, \beta, m) = 1 - \frac{\alpha}{\beta + \frac{\alpha-1}{m}} \tag{1.8}$$

since $\kappa(\alpha, \beta, m) \ge 0$. Here the condition (1.7) was introduced in [2] as the *stabilization property*, and the constant a_T is uniquely determined if a constant $\alpha \in (1, \beta)$ exists. We observe that $\kappa(1, \beta, m) = 1 - 1/\beta$, and $\kappa(\alpha, \beta, m)$ is strictly decreasing with respect to m only if $\alpha > 1$. This means that the order of regularity loss is smaller as $a(t)$ is more regular and stabilized in the senses of (1.6) and (1.7). The optimality of the estimate (1.2) with (1.8) is an open problem, but it is proved that there exists $a(t) \in C^\infty([0, T))$ satisfying (1.6) for any m and (1.7) such that the estimate (1.2) with $\mu(r) = r^\kappa$ does not holds in general for any $\kappa < 1 - \alpha/\beta$.

The results for strictly hyperbolic problems as above can be generalized to ones for weakly hyperbolic problems if the coefficients are singular only at $t = T$. For a function $\lambda(t) \in C^1([0, T])$ satisfying

$$\lambda'(t) \le 0, \;\; \lambda(t) > 0 \;\text{ on }\; [0, T) \;\text{ and }\; \lambda(T) \ge 0 \tag{1.9}$$

we define positive monotone decreasing functions $\Lambda(t)$ and $\Theta(t)$ as follows:

$$\Lambda(t) = \int_t^T \lambda(s)\, ds \tag{1.10}$$

and

$$\Theta(t) = \int_t^T |a(s) - \lambda(s)|\, ds. \tag{1.11}$$

Then we have the following result:

Theorem 1.1 ([13])**.** *Let $m \geq 2$ and $a(t) \in C^m([0,T))$. Assume that there exists a function $\lambda(t) \in C^1([0,T])$ satisfying (1.9) such that the following conditions (H1)-(H3) are established:*

(H1) There exist positive constants M_0 and C_0 satisfying $1 \leq M_0 \leq C_0$ such that

$$C_0^{-1}\lambda(t) \leq a(t) \leq M_0\lambda(t). \tag{1.12}$$

(H2)

$$\Theta(t) = o(\Lambda(t)) \quad (t \to T). \tag{1.13}$$

(H3) There exist positive constants $M_1, \ldots, M_m$ and $\gamma \in [0,1) \cup (1,\infty)$ such that

$$\frac{\left|a^{(k)}(t)\right|}{\lambda(t)} \leq M_k \left(\lambda(t)\left(\frac{\Lambda(t)}{\Theta(t)}\right)^{-\frac{1}{m}}\left(\frac{1}{\Theta(t)}\right)^{\gamma}\right)^k \tag{1.14}$$

for $k = 1, \ldots, m$.

Then the estimate (1.2) with $\mu(r) = r^{1-1/\gamma}$ is established. In particular, if $a_0 > 0$, then $\gamma = 1$ is admissible.

Remark 1.1. Theorem 1.1 is a natural generalization of the previous works for strictly hyperbolic problems. If $a_0 > 0$, then (1.13) with $\lambda(t) \equiv a_T$ is a generalization of (1.7). Indeed, if we restrict ourselves to $\Theta(t) \simeq O((T-t)^\alpha) = O(\Lambda(t)^\alpha)$, then we see

$$\lambda(t)\left(\frac{\Lambda(t)}{\Theta(t)}\right)^{-\frac{1}{m}}\left(\frac{1}{\Theta(t)}\right)^{\gamma} \simeq (T-t)^{-\beta}, \quad \beta = \alpha\gamma - \frac{\alpha-1}{m}.$$

It follows that $\kappa(\alpha,\beta,m) = \kappa(\alpha, \alpha\gamma - (\alpha-1)/m, m) = 1 - 1/\gamma$.

Remark 1.2. It is obvious that $\Theta(t) = O(\Lambda(t))$ since (H1) is valid, but (H2) is not so; we shall call the non-trivial condition (1.11) with (H2) *the stabilization property*. If (H2) holds, then the smoothness of $a(t)$, that is, the size of m has some influence of the orders of the derivatives of $a(t)$ in (H3), and also in (H4) to be introduced below.

Let us generalize the condition (1.14) for $a(t) \in C^\infty([0,T))$ on the *ultradifferentiable class* to the following:

$$\frac{\left|a^{(k)}(t)\right|}{\lambda(t)} \leq M_k \rho(t)^k \quad (k = 0, 1, \ldots), \tag{1.15}$$

where $\{M_k\}$ is a sequence of positive real numbers, while $\rho(t) \in C^0([0,T))$ is a positive and strictly increasing function. The function $a(t)$ satisfying (1.15) on $[0,T)$ is called a function in the ultradifferentiable class; we shall denote the class of these functions by $C^*(\{M_k\})$. For the sequence $\{M_k\}$ we introduce the following condition, which is called *the logarithmical convexity*:

$$\frac{M_k}{kM_{k-1}} \leq \frac{M_{k+1}}{(k+1)M_k} \quad (k = 1, 2, \ldots). \tag{1.16}$$

Remark 1.3. If $\{M_k\}$ is a logarithmical convex sequence, then $\{M_k^{k-1}/M_{k-1}^k\}$ is strictly increasing and

$$\lim_{k\to\infty} \frac{M_k^{k-1}}{M_{k-1}^k} = \infty \tag{1.17}$$

if $\{M_k\}$ satisfies (1.16). The logarithmical convexity in usual meaning is not (1.16) but $M_k/M_{k-1} \le M_{k+1}/M_k$ (see [16]). In fact, due to the condition (1.16) the sequence $\{M_k\}$ is increasing at least factorial order.

$C^*(\{k!^\nu\})$ with $\nu > 1$ is the Gevrey class of order ν, and $C^*(\{k!\})$ coincides with a real analytic class. If $\{M_k\}$ satisfies (1.16) and if $\lim_{k\to\infty} k!^\nu/M_k = \infty$ for any ν, then $C^*(\{M_k\})$ is wider class than any order of the Gevrey classes. For the finite sequence $\{M_k\}_{k=0}^m$ we identify $C^*(\{M_k\})$ with C^m class. Here we introduce the associated function of the sequence $\{M_k\}$ as follows:

Definition 1.1 (Associated function of $\{M_k\}$)**.** For a sequence of positive real numbers $\{M_k\}_{k=0}^\infty$ satisfying (1.16) we define the associated function $\mathcal{M}(\tau)$ for $\tau > 0$ by

$$\mathcal{M}(\tau) = \mathcal{M}(\tau; \{M_k\}) := \sup_{k\ge 1} \left\{ \frac{\tau^k}{M_k} \right\}.$$

For the finite sequence $\{M_k\}_{k=0}^m$ satisfying (1.16) for $k \le m-1$ we define

$$\mathcal{M}(\tau) = \mathcal{M}(\tau; \{M_k\}) := \max_{1\le k\le m} \left\{ \frac{\tau^k}{M_k} \right\}.$$

Then we have the following lemma for the associated function $\mathcal{M}(\tau; \{M_k\})$:

Lemma 1.1. *$\mathcal{M}(\tau) = \mathcal{M}(\tau; \{M_k\})$ is continuous and strictly increasing on $(0, \infty)$. Moreover, $\mathcal{M}(\tau)$ and its inverse function $\mathcal{M}^{-1}(\tau)$ are represented as follows:*

$$\mathcal{M}(\tau) = \frac{\tau^k}{M_k} \qquad on \quad \left[\frac{M_k}{M_{k-1}}, \frac{M_{k+1}}{M_k}\right) \tag{1.18}$$

and

$$\mathcal{M}^{-1}(\tau) = M_k^{\frac{1}{k}} \tau^{\frac{1}{k}} \qquad on \quad \left[\frac{M_k^{k-1}}{M_{k-1}^k}, \frac{M_{k+1}^k}{M_k^{k+1}}\right) \tag{1.19}$$

for $k = 1, 2, \ldots$. For the finite sequence $\{M_k\}_{k=0}^m$ we have $\mathcal{M}(\tau) = \tau^m/M_m$ for $\tau \ge M_m/M_{m-1}$ and $\mathcal{M}^{-1}(\tau) = M_m^{1/m}\tau^{1/m}$ for $\tau \ge M_m^{m-1}/M_{m-1}^m$.

Proof. By $\lim_{k\to\infty} M_k/M_{k-1} = \infty$, for any given $\tau \in (M_1/M_0, \infty)$ there exists $k \in \mathbb{N}$ such that $\tau \in [M_k/M_{k-1}, M_{k+1}/M_k)$, and then

$$\frac{\tau^k}{M_k} = \begin{cases} \dfrac{\tau^{k-1}}{M_{k-1}} \dfrac{M_{k-1}}{M_k}\tau \ge \dfrac{\tau^{k-1}}{M_{k-1}}, \\[2ex] \dfrac{\tau^{k+1}}{M_{k+1}} \dfrac{M_{k+1}}{M_k}\tau^{-1} > \dfrac{\tau^{k+1}}{M_{k+1}}, \end{cases}$$

that is, (1.18). Moreover, we have

$$\frac{M_k^{k-1}}{M_{k-1}^k} = \mathcal{M}\left(\frac{M_k}{M_{k-1}}\right) \le \mathcal{M}\left(\frac{M_{k+1}}{M_k}\right) = \frac{M_{k+1}^k}{M_k^{k+1}}.$$

The continuity of $\mathcal{M}(\tau)$ and the representation (1.19) are evident from (1.18). □

Example 1.1. The following sequences satisfy (1.16):

(i) $\{k!^\nu\}_{k=0}^\infty$ with $\nu \ge 1$ satisfies (1.16). Then there exist positive constants δ_0 and δ_1 such that the associated function satisfies

$$\exp\left(\delta_0 \tau^{\frac{1}{\nu}}\right) \le \mathcal{M}(\tau) \le \exp\left(\delta_1 \tau^{\frac{1}{\nu}}\right)$$

by Stirling's formula.

(ii) $\{\prod_{j=1}^k \exp(j^s)\}_{k=0}^\infty$ with $s > 0$ satisfies (1.16). Then there exist positive constants δ_0 and δ_1 such that the associated function fulfills

$$\exp\left(\delta_0 (\log(1+\tau))^{1+\frac{1}{s}}\right) \le \mathcal{M}(\tau) \le \exp\left(\delta_1 (\log(1+\tau))^{1+\frac{1}{s}}\right). \tag{1.20}$$

Indeed, for $\tau \in [M_k/M_{k-1}, M_{k+1}/M_k) = [\exp(k^s), \exp((k+1)^s))$ and large k there exist positive constants ρ_0 and ρ_1 independent of k such that

$$\mathcal{M}(\tau) = \frac{\tau^k}{M_k} \begin{cases} \ge \frac{1}{M_k}\left(\frac{M_k}{M_{k-1}}\right)^k = \exp\left(k^{s+1}\left(1 - \frac{\sum_{j=1}^k j^s}{k^{s+1}}\right)\right) \ge \exp\left(\rho_0 k^{s+1}\right), \\ \le \frac{1}{M_k}\left(\frac{M_{k+1}}{M_k}\right)^k = \exp\left(k^{s+1}\left(\left(\frac{k+1}{k}\right)^s - \frac{\sum_{j=1}^k j^s}{k^{s+1}}\right)\right) \le \exp\left(\rho_1 k^{s+1}\right). \end{cases}$$

Therefore, noting $k \le (\log \tau)^{1/s} \le k+1$, we have (1.20).

(iii) For the finite sequence $\{M_k\}_{k=0}^m$ there exist positive constants δ_0 and δ_1 such that the associated function satisfies

$$\delta_0 \tau^m \le \mathcal{M}(\tau) \le \delta_1 \tau^m.$$

Our main purpose of the present paper is to generalize Theorem 1.1 for $a(t) \in C^m$ to the similar result in the ultradifferentiable class $C^*(\{M_k\})$. If we restrict ourselves to the finite sequence $\{M_k\}_{k=0}^m$, the assumption (1.14) of (H3), then the order of $\mu(r)$ in the estimate (1.2) are represented by (1.15) for $k = 0, 1, \ldots, m$ with

$$\rho(t) = \frac{\lambda(t)}{\mathcal{M}^{-1}\left(\frac{\Lambda(t)}{\Theta(t)}\right)} \eta\left(\frac{1}{\Theta(t)}\right), \tag{1.21}$$

where $\mathcal{M}^{-1}(\tau) \simeq \tau^{1/m}$, $\eta(\tau) = \tau^\gamma$, and

$$\mu(r) = \frac{r}{\eta^{-1}(r)} \tag{1.22}$$

respectively. Actually, our main theorem in the next section tells us that such a generalization is realized for ultradifferentiable classes.

Remark 1.4. We can restrict ourselves that $\rho(t)$ is strictly increasing and $\lim_{t\to T} \rho(t) = \infty$. Indeed, if $\rho(t)$ is bounded, then the estimate (1.2) with $\mu(r) = 1$ is obvious for $m = 1$ by the usual energy method.

2 Main results

Let us introduce the following condition corresponding to (H3) for Theorem 1.1:

(H4) There exists a sequence of positive real numbers $\{M_k\}_{k=0}^{\infty}$ satisfying (1.16) such that the estimate

$$\frac{\left|a^{(k)}(t)\right|}{\lambda(t)} \le M_k \left(\frac{\lambda(t)}{\mathcal{M}^{-1}\left(\frac{\Lambda(t)}{\Theta(t)}\right)} \eta\left(\frac{1}{\Theta(t)}\right) \right)^k$$

is valid for any $k \in \mathbb{N}$ with a positive, continuous and strictly increasing function $\eta(\tau)$ on $[0,\infty)$ satisfying that $\eta(\tau)/\tau$, and $\eta(\tau)/(\tau \log \tau)$ are monotonically increasing for $a_0 > 0$, and for $a_0 = 0$ respectively as $\tau \to \infty$.

Remark 2.1. The assumption to $\eta(\tau)$ in (H4) implies that $\mu(r)$ defined by (1.22) satisfies (1.3) for $a_0 > 0$, and (1.4) for $a_0 = 0$ respectively.

Then our main theorem is given as follows:

Theorem 2.1. *Let $a(t) \in C^\infty([0,T))$. If there is $\lambda(t) \in C^1([0,T])$ satisfying (1.9) such that (H1), (H2) and (H4) are valid, then there exists a positive constant C such that (1.2) with (1.22) is established.*

Remark 2.2. The energy estimate of solution to (1.1) for $a_0 > 0$ and $T = \infty$ with the coefficient in the Gevrey classes is studied in [14]. This result is corresponding to the case that $\lambda(t) = 1$, $\{M_k\} = \{k!^\nu\}$ with $\nu > 1$ and $\eta(r) = r$.

Example 2.1. Let us examine the orders of $\{M_k\}$ and $\rho(t)$ corresponding to the example Example 1.1.

(i) Let $\nu \ge 1$ and $M_k = k!^\nu$. Then by Example 1.1 (i) there exist positive constants δ_0 and δ_1 such that

$$\delta_1^{-\nu}(\log\tau)^\nu \le \mathcal{M}^{-1}(\tau) \le \delta_0^{-\nu}(\log\tau)^\nu. \tag{2.1}$$

It follows that

$$\rho(t) \le \delta_1^\nu \lambda(t) \left(\log\frac{\Lambda(t)}{\Theta(t)}\right)^{-\nu} \eta\left(\frac{1}{\Theta(t)}\right) =: \rho_1(t;\nu).$$

(ii) Let $s \ge 1$ and $M_k = \prod_{j=1}^k \exp(j^s)$. Then by Example 1.1 (ii) there exist positive constants δ_0 and δ_1 such that

$$\exp\left(\delta_1^{-\frac{s}{s+1}}(\log\tau)^{\frac{s}{s+1}}\right) \le \mathcal{M}^{-1}(\tau) \le \exp\left(\delta_0^{-\frac{s}{s+1}}(\log\tau)^{\frac{s}{s+1}}\right). \tag{2.2}$$

It follows that

$$\rho(t) \le \lambda(t)\exp\left(-\delta_1^{-\frac{s}{s+1}}\left(\log\frac{\Lambda(t)}{\Theta(t)}\right)^{\frac{s}{s+1}}\right)\eta\left(\frac{1}{\Theta(t)}\right) =: \rho_2(t;s).$$

(iii) For the finite sequence $\{M_k\}_{k=1}^m$ Then we get $\mathcal{M}(r) \simeq r^m$, which follows $\mathcal{M}^{-1}(\tau) \simeq \tau^{1/m}$. Then we have

$$\rho(t) \simeq \lambda(t)\left(\log\frac{\Lambda(t)}{\Theta(t)}\right)^{-\frac{1}{m}} \eta\left(\frac{1}{\Theta(t)}\right) =: \rho_3(t;m).$$

Now we see that

$$\rho_1(t;\nu) = o\left(\rho_1(t;\nu_0)\right), \quad \rho_2(t;s) = o\left(\rho_2(t;s_0)\right), \quad \rho_3(t;m) = o\left(\rho_3(t;m_0)\right)$$

for $m < m_0$, $s > s_0$, $\nu > \nu_0$, and that

$$\rho_2(t;s) = o(\rho_1(t;\nu)), \quad \rho_3(t;m) = o(\rho_2(t;s))$$

for any $m \geq 2$, $s \geq 1$ and $\nu \geq 1$ as $t \to T$. Here we note that the order of oscillating speed of $a(t)$ satisfying (1.15) as $t \to T$ is given by $M_1\lambda(t)\rho(t)$. Thus we observe that faster oscillation is admissible for the estimate (1.2) as the order of $\{M_k\}$ is smaller, that is, the coefficient is smoother.

Let us introduce concrete examples of $a(t)$, which can be applied for Theorem 2.1.

Example 2.2. Let $2 \leq m \leq \infty$ and $\omega(r) \in C^m(\mathbb{R})$ be a 1-periodic function satisfying $0 \leq \omega(r) \leq 1$, $\omega(r) \equiv 0$ near $r = 0$ and

$$\sup_{r\in\mathbb{R}} \left\{ \left| \omega^{(k)}(r) \right| \right\} \leq M_k \quad (k = 0, \ldots, m)$$

for a sequence of positive real numbers $\{M_k\}_{k=0}^m$ satisfying (1.16). For $0 < \delta < 1$, $p \in \mathbb{N}$ and $q > p$ we define $\{\tau_j\}_{j=0}^\infty$ and $a(t)$ by $\tau_j = T - \delta^j$ and

$$a(t) = \begin{cases} (T-t)^p \left(1 + \omega\left(\delta^{-jq}(t - \tau_j)\right)\right), & t \in [\tau_j, \tau_j + \delta^{jq}] =: I_j, \\ (T-t)^p, & t \in (\tau_j + \delta^{jq}, \tau_{j+1}] =: \tilde{I}_j \end{cases} \tag{2.3}$$

for $j = 0, 1, \ldots$. For $t \in I_j$ and $k = 1, \ldots, m$ we have

$$\begin{aligned} \left| a^{(k)}(t) \right| \leq & \sum_{l=0}^{k} \binom{k}{l} \frac{p!}{(p-l)!} (T-t)^{p-l} \delta^{-jq(k-l)} M_{k-l} \\ \leq & p^k \delta^{j(p-kq)} M_k \sum_{l=0}^{k} \binom{k}{l} p^{l-k} \delta^{jl(q-1)} \frac{M_{k-l}}{M_k} \\ \leq & (2p)^k M_k \delta^{j(p-kq)}. \end{aligned}$$

If we define $\lambda(t) = (T-t)^p$, then for $t \in I_j \cup \tilde{I}_j$ we have

$$\lambda(t) \simeq \delta^{jp}, \quad \Lambda(t) \simeq (T-t)^{p+1} \simeq \delta^{j(p+1)},$$

and

$$\Theta(t) \simeq \sum_{l=j}^{\infty} \delta^{l(p+q)} \simeq \delta^{j(p+q)},$$

which follows that $\delta^{-jq} \simeq \lambda(t)/\Theta(t)$ and $\Lambda(t)/\Theta(t) \simeq \delta^{-j(q-1)} \simeq \Theta(t)^{-(q-1)/(p+q)}$. Therefore we have

$$\frac{\left|a^{(k)}(t)\right|}{\lambda(t)} \leq M_k \left(C \frac{\lambda(t)}{\mathcal{M}^{-1}\left(\frac{\Lambda(t)}{\Theta(t)}\right)} \frac{1}{\Theta(t)} \mathcal{M}^{-1} \left(C_1 \left(\frac{1}{\Theta(t)} \right)^{\frac{q-1}{p+q}} \right) \right)^k,$$

where C and C_1 are positive constants. From now on, we shall denote universal positive constants by C and C_k $(k = 1, 2, \ldots)$. Let us examine the condition of η and the order of μ such that the estimate (1.15) for (1.21) is valid.

(i) Let $\nu \ge 1$ and $M_k = k!^\nu$. Noting the estimates (2.1), we have

$$\mathcal{M}^{-1}\left(C_1\tau^{\frac{q-1}{p+q}}\right) \le \delta_0^{-\nu}\left(\log\left(C_1\tau^{\frac{q-1}{p+q}}\right)\right)^\nu \le C_2\left(\log\tau\right)^\nu.$$

Therefore, setting $\eta(\tau) = C_2\tau(\log\tau)^\nu$, that is,

$$\mu(r) = \frac{r}{\eta^{-1}(r)} \simeq (\log r)^\nu,$$

we have the estimate (1.2).

(ii) Let $s > 0$ and $M_k = \prod_{j=1}^k \exp(j^s)$. Noting the estimates (2.2), we have

$$\mathcal{M}^{-1}\left(C_1\tau^{\frac{q-1}{p+q}}\right) \le \exp\left(\delta_0^{-\frac{s}{s+1}}\left(\log\left(C_1\tau^{\frac{q-1}{p+q}}\right)\right)^{\frac{s}{s+1}}\right) \le \exp\left(C_2\left(\log\tau\right)^{\frac{s}{s+1}}\right).$$

Therefore, setting $\eta(\tau) = C_0\tau\exp(C_2(\log\tau)^{\frac{s}{s+1}})$, that is,

$$\mu(r) = \frac{r}{\eta^{-1}(r)} \simeq \exp\left(C\left(\log r\right)^{\frac{s}{s+1}}\right),$$

we have the estimate (1.2).

(iii) Let $m < \infty$. Noting $\mathcal{M}^{-1}(\tau) \simeq r^{\frac{1}{m}}$, we have

$$\frac{\left|a^{(k)}(t)\right|}{\lambda(t)} \le \left(C\frac{\lambda(t)}{\mathcal{M}^{-1}\left(\frac{\Lambda(t)}{\Theta(t)}\right)}\left(\frac{1}{\Theta(t)}\right)^{\gamma_m}\right)^k$$

for $k = 1,\dots,m$, where $\gamma_m = 1 + (q-1)/(m(p+q))$. Therefore, setting $\eta(\tau) = C\tau^{\gamma_m}$, that is,

$$\mu(r) = C^{\frac{1}{\gamma_m}}r^{1-\frac{1}{\gamma_m}},$$

we have the estimate (1.2). Here we note that γ_m is strictly increasing since $q > 1$ and $\lim_{m\to\infty}\gamma_m = 1$.

Remark 2.3. In [3, 10] the authors consider the relation between the smoothness of the coefficient and the order of μ for the estimate (1.2). Their classes of the coefficient lie between C^∞ and real analytic class, and one can identify them with the ultradifferentiable classes. Actually, the coefficients in [3, 10] can have many zeros and the smoothness is uniform on $[0,T]$. On the other hand, the coefficient $a(t)$ in our theorem may has a zero only at $t = T$, however it can be singular at $t = T$.

Remark 2.4. The similar problem for a generalization of the equation in (1.1) to the following second order weakly hyperbolic equation with first order terms:

$$\left(\partial_t^2 - \alpha(t,\partial_x) + \beta(t,\partial_x)\right)u = 0, \tag{2.4}$$

where

$$\alpha(t,\xi) = \sum_{1\le j,k\le n}\alpha_{jk}(t)\xi_j\xi_k, \quad \beta(t,\xi) = \sum_{j=1}^n \beta_j(t)\xi,$$

is open. It essentially needs to introduce some suitable Levi condition and stabilization property for $\alpha(t,\xi)/|\xi|^2$ corresponding to (1.11); however, the both of them are non-trivial. We will try to find them in forthcoming papers.

3 Proof of the theorem

3.1 Strategy of the proof

By partial Fourier transformation with respect to x the Cauchy problem (1.1) is reduced to the following problem:

$$\begin{cases} \left(\partial_t^2 + a(t)^2|\xi|^2\right) v = 0, & (t,\xi) \in (0,T] \times \mathbf{R}^n, \\ (v(0,\xi), v_t(0,\xi)) = (v_0(\xi), v_1(\xi)), & \xi \in \mathbf{R}^n. \end{cases} \tag{3.1}$$

A basic idea of the proof is appropriate division of the phase space into several zones and deriving estimates of the solution in different ways in the each zone. We represent the properties of the solution by three characters; *smoothness*, *degeneration*, and *stabilization*, which are described by the orders of $\{M_k\}$, $\lambda(t)$, and $\Theta(t)$ respectively. These properties are possible to make some influence to the loss of regularity of the solution, however they are not uniform in the phase space. For instance, the smoothness of the coefficient is not important for the estimate in the low frequency zone, but the order of the stabilization is crucial for the estimate in there. On the other hand, in the high frequency zone, the smoothness essentially contributes to precise estimate of the solution, but the degeneration and stabilization do not give so big influence. We essentially divide the phase space into three zones: $Z_{\Psi,0}$, $Z_{\Psi,2}$ and Z_H, which will be introduced below, and the intermediate zones of them. Let us make preparations for the definitions of the zones.

Let N be a large constant; more precisely, it is chosen like

$$N \ge \max\left\{ \frac{1}{\eta\left(T\lambda(0)\right)}, \frac{2M_0\lambda(0)}{\sqrt{3}M_1\rho(0)}, 64\nu^2\kappa^2\left(\frac{4\pi^2 M_0}{3}\right)^3 \right\},$$

where κ and ν are constants, which will be defined by (3.13).

Definition 3.1. For $r \in [1,\infty)$ we define $\tau_0 = \tau_0(r)$, $\tau_1 = \tau_1(r)$ and $\tau_2 = \tau_2(r)$ as follows:

$$\begin{aligned}
\tau_0(r) &= \begin{cases} 0, & r \in \left[1, N\eta\left(\frac{1}{T\lambda(0)}\right)\right), \\ \max\left\{t \in [0,T]\,;\, N\eta\left(\frac{1}{(T-t)\lambda(t)}\right) = r\right\}, & r \in \left[N\eta\left(\frac{1}{T\lambda(0)}\right), \infty\right), \end{cases} \\
\tau_1(r) &= \begin{cases} 0, & r \in \left[1, N\eta\left(\frac{1}{\Lambda(0)}\right)\right), \\ \max\left\{t \in [0,T]\,;\, N\eta\left(\frac{1}{\Lambda(t)}\right) = r\right\}, & r \in \left[N\eta\left(\frac{1}{\Lambda(0)}\right), \infty\right), \end{cases} \\
\tau_2(r) &= \begin{cases} 0, & r \in \left[1, N\eta\left(\frac{1}{\Theta(0)}\right)\right), \\ \max\left\{t \in [0,T]\,;\, N\eta\left(\frac{1}{\Theta(t)}\right) = r\right\}, & r \in \left[N\eta\left(\frac{1}{\Theta(0)}\right), \infty\right). \end{cases}
\end{aligned} \tag{3.2}$$

It is easy to verify that

$$(T-\tau_0)\lambda(\tau_0) = \Lambda(\tau_1) = \Theta(\tau_2) = \frac{1}{\eta^{-1}\left(\frac{r}{N}\right)} \tag{3.3}$$

since τ_0, τ_1 and τ_2 are positive.

Definition 3.2. For $r \in [1,\infty)$ we define $t_l = t_l(r)$ $(l = 0,1,\ldots)$ as follows:

$$t_l(r) = \begin{cases} 0, & r \in \left[1, N\frac{M_{l+1}}{M_l}\frac{\rho(0)}{\lambda(0)}\right), \\ \max\left\{t \in [0,T]\,;\, N\frac{M_{l+1}}{M_l}\frac{\rho(t)}{\lambda(t)} = r\right\}, & r \in \left[N\frac{M_{l+1}}{M_l}\frac{\rho(0)}{\lambda(0)}, \infty\right). \end{cases} \tag{3.4}$$

Definition 3.3. We define $l_0 \in \mathbb{N}$ by

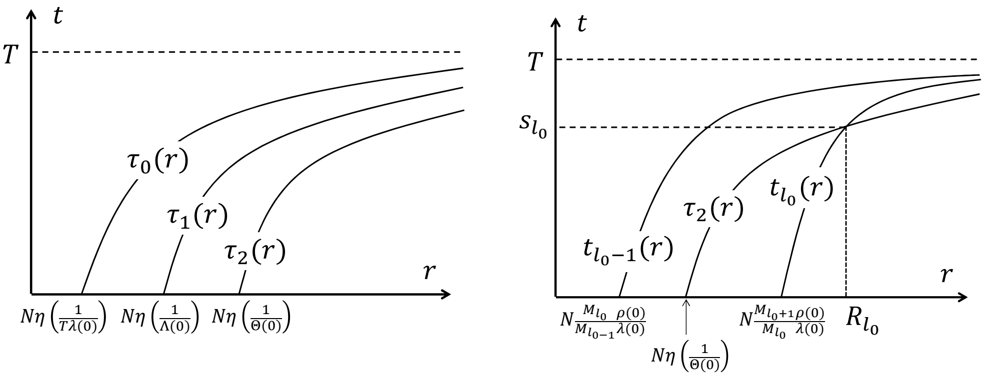
$$l_0 = \max\left\{2, \min\left\{l \in \mathbb{N}\,;\, \eta\left(\frac{1}{\Theta(0)}\right) \le \frac{M_{l+1}}{M_l}\frac{\rho(0)}{\lambda(0)}\right\}\right\}. \tag{3.5}$$

Moreover, for $l \ge l_0$ we define $s_l \in [0,T)$ and $R_l \in [1,\infty)$ by

$$s_l = \max\left\{t \in (0,T)\,;\, \frac{\Lambda(t)}{\Theta(t)} = \frac{M_{l+1}^l}{M_l^{l+1}}\right\} \tag{3.6}$$

and

$$R_l = N\eta\left(\frac{1}{\Theta(s_l)}\right). \tag{3.7}$$



From the above definitions the following properties are immediately valid:

Lemma 3.1. *(i) $\tau_j(r)$ $(j = 0,1,2)$ and $t_l(r)$ $(r = 0,1,\ldots)$ are monotonically increasing and tend to T as $r \to \infty$.*
(ii) $\tau_0(r) \ge \tau_1(r) \ge \tau_2(r)$ for any $r \in [1,\infty)$.
(iii) If $t_{l+1}(r) > 0$, then $t_l(r) > t_{l+1}(r)$.
(iv) $s_l = \tau_2(R_l) = t_l(R_l)$, $R_l \ge 2/\sqrt{3}$ and $\tau_2(r) < t_l(r)$ for $r > R_l$.

Proof. (i), (ii) and (iii) are trivial. By (1.19), (1.21), (3.2), (3.4), the definitions of s_l and R_l we have $\eta(1/\Theta(s_l)) = R_l/N = \eta(1/\Theta(\tau_2(R_l)))$ and

$$\frac{\rho(s_l)}{\lambda(s_l)} = \frac{\eta\left(\frac{1}{\Theta(s_l)}\right)}{\mathcal{M}^{-1}\left(\frac{\Lambda(s_l)}{\Theta(s_l)}\right)} = \frac{R_l}{N\mathcal{M}^{-1}\left(\frac{M_{l+1}^l}{M_l^{l+1}}\right)} = \frac{R_l}{N}\frac{M_l}{M_{l+1}} = \frac{\rho(t_l(R_l))}{\lambda(t_l(R_l))}.$$

Hence, noting Remark 1.4, we have $s_l = \tau_2(R_l) = t_l(R_l)$ and

$$R_l = N\frac{M_{l+1}}{M_l}\frac{\rho(t_l(R_l))}{\lambda(t_l(R_l))} \ge N\frac{M_1}{M_0}\frac{\rho(0)}{\lambda(0)} \ge \frac{2}{\sqrt{3}}.$$

Moreover, for $r > R_l$ we have

$$\eta\left(\frac{1}{\Theta(\tau_2(r))}\right) = \frac{r}{N} = \frac{M_{l+1}}{M_l}\frac{\eta\left(\frac{1}{\Theta(t_l(r))}\right)}{\mathcal{M}^{-1}\left(\frac{\Lambda(t_l(r))}{\Theta(t_l(r))}\right)} < \frac{M_{l+1}}{M_l}\frac{\eta\left(\frac{1}{\Theta(t_l(r))}\right)}{\mathcal{M}^{-1}\left(\frac{\Lambda(t_l(R_l))}{\Theta(t_l(R_l))}\right)}$$

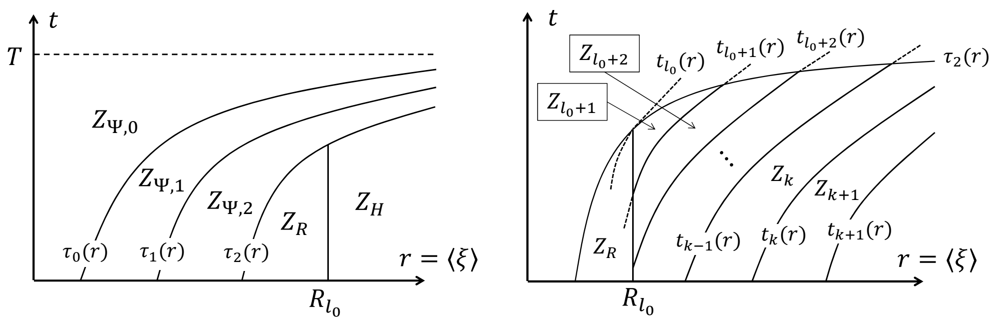
$$= \frac{M_{l+1}}{M_l}\frac{\eta\left(\frac{1}{\Theta(t_l(r))}\right)}{\mathcal{M}^{-1}\left(\frac{\Lambda(s_l)}{\Theta(s_l)}\right)} = \eta\left(\frac{1}{\Theta(t_l(r))}\right),$$

it follows that $\tau_2(r) < t_l(r)$. $\square$

Let us define the zones of the phase space as follows:

$$\begin{aligned} Z_{\Psi,0} &= \{\tau_0(\langle\xi\rangle) \le t \le T\}, \\ Z_{\Psi,1} &= \{\tau_1(\langle\xi\rangle) \le t < \tau_0(\langle\xi\rangle)\}, \\ Z_{\Psi,2} &= \{\tau_2(\langle\xi\rangle) \le t < \tau_1(\langle\xi\rangle)\}, \\ Z_R &= \{0 \le t < \tau_2(\langle\xi\rangle)\} \cap \{\langle\xi\rangle \le R_{l_0}\}, \\ Z_H &= \{0 \le t < \tau_2(\langle\xi\rangle)\} \setminus Z_R. \end{aligned} \tag{3.8}$$

Moreover, for $k \ge l_0$ we define the subzones $Z_{H,k}$ of Z_H by



$$Z_{H,k} = \{(t,\xi) \in Z_H \,;\, t_k(\langle\xi\rangle) \le t < t_{k-1}(\langle\xi\rangle)\}.$$

Thanks to (1.17) we see that $Z_H = \bigcup_{k=l_0}^{\infty} Z_{H,k}$.

3.2 Estimate in $Z_{\Psi,0}$

Let $(t,\xi) \in Z_{\Psi,0}$. We define $\mathcal{E}_0(t,\xi)$ by

$$\mathcal{E}_0(t,\xi) = \frac{1}{2}\left((T-\tau_0)^{-2}\Lambda(\tau_0)^2|\xi|^2|v(t,\xi)|^2 + |v_t(t,\xi)|^2\right).$$

In view of (1.12) we have

$$\begin{aligned} \partial_t\mathcal{E}_0(t,\xi) &= \left((T-\tau_0)^{-2}\Lambda(\tau_0)^2 - a(t)^2\right)|\xi|^2\Re\{v\overline{v_t}\} \\ &\le \frac{(T-\tau_0)^{-2}\Lambda\left(\tau_0\right)^2 + M_0^2\lambda(t)^2}{(T-\tau_0)^{-1}\Lambda\left(\tau_0\right)}|\xi|\mathcal{E}_0(t,\xi) \\ &\le \left((T-\tau_0)^{-1}\Lambda(\tau_1) + \frac{M_0^2(T-\tau_0)\lambda(\tau_0)}{\Lambda(\tau_0)}\lambda(t)\right)\langle\xi\rangle\mathcal{E}_0(t,\xi). \end{aligned}$$

Therefore, by Gronwall's inequality and (3.3) we have

$$\begin{aligned} \mathcal{E}_0(t,\xi) &\le \mathcal{E}_0(\tau_0,\xi)\exp\left(\int_{\tau_0}^{t}\left((T-\tau_0)^{-1}\Lambda(\tau_1) + \frac{M_0^2(T-\tau_0)\lambda(\tau_0)}{\Lambda(\tau_0)}\lambda(s)\right)ds\langle\xi\rangle\right) \\ &\le \mathcal{E}_0(\tau_0,\xi)\exp\left(\Lambda(\tau_1)\langle\xi\rangle + M_0^2(T-\tau_0)\lambda(\tau_0)\langle\xi\rangle\right) \\ &= \mathcal{E}_0(\tau_0,\xi)\exp\left(\frac{\left(1+M_0^2\right)\langle\xi\rangle}{\eta^{-1}\left(\frac{\langle\xi\rangle}{N}\right)}\right) \\ &= \mathcal{E}_0(\tau_0,\xi)\exp\left(N\left(1+M_0^2\right)\mu\left(\frac{\langle\xi\rangle}{N}\right)\right) \end{aligned}$$

uniformly in $Z_{\Psi,0}$.

3.3 Estimate in $Z_{\Psi,1}$

Let $(t,\xi) \in Z_{\Psi,1}$. We define $\mathcal{E}_1(t,\xi)$ by

$$\mathcal{E}_1(t,\xi) = \frac{1}{2}\left(\frac{\lambda(t)^2\Lambda(t)}{\Lambda(\tau_1)}|\xi|^2|v(t,\xi)|^2 + |v_t(t,\xi)|^2\right).$$

Since $\lambda(t)^2\Lambda(t)$ is monotonically decreasing, we have

$$\begin{aligned}
\partial_t\mathcal{E}_1(t,\xi) =& \frac{1}{2}\partial_t\left(\frac{\lambda(t)^2\Lambda(t)}{\Lambda(\tau_1)}\right)|\xi|^2|v(t,\xi)|^2 + \left(\frac{\lambda(t)^2\Lambda(t)}{\Lambda(\tau_1)} - a(t)^2\right)|\xi|^2\Re\{v\overline{v_t}\} \\
\le & \left(\frac{\frac{\lambda(t)^2\Lambda(t)}{\Lambda(\tau_1)} + M_0^2\lambda(t)^2}{\frac{\lambda(t)\Lambda(t)^{\frac{1}{2}}}{\Lambda(\tau_1)^{\frac{1}{2}}}}\langle\xi\rangle\right)\mathcal{E}_1(t,\xi) \\
=& -\partial_t\left(\frac{2\Lambda(t)^{\frac{3}{2}}}{3\Lambda(\tau_1)^{\frac{1}{2}}} + 2M_0^2\Lambda(\tau_1)^{\frac{1}{2}}\Lambda(t)^{\frac{1}{2}}\langle\xi\rangle\right)\mathcal{E}_1(t,\xi).
\end{aligned}$$

Therefore, by Gronwall's inequality we have

$$\begin{aligned}
\mathcal{E}_1(t,\xi) \le & \exp\left(\left(\frac{2}{3} + 2M_0^2\right)\Lambda(\tau_1)\langle\xi\rangle\right)\mathcal{E}_1(\tau_1,\xi). \\
=& \exp\left(N\left(\frac{2}{3} + 2M_0^2\right)\mu\left(\frac{\langle\xi\rangle}{N}\right)\right)\mathcal{E}_1(\tau_1,\xi)
\end{aligned}$$

uniformly in $Z_{\Psi,1}$.

3.4 Estimate in $Z_{\Psi,2}$

Let $(t,\xi) \in Z_{\Psi,2}$. We define $\mathcal{E}_2(t,\xi)$ by

$$\mathcal{E}_2(t,\xi) = \frac{1}{2}\left(\lambda(t)^2|\xi|^2|v(t,\xi)|^2 + |v_t(t,\xi)|^2\right).$$

On account of (1.12) we have

$$\begin{aligned}
\partial_t\mathcal{E}_2(t,\xi) \le & \frac{(\lambda(t) + a(t))\,|\lambda(t) - a(t)|}{\lambda(t)}|\xi|\mathcal{E}_2(t,\xi) \\
\le & (1 + M_0)\,|\lambda(t) - a(t)|\langle\xi\rangle\mathcal{E}_2(t,\xi).
\end{aligned}$$

Therefore, by Gronwall's inequality we have

$$\begin{aligned}
\mathcal{E}_2(t,\xi) \le & \exp\left((1 + M_0)\int_{\tau_2}^t|\lambda(s) - a(s)|\,ds\langle\xi\rangle\right)\mathcal{E}_2(\tau_2,\xi) \\
\le & \exp\left((1 + M_0)\,\Theta(\tau_2)\langle\xi\rangle\right)\mathcal{E}_2(\tau_2,\xi) \\
=& \exp\left(N\,(1 + M_0)\,\mu\left(\frac{\langle\xi\rangle}{N}\right)\right)\mathcal{E}_2(\tau_2,\xi)
\end{aligned}$$

uniformly in $Z_{\Psi,2}$.

3.5 Estimate in Z_R

Let $(t,\xi)\in Z_R$. In the same way as the estimate in $Z_{\Psi,2}$ we have

$$\mathcal{E}_2(t,\xi)\le \exp\left((1+M_0)^2\,\Lambda(0)R_{l_0}\right)\mathcal{E}_2(0,\xi)$$

uniformly in Z_R.

3.6 Estimate in $Z_{H,m}$

3.6.1 Refined diagonalization procedure

Let us reduce the equation in (3.1) to the following first order system:

$$\partial_t V_1 = A_1 V_1, \tag{3.9}$$

where

$$V_1=\begin{pmatrix} v_t+\mathrm{i}a(t)|\xi|v \\ v_t-\mathrm{i}a(t)|\xi|v\end{pmatrix},\quad A_1=\begin{pmatrix}\phi_1 & \overline{b_1}\\ b_1 & \overline{\phi_1}\end{pmatrix},$$

$$b_1=\overline{b_1}=-\frac{a'(t)}{2a(t)}\quad\text{and}\quad \phi_1=\frac{a'(t)}{2a(t)}+\mathrm{i}a(t)|\xi|. \tag{3.10}$$

Let us denote

$$\phi_{1\Re}=\Re\{\phi_1\}=\frac{d}{dt}\log\sqrt{a(t)},\quad \phi_{1\Im}=\Im\{\phi_1\}=a(t)|\xi|,$$

$$\lambda_1=\phi_{1\Re}+\mathrm{i}\sqrt{\phi_{1\Im}^2-|b_1|^2}$$

and

$$\theta_1=\frac{\lambda_1-\phi_1}{\overline{b_1}}=-\mathrm{i}\frac{\phi_{1\Im}}{\overline{b_1}}\left(1-\sqrt{1-\frac{|b_1|^2}{\phi_{1\Im}^2}}\right).$$

The eigenvalues $\{\lambda_{1+},\lambda_{1-}\}$ of A_1 and their corresponding eigenvectors are given by $\lambda_{1+}=\lambda_1$, $\lambda_{1-}=\overline{\lambda_1}$, and $\{{}^t(1,\theta_1),{}^t(\overline{\theta_1},1)\}$ respectively. Therefore, if $|\theta_1|<1$, then A_1 is diagonalized by the diagonalizer Ξ_1 as follows:

$$\Xi_1^{-1}A_1\Xi_1=\begin{pmatrix}\lambda_1 & 0\\ 0 & \overline{\lambda_1}\end{pmatrix},\quad \Xi_1=\begin{pmatrix}1 & \overline{\theta_1}\\ \theta_1 & 1\end{pmatrix}.$$

Let us define $V_2=\Xi_1^{-1}V_1$. Then V_2 solves the following system:

$$\partial_t V_2 = A_2 V_2, \tag{3.11}$$

$$A_2=\begin{pmatrix}\lambda_1 & 0\\ 0 & \overline{\lambda_1}\end{pmatrix}-\Xi_1^{-1}(\partial_t\Xi_1)=\begin{pmatrix}\phi_2 & \overline{b_2}\\ b_2 & \overline{\phi_2}\end{pmatrix},$$

where

$$b_2=-\frac{(\theta_1)_t}{1-|\theta_1|^2}\quad\text{and}\quad \phi_2=\phi_{2\Re}+\mathrm{i}\phi_{2\Im},$$

$$\phi_{2\Re}=\phi_{1\Re}-\partial_t\log\sqrt{1-|\theta_1|^2}$$

and

$$\phi_{2\Im}=\sqrt{\phi_{1\Im}^2-|b_1|^2}-\Im\left\{\overline{\theta_1}b_2\right\}.$$

Generally, we have the following lemma:

Lemma 3.2. *Let V_k be a solution to the following system:*

$$\partial_t V_k = A_k V_k, \quad A_k = \begin{pmatrix} \phi_k & \overline{b_k} \\ b_k & \overline{\phi_k} \end{pmatrix},$$

and the matrix Ξ_k be defined by

$$\Xi_k = \begin{pmatrix} 1 & \overline{\theta_k} \\ \theta_k & 1 \end{pmatrix}, \quad \theta_k = -\mathrm{i}\frac{\phi_{k\Im}}{\overline{b_k}} \left(1 - \sqrt{1 - \frac{|b_k|^2}{\phi_{k\Im}^2}} \right),$$

where $\phi_{k\Re} = \Re\{\phi_k\}$ and $\phi_{k\Im} = \Im\{\phi_k\}$. If $|\theta_k| < 1$, then $V_{k+1} = \Xi_k^{-1} V_k$ solves the following system:

$$\partial_t V_{k+1} = A_{k+1} V_{k+1},$$

where

$$A_{k+1} = \begin{pmatrix} \phi_{k+1} & \overline{b_{k+1}} \\ b_{k+1} & \overline{\phi_{k+1}} \end{pmatrix}, \quad b_{k+1} = -\frac{(\theta_k)_t}{1 - |\theta_k|^2}$$

and

$$\phi_{k+1} = \phi_k - \partial_t \log \sqrt{1 - |\theta_k|^2} + \mathrm{i} \left(-\phi_{k\Im} + \sqrt{\phi_{k\Im}^2 - |b_k|^2} - \Im\{\overline{\theta_k} b_{k+1}\} \right).$$

Proof. The proof is straightforward. □

Let $(t, \xi) \in Z_{H,k}$, and denote $V_k = {}^t(V_{k,1}, V_{k,2})$. Then we obtain

$$\begin{aligned} \partial_t |V_k|^2 =& 2\Re \left(A_k V_k, V_k \right)_{\mathbb{C}^2} = 2\phi_{k\Re} |V_k|^2 + 4\Re\{b_k V_{k,1} \overline{V_{k,2}}\} \\ \leq& 2 \left(\phi_{k\Re} + |b_k| \right) |V_k|^2, \end{aligned}$$

which follows that

$$|V_k(t, \xi)|^2 \leq \exp \left(2 \int_\tau^t \left(\phi_{k\Re}(s, \xi) + |b_k(s, \xi)| \right) ds \right) |V_k(\tau, \xi)|^2 \tag{3.12}$$

for $t_k \leq \tau < t \leq t_{k-1}$. Thus we must consider the invertibility of Ξ_k, the estimates for $\int_\tau^t \phi_{k\Re}(s, \xi)\, ds$ and $\int_\tau^t |b_k(t, \xi)|\, ds$ if we derive suitable estimates in $Z_{H,k}$ corresponding to the estimates in the zones $Z_{\Psi,0}$, $Z_{\Psi,1}$, $Z_{\Psi,2}$ and Z_R.

3.6.2 Symbol class in $Z_{H,m}$

We define the constants κ_1, κ and ν by

$$\kappa_1 = \frac{4\pi^2 M_0}{3}, \quad \kappa = 16\kappa_1^4 \text{ and } \nu = \mathrm{e}^2 C_0 \kappa_1. \tag{3.13}$$

Let us fix a positive integer m. For integers p, q and r with $0 \leq p \leq m$, and positive real numbers K and N, the symbol class $S^{(p)}\{q, r; K, N\}$ is the set of functions satisfying

$$\left| \partial_t^k f(t, \xi) \right| \leq K \frac{M_{r+k}}{(r + k + 1)^2} \left(\lambda(t) \langle \xi \rangle \right)^q \left(\nu \rho(t) \right)^{r+k}$$

for $k = 0, \ldots, p$ in $Z_{H,m}$. Here we introduce the notation

$$S^{(p)}\{q, r; K, N\} = S^{(p)}\{q, r; K\} \text{ and } S^{(p)}\{q, r; 1\} = S^{(p)}\{q, r\}$$

without any confusion. From the definition we immediately know that $S^{(p_1)}\{q, r; K\} \subset S^{(p_2)}\{q, r; K\}$ for any $p_1 > p_2$. Moreover, we have the following lemma:

Lemma 3.3. *The following properties are established in $Z_{H,m}$:*
(i) If $f \in S^{(p)}\{q,r;K\}$ and $p \geq 1$, then $\partial_t f \in S^{(p-1)}\{q,r+1;K\}$.
(ii) If $f_1 \in S^{(p)}\{q,r;K_1\}$ and $f_2 \in S^{(p)}\{q,r;K_2\}$, then $f_1+f_2 \in S^{(p)}\{q,r;K_1+K_2\}$.
(iii) If $f \in S^{(p)}\{q,r;K_1\}$ and $K_2>0$, then $K_2 f \in S^{(p)}\{q,r;K_1K_2\}$.
(iv) If $f_1 \in S^{(p)}\{q_1,r_1;K_1\}$ and $f_2 \in S^{(p)}\{q_2,r_2;K_2\}$, then $f_1f_2 \in S^{(p)}\{q_1+q_2,r_1+r_2;\kappa_1K_1K_2\}$.
(v) If $f \in S^{(p)}\{q,r;K,N\}$, then $f \in S^{(\min\{p,m-r\})}\{q+l,r-l;K(\nu N^{-1})^l,N\}$ for any $l \leq r \leq m$.
(vi) If $f \in S^{(p)}\{q,r;K_1,N\}$ and $g \in S^{(p)}\{-r,r;K_2,N\}$, then $fg \in S^{(p)}\{q,r;\kappa_1K_1K_2(2\nu N^{-1})^r,N\}$ for any $p \leq m$ and $r \leq m$.

Proof. (i), (ii) and (iii) are evident from the definition of the symbol classes.
(iv): Let $k \in \mathbb{N}$ and assume that $r_1 \leq r_2$ without loss of generality. Because of Leibniz rule, Lemma 4.1 and Lemma 4.2 we have

$$\begin{aligned}
\left|\partial_t^k(f_1f_2)\right| \leq & K_1K_2\frac{M_{r_1+r_2}}{(r_1+r_2+k+1)^2}\left(\lambda(t)\langle\xi\rangle\right)^{q_1+q_2}\left(\nu\rho(t)\right)^{r_1+r_2+k} \\
& \times\sum_{j=0}^{k}\binom{k}{j}\frac{M_{r_1+j}M_{r_2+k-j}}{M_{r_1+r_2+k}}\left(\frac{r_1+r_2+k+j}{(r_1+j+1)(r_2+k-j+1)}\right)^2. \\
\leq & \frac{4\pi^2M_0}{3}K_1K_2\frac{M_{r_1+r_2}}{(r_1+r_2+k+1)^2}\left(\lambda(t)\langle\xi\rangle\right)^{q_1+q_2}\left(\nu\rho(t)\right)^{r_1+r_2+k}.
\end{aligned}$$

(v): Let $0 \leq k \leq \min\{p,m-r\}$. Noting (1.16) and that $\rho(t)/\lambda(t)$ is strictly increasing, we obtain

$$\begin{aligned}
\left|\partial_t^k f\right| \leq & K\frac{M_{r+k}}{(r+k+1)^2}(\lambda(t)\langle\xi\rangle)^{-r}(\nu\rho(t))^{r+k} \\
= & K\left(\frac{\nu\rho(t)}{\lambda(t)\langle\xi\rangle}\right)^r\frac{M_{r+k}}{M_k}\frac{(k+1)^2}{(r+k+1)^2}\frac{M_k}{(k+1)^2}(\nu\rho(t))^k \\
\leq & K\left(\frac{\nu\rho(t_{m-1})}{\lambda(t_{m-1})\langle\xi\rangle}\right)^r\frac{M_{r+k}}{M_k}\frac{M_k}{(k+1)^2}(\nu\rho(t))^k \\
= & K\left(\frac{\nu}{N}\right)^r\frac{(k+r)\cdots(k+1)}{m^r}\frac{\frac{M_{k+r}}{(k+r)M_{k+r-1}}\cdots\frac{M_{k+1}}{(k+1)M_k}}{\left(\frac{M_m}{mM_{m-1}}\right)^r}\frac{M_k}{(k+1)^2}(\nu\rho(t))^k \\
\leq & K\left(\frac{\nu}{N}\right)^r\frac{M_k}{(k+1)^2}(\nu\rho(t))^k.
\end{aligned}$$

(vi): Let $0 \leq k \leq p$. By Lemma 4.1 and Lemma 4.2

$$\begin{aligned}
\left|\partial_t^k(fg)\right| \leq & K_1K_2\left(\frac{\nu\rho(t)}{\lambda(t)\langle\xi\rangle}\right)^r\frac{M_{r+k}}{(r+k+1)^2}\left(\lambda(t)\langle\xi\rangle\right)^q\left(\nu\rho(t)\right)^{r+k} \\
& \times\sum_{j=0}^{k}\binom{k}{j}\frac{(r+k+1)^2}{(r+j+1)^2(r+k-j+1)^2}\frac{M_{r+j}M_{r+k-j}}{M_{r+k}} \\
\leq & \kappa_1K_1K_2\frac{M_{r+k}}{(r+k+1)^2}\left(\lambda(t)\langle\xi\rangle\right)^q\left(\nu\rho(t)\right)^{r+k} \\
& \times\left(\frac{\nu\rho(t_{m-1})}{\lambda(t_{m-1})\langle\xi\rangle}\right)^r\frac{(r+k-1)!}{(k-1)!r!}\frac{M_r}{M_0}.
\end{aligned}$$

Noting the estimates

$$\left(\frac{\nu\rho(t_{m-1})}{\lambda(t_{m-1})\langle\xi\rangle}\right)^r \frac{(r+k-1)!}{(k-1)!r!}\frac{M_r}{M_0} \le \left(\frac{\nu}{N}\right)^r \frac{(2m)^r}{r!} \frac{\frac{M_r}{M_{r-1}}\cdots\frac{M_1}{M_0}}{\left(\frac{M_m}{M_{m-1}}\right)^r}$$
$$= \left(\frac{2\nu}{N}\right)^r \frac{\frac{M_r}{rM_{r-1}}\cdots\frac{M_1}{1M_0}}{\left(\frac{M_m}{mM_{m-1}}\right)^r} \le \left(\frac{2\nu}{N}\right)^r$$

for $k \le 2m-r+1$ and $r \le m$, we complete the proof. □

Moreover, we show the following lemma:

Lemma 3.4. *Let* m*,* p *and* r *be positive integers satisfying* $\max\{p,r\} \le m$*,* $f \in S^{(p)}\{-r,r;K,N\}$ *and* $N \ge 4K\kappa_1\nu$*. Then there exist* $g_1, g_2 \in S^{(\min\{p,m-r\})}\{-r,r;2K,N\}$ *such that*

$$\frac{1}{1-f} = 1 + g_1 \tag{3.14}$$

and

$$1 - \sqrt{1-f} = \frac{1}{2}f(1+g_2). \tag{3.15}$$

Proof. Let us denote $p_0 = \min\{p, m-r\}$. By Lemma 3.3 (v) we have $f \in S^{(p_0)}\{0,0;K(\nu N^{-1})^r, N\}$, it follows that $|f| \le K(\nu N^{-1})^r < 1$. Moreover, by Lemma 3.3 (vi) we see

$$f^l \in S^{(p_0)}\left\{-r, r; K\left(K\kappa_1\left(2\nu N^{-1}\right)^r\right)^{l-1}, N\right\} \tag{3.16}$$

for $l = 2, 3, \ldots$. Therefore, by Lemma 3.3 (ii) and noting $\sum_{l=1}^{\infty}(K\kappa_1(2\nu N^{-1})^r)^{l-1} \le 2$ for $N \ge 4K\kappa_1\nu$ we have

$$g_1 = \sum_{l=1}^{\infty} f^l \in S^{(p_0)}\left\{-r, r; 2K, N\right\};$$

thus (3.14) is proved. Moreover, thanks to the representation

$$g_2 = 2\sum_{l=1}^{\infty}\binom{1/2}{l+1}(-f)^l$$

and the inequality $|\binom{1/2}{l+1}| \le 1/2$ for any $l \ge 0$ we have (3.15). □

For $f \in S^{(p)}\{q,r;K\}$ we introduce the following notation for convenience:

$$f \lesssim K\sigma^{(p)}\{q,r\} = K\sigma^{(p)}\{q,r\}(t,\xi).$$

In particular, we denote $1\sigma^{(p)}\{q,r\} = \sigma^{(p)}\{q,r\}$, that is, $\sigma^{(p)}\{q,r\}$ stands for any function in the symbol class $S^{(p)}\{q,r;1\}$. Moreover, we introduce the following notation:

- (Scalar product) For $K_1 > 0$ we define

$$K_1(K\sigma^{(p)}\{q,r\}) = (K_1K)\sigma^{(p)}\{q,r\}.$$

- (Summation) For $K_1, K_2 > 0$ we define

$$K_1\sigma^{(p)}\{q,r\} + K_2\sigma^{(p)}\{q,r\} = (K_1+K_2)\sigma^{(p)}\{q,r\};$$

  this notation is meaningful by Lemma 3.3 (ii).

- (Inclusion)

$$K_1\sigma^{(p_1)}\{q_1,r_1\} \lesssim K_2\sigma^{(p_2)}\{q_2,r_2\} \;\Leftrightarrow\; S^{(p_1)}\{q_1,r_1;K_1\} \subseteq S^{(p_2)}\{q_2,r_2;K_2\}.$$

- (Multiplication) For $K_1, K_2 > 0$ we define

$$\left(K_1\sigma^{(p)}\{q_1,r_1\}\right)\left(K_2\sigma^{(p)}\{q_2,r_2\}\right) \lesssim \kappa_1 K_1 K_2 \sigma^{(p)}\{q_1+q_2, r_1+r_2\};$$

  this notation is meaningful by Lemma 3.3 (iv).

By use of the above notation the properties of Lemma 3.3 and Lemma 3.4 are expressed as follows:

Lemma 3.5. *Let $0 \le r \le m$. Then the following properties are established in $Z_{H,m}$:*
(i) If $p_1 > p_2$, then $\sigma^{(p_1)}\{q,r\} \lesssim \sigma^{(p_2)}\{q,r\}$.
(ii) If $p \ge 1$ and $f \lesssim \sigma^{(p)}\{q,r\}$, then $\partial_t f \lesssim \sigma^{(p-1)}\{q,r+1\}$.
(iii) $\sigma^{(p)}\{q,r\} \lesssim (\nu N^{-1})^l \sigma^{(\min\{p,m-r\})}\{q+l, r-l\}$ for $l \le r \le m$.
(iv) $\sigma^{(p)}\{q_1,r_1\}\sigma^{(p)}\{q_2,r_2\} \lesssim \kappa_1\sigma^{(p)}\{q_1+q_2,r_1+r_2\}$, and $\sigma^{(p)}\{q,r\}\sigma^{(p)}\{-r,r\} \lesssim \kappa_1(2\nu N^{-1})^r\sigma^{(p)}\{q,r\}$ for any $p \le m$ and $r \le m$.
(v) If $f \lesssim K\sigma^{(p)}\{-r,r\}$, $N \ge 4K\kappa_1\nu$ and $p, r \le m$, then $2(1-\sqrt{1-f})/f \lesssim 1+2K\sigma^{(\min\{p,m-r\})}\{-r,r\}$ for $f \ne 0$ and $1/(1-f) \lesssim 1 + 2K\sigma^{(\min\{p,m-r\})}\{-r,r\}$ due to $|f| < 1$.

3.6.3 Symbol calculus of the coefficients

Let $(t,\xi) \in Z_{H,m}$. Then we have the following lemmata:

Lemma 3.6. *For any $k = 0, 1, \dots$ the following estimates are established:*

$$\left|a^{(k)}(t)\right| \le \lambda(t)\frac{M_k}{(k+1)^2}\left(\mathrm{e}^2\rho(t)\right)^k. \tag{3.17}$$

Proof. The proof is straightforward as below:

$$\begin{aligned}\left|a^{(k)}(t)\right| \le& \lambda(t)M_k\rho(t)^k = \lambda(t)\frac{M_k}{(k+1)^2}\left((k+1)^{\frac{2}{k}}\rho(t)\right)^k\\ \le& \lambda(t)\frac{M_k}{(k+1)^2}\left(\mathrm{e}^2\rho(t)\right)^k.\end{aligned}$$

□

Lemma 3.7. *Let b_1 be given by (3.10). Then the following estimates are established:*

$$b_1 \lesssim \frac{C_0}{2M_0}\sigma^{(m-1)}\{0,1\} \tag{3.18}$$

and

$$\frac{b_1}{\phi_{1\Im}} \lesssim \frac{\kappa_1 C_0^2}{M_0^2}\sigma^{(m-1)}\{-1,1\}. \tag{3.19}$$

Proof. We note that the following estimates are valid for any $j \in \mathbb{N}$:

$$\left|\left(\frac{1}{a}\right)^{(j)}\right| \leq \frac{C_0}{M_0}\lambda(t)^{-1}\frac{M_j}{(j+1)^2}(\nu\rho(t))^j. \tag{3.20}$$

Indeed, if (3.20) is valid for any $0 \leq j \leq l$, then by virtue of Leibniz rule we see

$$0 = \left|(1)^{(l+1)}\right| = \sum_{j=0}^{l+1}\binom{l+1}{j}a^{(j)}\left(\frac{1}{a}\right)^{(l-j+1)},$$

which follows that

$$\left(\frac{1}{a}\right)^{(l+1)} = -\frac{1}{a}\sum_{j=1}^{l+1}\binom{l+1}{j}a^{(j)}\left(\frac{1}{a}\right)^{(l-j+1)}.$$

By Lemma 3.6, Lemma 4.1 and Lemma 4.2 we have

$$\begin{aligned}
\left|\left(\frac{1}{a}\right)^{(l+1)}\right| \leq& \frac{C_0^2}{M_0\lambda(t)}\sum_{j=1}^{l+1}\binom{l+1}{j}\frac{M_j}{(j+1)^2}\left(e^2\rho(t)\right)^j\frac{M_{l-j+1}}{(l-j+2)^2}(\nu\rho(t))^{l-j+1} \\
\leq& \frac{C_0^2}{M_0\lambda(t)}\frac{M_{l+1}}{(l+2)^2}(\nu\rho(t))^{l+1}\sum_{j=1}^{l+1}\binom{l+1}{j}\frac{M_jM_{l-j+1}}{M_{l+1}}\left(\frac{e^2}{\nu}\right)^j\frac{(l+2)^2}{(j+1)^2(l-j+2)^2} \\
\leq& \frac{4\pi^2e^2C_0^2}{3\nu\lambda(t)}\frac{M_{l+1}}{(l+2)^2}(\nu\rho(t))^{l+1} \\
=& \frac{C_0}{M_0}\lambda(t)^{-1}\frac{M_{l+1}}{(l+2)^2}(\nu\rho(t))^{l+1}
\end{aligned}$$

for $\nu = e^2C_0\kappa_1$. Thus the estimate (3.20) is valid for $j = l+1$. Consequently, by the estimates (3.17) and (3.20) we obtain

$$\begin{aligned}
\left|b_1^{(k-1)}(t)\right| \leq& \frac{C_0^2}{2M_0}\frac{M_k}{(k+1)^2}(\nu\rho(t))^k\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(k+1)^2}{(j+2)^2(k-j)^2}\left(\frac{e^2}{\nu}\right)^{j+1}\frac{M_{k-j-1}M_{j+1}}{M_k} \\
\leq& \frac{C_0}{2M_0}\frac{M_k}{(k+1)^2}(\nu\rho(t))^k,
\end{aligned}$$

that is, (3.18) is valid. By noting $\langle\xi\rangle \geq 2/\sqrt{3}$ in Z_H, it follows that $|\xi| \geq \langle\xi\rangle/2$, and so that

$$\frac{1}{\phi_{1\Im}} = \frac{1}{a(t)|\xi|} \lesssim \frac{2C_0}{M_0}\sigma^{(m)}\{-1,0\}.$$

Therefore, by Lemma 3.5 (iv) we know (3.19). □

Lemma 3.8. *Let $\kappa_2 = \kappa_1 C_0^2/M_0^2 (> \kappa_1)$. Then the following estimates are established in $Z_{H,m}$:*

$$\theta_1 \lesssim \kappa_2\sigma^{(m-1)}\{-1,1\} \quad \text{and} \quad \det\Xi_1 \geq 1 - 1/\kappa_2^2 > 0. \tag{3.21}$$

Proof. By Lemma 3.5 (iv) and (3.19) we have

$$\frac{|b_1|^2}{\phi_{1\Im}^2} \lesssim 2\kappa_1\kappa_2^2\nu N^{-1}\sigma^{(m-1)}\{-1,1\}.$$

Therefore, applying Lemma 3.5 (iv), (v) for $K = 2\kappa_1\kappa_2^2\nu N^{-1}$ and $r = 1$, we observe

$$\begin{aligned}\theta_1 =& \frac{-\frac{ib_1}{\phi_{1\Im}}\left(1-\sqrt{1-\frac{|b_1|^2}{\phi_{1\Im}^2}}\right)}{\frac{|b_1|^2}{\phi_{1\Im}^2}} \lesssim \kappa_2\sigma^{(m-1)}\{-1,1\}\left(\frac{1}{2}+2\kappa_1\kappa_2^2\nu N^{-1}\sigma^{(m)}\{-1,1\}\right) \\ \lesssim& \left(\frac{\kappa_2}{2}+\frac{2^2\kappa_1^2\kappa_2^3\nu^2}{N^2}\right)\sigma^{(m-1)}\{-1,1\} \lesssim \kappa_2\sigma^{(m-1)}\{-1,1\}\end{aligned} \tag{3.22}$$

for $N \geq N_1 = 2\sqrt{2}\kappa_1\kappa_2\nu$. Moreover, by Lemma 3.5 (iii), (iv) we get

$$\begin{aligned}|\theta_1|^2 \lesssim& \frac{2\kappa_1\kappa_2^2\nu}{N}\sigma^{(m-1)}\{-1,1\} \\ \lesssim& \frac{2\kappa_1\kappa_2^2\nu^2}{N^2}\sigma^{(m-1)}\{0,0\} \lesssim (4\kappa_1)^{-1}\sigma^{(m-1)}\{0,0\}\end{aligned}$$

for $N \geq N_1$. It follows that $\det \Xi_1 = 1 - |\theta_1|^2 \geq 1 - (4\kappa_1)^{-2} > 0$. □

By Lemma 3.8 we can reduce the equation (3.9) to (3.11) by Ξ_1 in $Z_{H,m}$. Up to now, we can carry out the diagonalization procedures of Lemma 3.2 for $k \leq m-1$ in $Z_{H,m}$; indeed the following proposition ensures that.

Proposition 3.1. *Let V_1 be a solution to (3.9). Then the diagonalization procedures of Lemma 3.2 can be carried out for $k = 1, \dots, m-1$ in $Z_{H,m}$. Moreover, we have*

$$b_k \lesssim \kappa^k\sigma^{(m-k)}\{-k+1,k\} \tag{3.23}$$

for $k = 1, \dots, m$.

For a preparation to prove Proposition 3.1 we introduce the following lemma:

Lemma 3.9. *Let $(t,\xi) \in Z_{H,m}$ and $1 \leq k \leq m-1$. If $b_k \lesssim \kappa^k\sigma^{(m-k)}\{-k+1,k\}$ and $\theta_k \lesssim \kappa^k\sigma^{(m-k)}\{-k,k\}$, then $\det \Xi_k > 0$, $b_{k+1} \lesssim 2\kappa_1\kappa^k\sigma^{(m-k-1)}\{-k,k+1\}$ and $\phi_{1\Im}/\phi_{(k+1)\Im} - 1 \lesssim \sigma^{(m-k-1)}\{0,0\}$ for $N \geq N_2 = 32\nu^2\kappa_1^3\kappa^2 (\geq N_1)$.*

Proof. Suppose that $\theta_k \lesssim \kappa^k\sigma^{(m-k)}\{-k,k\}$. Then by Lemma 3.5 (iii) and (iv) we have

$$\begin{aligned}|\theta_k|^2 \lesssim& \kappa_1\left(\frac{2\nu\kappa^2}{N}\right)^k\sigma^{(m-k)}\{-k,k\} \lesssim \kappa_1\left(\frac{2\nu^2\kappa^2}{N^2}\right)^k\sigma^{(m-k)}\{0,0\} \\ \lesssim& \kappa_1\left(\frac{1}{4\kappa_1^2}\right)^k\sigma^{(m-k)}\{0,0\} \lesssim \frac{1}{4\kappa_1}\sigma^{(m-k)}\{0,0\} \\ \lesssim& \frac{1}{4}\sigma^{(m-k)}\{0,0\}\end{aligned}$$

for $N \geq N_2$. It follows that $\det \Xi_k > 0$. Moreover, applying Lemma 3.5 (v) with $K = 1/4$ and $r = 0$, we have

$$\frac{1}{1 \pm |\theta_k|^2} \lesssim 1 + \frac{1}{2}\sigma^{(m-k)}\{0,0\} \lesssim 2\sigma^{(m-k)}\{0,0\}.$$

Therefore, due to Lemma 3.5 (ii) and the representation of Lemma 3.2 we obtain

$$b_{k+1} \lesssim \left(\kappa^k\sigma^{(m-k-1)}\{-k,k+1\}\right)\left(2\sigma^{(m-k)}\{0,0\}\right) \lesssim 2\kappa_1\kappa^k\sigma^{(m-k-1)}\{-k,k+1\}$$

and

$$\frac{|b_k|^2}{\phi_{k\Im}^2} = \left(\frac{2|\theta_k|}{1+|\theta_k|^2}\right)^2 \lesssim \left(\left(2\kappa^k\sigma^{(m-k)}\{-k,k\}\right)\left(2\sigma^{(m-k)}\{0,0\}\right)\right)^2$$
$$\lesssim \left(4\kappa_1\kappa^k\sigma^{(m-k)}\{-k,k\}\right)^2 \lesssim 16\kappa_1^3\left(\frac{2\nu\kappa^2}{N}\right)^k\sigma^{(m-k)}\{-k,k\}$$
$$\lesssim 16\kappa_1^3\left(\frac{2\nu^2\kappa^2}{N^2}\right)^k\sigma^{(m-k)}\{0,0\}.$$

We denote

$$\alpha_k = -1+\sqrt{1-\frac{|b_k|^2}{\phi_{k\Im}^2}} \quad \text{and} \quad \beta_k = -\frac{\Im\{\overline{\theta_k}b_{k+1}\}}{\phi_{1\Im}}.$$

Applying Lemma 3.5 (v) for $K = 16\kappa_1^3(2\nu^2\kappa^2N^{-2})^k$ and $r=0$, we have

$$\alpha_k \lesssim 16\kappa_1^3\left(\frac{2\nu^2\kappa^2}{N^2}\right)^k\sigma^{(m-k)}\{0,0\}\left(\frac{1}{2}+16\kappa_1^3\left(\frac{2\nu^2\kappa^2}{N^2}\right)^k\sigma^{(m-k)}\{0,0\}\right)$$
$$\lesssim 16\kappa_1^3\left(\frac{2\nu^2\kappa^2}{N^2}\right)^k\sigma^{(m-k)}\{0,0\}\left(\frac{1}{2}+\frac{1}{2\kappa_1}\sigma^{(m-k)}\{0,0\}\right)$$
$$\lesssim 16\kappa_1^3\left(\frac{2\nu^2\kappa^2}{N^2}\right)^k\sigma^{(m-k)}\{0,0\} \lesssim N^{-k}\sigma^{(m-k)}\{0,0\}$$

and

$$\beta_k \lesssim \left(\kappa^k\sigma^{(m-k)}\{-k,k\}\right)\left(2\kappa_1\kappa^k\sigma^{(m-k-1)}\{-k,k+1\}\right)\left(\frac{2C_0}{M_0}\sigma^{(m)}\{-1,0\}\right)$$
$$\lesssim \frac{4\kappa_1^3C_0}{M_0}\left(\frac{2\nu\kappa^2}{N}\right)^k\left(\sigma^{(m-k-1)}\{-k-1,k+1\}\right)$$
$$\lesssim \frac{4\nu\kappa_1^3C_0}{M_0N}\left(\frac{2\nu^2\kappa^2}{N^2}\right)^k\sigma^{(m-k-1)}\{0,0\} \lesssim N^{-k}\sigma^{(m-k)}\{0,0\}$$

for $N \geq N_2$. We stand for $\sigma_0 = \sigma^{(m-k-1)}\{0,0\}$ and

$$\psi_k = \prod_{l=1}^{k}(1+\alpha_l) + \sum_{j=1}^{k-1}\beta_j\prod_{l=j+1}^{k}(1+\alpha_l) + \beta_k.$$

Owing to the representation in Lemma 3.2, we have

$$\phi_{(k+1)\Im} = \phi_{1\Im}\psi_k,$$

and that

$$\psi_k \lesssim \left(1+\sum_{j=1}^{k}N^{-j}\sigma_0\right)\prod_{j=1}^{k}\left(1+N^{-j}\sigma_0\right) \lesssim \left(1+\sum_{j=1}^{k}N^{-j}\sigma_0\right)\left(1+\sum_{j=1}^{\infty}\left(N^{-1}\sigma_0\right)^j\right)^2$$
$$\lesssim \left(1+\sum_{j=1}^{k}N^{-j}\sigma_0\right)\left(1+\sum_{j=1}^{\infty}\left(\kappa_1N^{-1}\right)^j\sigma_0\right)^2 \lesssim \left(1+\frac{\kappa_1N^{-1}}{1-\kappa_1N^{-1}}\sigma_0\right)^3$$
$$\lesssim \left(1+2\kappa_1N^{-1}\sigma_0\right)^3 = 1+6\kappa_1N^{-1}\sigma_0+12\kappa_1^2N^{-2}\sigma_0^2+8\kappa_1^3N^{-3}\sigma_0^3$$
$$\lesssim 1+2N^{-1}\left(3\kappa_1+6\kappa_1^3+4\kappa_1^5\right)\sigma_0.$$

In the sequel, for $N \geq (N_2 \geq) 4\kappa_1(3\kappa_1 + 3\kappa_1^2 + \kappa_1^3)$ we have

$$\frac{\phi_{1\Im}}{\phi_{(k+1)\Im}} - 1 = \frac{1-\psi_k}{\psi_k} = \sum_{j=1}^{\infty}(1-\psi_k)^j \lesssim \sum_{j=1}^{\infty}\left(\frac{1}{2\kappa_1}\sigma_0\right)^j \lesssim \sum_{j=1}^{\infty}\left(\frac{1}{2}\right)^j \sigma_0 = \sigma_0.$$

Thus we complete the proof. □

Proof of Proposition 3.1. By noting $\kappa \geq \kappa_2$, the assumptions of Lemma 3.9 are satisfied for $k = 1$ because of (3.18) and (3.22). Assume that $b_{k+1} \lesssim 2\kappa_1\kappa^k\sigma^{(m-k-1)}\{-k, k+1\}$ and $\phi_{1\Im}/\phi_{(k+1)\Im} - 1 \lesssim \sigma^{(m-k-1)}\{0,0\}$. Then we have

$$\begin{aligned}
\frac{b_{k+1}}{\phi_{(k+1)\Im}} =& \frac{b_{k+1}}{\phi_{1\Im}}\left(1 + \frac{\phi_{1\Im}}{\phi_{(k+1)\Im}} - 1\right) \\
\lesssim& \left(\frac{4C_0\kappa_1^2}{M_0}\kappa^k\sigma^{(m-k-1)}\{-k-1, k+1\}\right)\left(1 + \sigma^{(m-k-1)}\{0,0\}\right) \\
\lesssim& \frac{8C_0\kappa_1^3}{M_0}\kappa^k\sigma^{(m-k-1)}\{-k-1, k+1\},
\end{aligned}$$

and hence

$$\begin{aligned}
\frac{|b_{k+1}|^2}{\phi_{(k+1)\Im}^2} \lesssim& \frac{64C_0^2\kappa_1^7}{M_0^2\kappa^2}\left(\frac{2\nu\kappa^2}{N}\right)^{k+1}\sigma^{(m-k-1)}\{-k-1, k+1\} \\
\lesssim& \frac{64C_0^2\kappa_1^7}{M_0^2\kappa^2}\left(\frac{2\nu^2\kappa^2}{N^2}\right)^{k+1}\sigma^{(m-k-1)}\{0,0\}.
\end{aligned}$$

Therefore, in view of Lemma 3.5 (v) we obtain

$$\begin{aligned}
\theta_{k+1} =& -\mathrm{i}\frac{b_{k+1}}{\phi_{(k+1)\Im}}\frac{1 - \sqrt{1 - \frac{|b_{k+1}|^2}{\phi_{(k+1)\Im}^2}}}{\frac{|b_{k+1}|^2}{\phi_{(k+1)\Im}^2}} \\
\lesssim& \frac{8C_0\kappa_1^3}{M_0}\kappa^k\sigma^{(m-k-1)}\{-k-1, k+1\}\left(\frac{1}{2} + \frac{64C_0^2\kappa_1^7}{M_0^2\kappa^2}\left(\frac{2\nu^2\kappa^2}{N^2}\right)^{k+1}\sigma^{(m-k-1)}\{0,0\}\right) \\
\lesssim& \frac{8C_0\kappa_1^4}{M_0}\kappa^k\sigma^{(m-k-1)}\{-k-1, k+1\}
\end{aligned}$$

for $N \geq (N_2 \geq)(512C_0^2\nu^4\kappa_1^7\kappa^2/M_0^2)^{1/4}$. It follows that $\theta_{k+1} \lesssim \kappa^{k+1}\sigma^{(m-k-1)}\{-k-1, k+1\}$ and $b_{k+1} \lesssim \kappa^{k+1}\sigma^{(m-k-1)}\{-k, k+1\}$ for $\kappa = 16\kappa_1^4 (= \max\{16\kappa_1^4, \kappa_2\})$ and $N \geq N_2$. Thus the proposition is proved by applying Lemma 3.9. □

By Proposition 3.1 we have the following lemma:

Lemma 3.10. *Let $m_0, m \in \mathbb{N}$ satisfy $m \geq l_0 (\geq 2)$, $t_{m_0} \leq \tau_2 \leq t_{m_0-1}$ and $t_{m-1} \leq \tau_2$, that is, $m - 1 \leq m_0$. Then the following estimates are established:*

$$\int_{t_m}^{t_{m-1}} |b_m(\tau, \xi)|\, d\tau \leq N\left(\frac{\kappa}{N}\right)^m \mu(\langle\xi\rangle) \tag{3.24}$$

and

$$\int_{t_{m_0}}^{\tau_2} |b_{m_0}(\tau, \xi)|\, d\tau \leq N\left(\frac{\kappa}{N}\right)^{m_0} \mu(\langle\xi\rangle). \tag{3.25}$$

Proof. Let $m \geq 2$. We note that $\eta(r)/r$ and $r/\mathcal{M}^{-1}(r)$ are monotonically increasing, and so that $\rho(t)\Lambda(t)/\lambda(t)$ is monotonically increasing. Therefore we have

$$\left(\frac{\rho(\tau)}{\lambda(\tau)}\right)^m \leq \left(\frac{\rho(t_{m-1})\Lambda(t_{m-1})}{\lambda(t_{m-1})}\right)^m \Lambda(\tau)^{-m} \quad (t_m \leq t \leq t_{m-1})$$

and

$$\left(\frac{\rho(\tau)}{\lambda(\tau)}\right)^{m_0} \leq \left(\frac{\Lambda(\tau_2)\rho(\tau_2)}{\lambda(\tau_2)}\right)^{m_0} \Lambda(\tau)^{-m_0} \quad (t_{m_0} \leq t \leq \tau_2).$$

Moreover, by noting

$$\frac{M_{m_0+1}\rho(t_{m_0})}{M_{m_0}\lambda(t_{m_0})} = \frac{M_{m_0}\rho(t_{m_0-1})}{M_{m_0-1}\lambda(t_{m_0-1})} \geq \frac{M_{m_0}\rho(\tau_2)}{M_{m_0-1}\lambda(\tau_2)},$$

it follows that

$$\frac{\frac{\rho(t_{m_0})}{\lambda(t_{m_0})}}{\frac{\rho(\tau_2)}{\lambda(\tau_2)}} \frac{M_{m_0+1}}{M_{m_0}} \in \left[\frac{M_{m_0}}{M_{m_0-1}}, \frac{M_{m_0+1}}{M_{m_0}}\right],$$

and the equalities

$$1 = \frac{NM_{m_0+1}}{\langle\xi\rangle M_{m_0}} \frac{\frac{\rho(t_{m_0})}{\lambda(t_{m_0})}}{\frac{\rho(\tau_2)}{\lambda(\tau_2)}} \frac{\eta\left(\frac{1}{\Theta(\tau_2)}\right)}{\mathcal{M}^{-1}\left(\frac{\Lambda(\tau_2)}{\Theta(\tau_2)}\right)} = \frac{M_{m_0+1}}{M_{m_0}} \frac{\frac{\rho(t_{m_0})}{\lambda(t_{m_0})}}{\frac{\rho(\tau_2)}{\lambda(\tau_2)}} \frac{1}{\mathcal{M}^{-1}\left(\frac{\Lambda(\tau_2)}{\Theta(\tau_2)}\right)}.$$

Hence we have

$$\frac{\Lambda(\tau_2)}{\Theta(\tau_2)} = \mathcal{M}\left(\frac{M_{m_0+1}}{M_{m_0}} \frac{\frac{\rho(t_{m_0})}{\lambda(t_{m_0})}}{\frac{\rho(\tau_2)}{\lambda(\tau_2)}}\right) = \frac{M_{m_0+1}^{m_0}}{M_{m_0}^{m_0+1}} \left(\frac{\frac{\rho(t_{m_0})}{\lambda(t_{m_0})}}{\frac{\rho(\tau_2)}{\lambda(\tau_2)}}\right)^{m_0}. \tag{3.26}$$

Consequently, we see

$$\begin{aligned}
\frac{M_m}{\langle\xi\rangle^{m-1}} \left(\frac{\rho(t_{m-1})}{\lambda(t_{m-1})}\right)^m \Lambda(t_{m-1}) =& N^{-m+1} M_m \left(\frac{N\rho(t_{m-1})}{\langle\xi\rangle\lambda(t_{m-1})}\right)^{m-1} \frac{\rho(t_{m-1})\Lambda(t_{m-1})}{\lambda(t_{m-1})} \\
\leq& N^{-m+1} M_m \left(\frac{N\rho(t_{m-1})}{\langle\xi\rangle\lambda(t_{m-1})}\right)^{m-1} \frac{\rho(\tau_2)\Lambda(\tau_2)}{\lambda(\tau_2)} \\
=& \frac{N^{-m+1}}{\eta^{-1}\left(\frac{\langle\xi\rangle}{N}\right)} \frac{M_{m-1}^{m-1}}{M_m^{m-2}} \frac{\rho(\tau_2)}{\lambda(\tau_2)} \frac{\Lambda(\tau_2)}{\Theta(\tau_2)} \\
\leq& \frac{N^{-m+1}}{\eta^{-1}\left(\frac{\langle\xi\rangle}{N}\right)} \frac{M_{m-1}^{m-1}}{M_m^{m-2}} \frac{\rho(t_{m_0})}{\lambda(t_{m_0})} \frac{M_{m_0+1}^{m_0}}{M_{m_0}^{m_0+1}} \left(\frac{\frac{\rho(t_{m_0})}{\lambda(t_{m_0})}}{\frac{\rho(\tau_2)}{\lambda(\tau_2)}}\right)^{m_0-1} \\
\leq& \frac{N^{-m}\langle\xi\rangle}{\eta^{-1}\left(\frac{\langle\xi\rangle}{N}\right)} \frac{M_{m-1}^{m-1}}{M_m^{m-2}} \frac{M_{m_0+1}^{m_0-1}}{M_{m_0}^{m_0}} \\
\leq& N^{-m+1}\mu\left(\frac{\langle\xi\rangle}{N}\right) \leq N^{-m+1}\mu\left(\langle\xi\rangle\right)
\end{aligned}$$

due to $m-1 \le m_0$, and

$$\begin{aligned}\frac{M_{m_0}}{\langle\xi\rangle^{m_0-1}}\left(\frac{\rho(\tau_2)}{\lambda(\tau_2)}\right)^{m_0}\Lambda(\tau_2) =& N^{-m_0}M_{m_0}\left(\frac{N\rho(t_{m_0})}{\langle\xi\rangle\lambda(t_{m_0})}\right)^{m_0}\left(\frac{\frac{\rho(\tau_2)}{\lambda(\tau_2)}}{\frac{\rho(t_{m_0})}{\lambda(t_{m_0})}}\right)^{m_0}\Lambda(\tau_2)\langle\xi\rangle\\ =& N^{-m_0+1}\frac{M_{m_0}^{m_0+1}}{M_{m_0+1}^{m_0}}\left(\frac{\frac{\rho(\tau_2)}{\lambda(\tau_2)}}{\frac{\rho(t_{m_0})}{\lambda(t_{m_0})}}\right)^{m_0}\frac{\Lambda(\tau_2)}{\Theta(\tau_2)}\frac{\frac{\langle\xi\rangle}{N}}{\eta^{-1}\left(\frac{\langle\xi\rangle}{N}\right)}\\ \le& N^{-m_0+1}\mu\left(\frac{\langle\xi\rangle}{N}\right) \le N^{-m_0+1}\mu\left(\langle\xi\rangle\right).\end{aligned}$$

Therefore, by Proposition 3.1 we have

$$\begin{aligned}\int_{t_m}^{t_{m-1}}|b_m(\tau,\xi)|\,d\tau \le& \frac{\kappa^m M_m}{(m+1)^2\langle\xi\rangle^{m-1}}\int_{t_m}^{t_{m-1}}\left(\frac{\rho(\tau)}{\lambda(\tau)}\right)^m\lambda(\tau)\,d\tau\\ \le& \frac{\kappa^m M_m}{\langle\xi\rangle^{m-1}}\left(\frac{\Lambda(t_{m-1})\rho(t_{m-1})}{\lambda(t_{m-1})}\right)^m\int_{t_m}^{t_{m-1}}\lambda(\tau)\Lambda(\tau)^{-m}\,d\tau\\ \le& \frac{\kappa^m M_m}{\langle\xi\rangle^{m-1}}\left(\frac{\rho(t_{m-1})}{\lambda(t_{m-1})}\right)^m\Lambda(t_{m-1})\\ \le& N\left(\frac{\kappa}{N}\right)^m\mu\left(\langle\xi\rangle\right)\end{aligned}$$

and

$$\begin{aligned}\int_{t_{m_0}}^{\tau_2}|b_{m_0}(\tau,\xi)|\,d\tau \le& \frac{\kappa^{m_0}M_{m_0}}{(m_0+1)^2\langle\xi\rangle^{m_0-1}}\int_{t_{m_0}}^{\tau_2}\left(\frac{\rho(\tau)}{\lambda(\tau)}\right)^{m_0}\lambda(\tau)\,d\tau\\ \le& \frac{\kappa^{m_0}M_{m_0}}{\langle\xi\rangle^{m_0-1}}\left(\frac{\rho(\tau_2)}{\lambda(\tau_2)}\right)^{m_0}\Lambda(\tau_2)\\ \le& N\left(\frac{\kappa}{N}\right)^{m_0}\mu\left(\langle\xi\rangle\right).\end{aligned}$$

Thus the proof of the lemma is completed. □

3.6.4 Uniform estimate in $Z_{H,m}$

Let $(t,\xi) \in Z_{H,m}$, $m_0 \in \mathbb{N}$ be defined in Lemma 3.10, and $m_1 \in \mathbb{N}$ satisfy

$$\frac{NM_{m_1}\rho(0)}{M_{m_1-1}\lambda(0)} \le \langle\xi\rangle \le \frac{NM_{m_1+1}\rho(0)}{M_{m_1}\lambda(0)}.$$

Here we note that $(0,\xi) \in Z_{H,m_1}$. Let $m_1 \le k \le m_0$ and $(t,\xi) \in Z_{H,k}$. Then by the representation

$$\phi_{k\Re} = \partial_t\left(\log\sqrt{a(t)} - \sum_{j=1}^{k-1}\log\sqrt{1-|\theta_j|^2}\right),$$

Lemma 3.10 and (3.12), we get

$$|V_k(t)|^2 \le \exp\left(2N\left(\frac{\kappa}{N}\right)^k\mu\left(\langle\xi\rangle\right)\right)\frac{a(t)}{a(\tau)}\left(\prod_{j=1}^{k-1}\frac{1-|\theta_j(\tau)|^2}{1-|\theta_j(t)|^2}\right)|V_k(\tau)|^2$$

for $t_k \le \tau < t \le t_{k-1}$. Moreover, noting

$$\begin{aligned}|V_{k+1}|^2 = \left|\Xi_k^{-1} V_k\right|^2 = & \frac{1}{\left(1-|\theta_k|^2\right)^2}\left(\left(1+|\theta_k|^2\right)|V_k|^2 - 4\Re\left(\theta_k V_{k,1}, V_{k,2}\right)_{\mathbb{C}^2}\right) \\ \lesseqgtr & \frac{1}{\left(1 \mp |\theta_k|\right)^2}|V_k|^2,\end{aligned}$$

and denoting

$$\mu_k = 2N\left(\frac{\kappa}{N}\right)^k \mu\left(\langle \xi \rangle\right),$$

we have

$$|V_k(t_{k-1})|^2 \le \mathrm{e}^{\mu_k} \frac{a(t_{k-1})}{a(t_k)} \frac{\prod_{j=1}^{k-1}\left(1-|\theta_j(t_k)|^2\right)}{\prod_{j=1}^{k-1}\left(1-|\theta_j(t_{k-1})|^2\right)}\left(1+|\theta_k(t_k)|\right)^2 |V_{k+1}(t_k)|^2. \tag{3.27}$$

Applying the estimate (3.27) from $k = m+1$ to $k = m_1$, we obtain

$$\begin{aligned}|V_{m+1}(t_m)|^2 \le & \mathrm{e}^{\mu_{m+1}} \frac{a(t_m)}{a(t_{m+1})} \frac{\prod_{j=1}^{m}\left(1-|\theta_j(t_{m+1})|^2\right)}{\prod_{j=1}^{m}\left(1-|\theta_j(t_m)|^2\right)}\left(1+|\theta_{m+1}(t_{m+1})|\right)^2 |V_{m+2}(t_{m+1})|^2 \\ \le & \mathrm{e}^{\mu_{m+1}+\mu_{m+2}} \frac{a(t_m)}{a(t_{m+2})} \frac{\left(1+|\theta_{m+1}(t_{m+1})|\right)}{\left(1-|\theta_{m+1}(t_{m+1})|\right)} \frac{\left(1+|\theta_{m+2}(t_{m+2})|\right)}{\left(1-|\theta_{m+2}(t_{m+2})|\right)} \\ & \times \frac{\prod_{j=1}^{m+2}\left(1-|\theta_j(t_{m+2})|^2\right)}{\prod_{j=1}^{m}\left(1-|\theta_j(t_m)|^2\right)}|V_{m+3}(t_{m+2})|^2 \\ & \vdots \\ \le & \exp\left(\sum_{k=m+1}^{m_1-1}\mu_k\right) \frac{a(t_m)}{a(t_{m_1-1})} \prod_{k=m+1}^{m_1-1} \frac{\left(1+|\theta_k(t_k)|\right)}{\left(1-|\theta_k(t_k)|\right)} \\ & \times \frac{\prod_{j=1}^{m_1-1}\left(1-|\theta_j(t_{m_1-1})|^2\right)}{\prod_{j=1}^{m}\left(1-|\theta_j(t_m)|^2\right)}|V_{m_1}(t_{m_1-1})|^2.\end{aligned}$$

Moreover, we have

$$\begin{aligned}|V_{m_1}(t_{m_1-1})|^2 \le & \mathrm{e}^{\mu_{m_1}} \frac{a(t_{m_1-1})}{a(0)} \frac{\prod_{j=1}^{m_1-1}\left(1-|\theta_j(0)|^2\right)}{\prod_{j=1}^{m_1-1}\left(1-|\theta_j(t_{m_1-1})|^2\right)}|V_{m_1}(0)|^2 \\ \le & \mathrm{e}^{\mu_{m_1}} \frac{a(t_{m_1-1})}{a(0)} \frac{\prod_{j=1}^{m_1-1}\left(1-|\theta_j(0)|^2\right)}{\prod_{j=1}^{m_1-1}\left(1-|\theta_j(t_{m_1-1})|^2\right)} \frac{1}{\left(1-|\theta_{m_1-1}(0)|\right)^2}|V_{m_1-1}(0)|^2 \\ & \vdots \\ \le & \mathrm{e}^{\mu_{m_1}} \frac{a(t_{m_1-1})}{a(0)} \frac{\prod_{j=1}^{m_1-1}\left(1-|\theta_j(0)|^2\right)}{\prod_{j=1}^{m_1-1}\left(1-|\theta_j(t_{m_1-1})|^2\right)} \frac{1}{\prod_{j=1}^{m_1-1}\left(1-|\theta_j(0)|\right)^2}|V_1(0)|^2 \\ = & \mathrm{e}^{\mu_{m_1}} \frac{a(t_{m_1-1})}{a(0)} \frac{1}{\prod_{j=1}^{m_1-1}\left(1-|\theta_j(t_{m_1-1})|^2\right)} \frac{\prod_{j=1}^{m_1-1}\left(1+|\theta_j(0)|\right)}{\prod_{j=1}^{m_1-1}\left(1-|\theta_j(0)|\right)}|V_1(0)|^2,\end{aligned}$$

$$|V_m(t)|^2 \le \mathrm{e}^{\mu_m} \frac{a(t)}{a(t_m)} \frac{\prod_{j=1}^{m-1}\left(1-|\theta_j(t_m)|^2\right)}{\prod_{j=1}^{m-1}\left(1-|\theta_j(t)|^2\right)}\left(1+|\theta_m(t_m)|\right)^2 |V_{m+1}(t_m)|^2,$$

and

$$|V_m(t)|^2 \geq \frac{1}{\left(1+|\theta_{m-1}(t)|\right)^2}|V_{m-1}(t)|^2 \geq \cdots \geq \frac{1}{\prod_{j=1}^{m-1}\left(1+|\theta_j(t)|\right)^2}|V_1(t)|^2.$$

Summing up the above estimates, we obtain

$$\begin{aligned}
|V_1(t)|^2 \leq & \prod_{j=1}^{m-1}\left(1+|\theta_j(t)|\right)^2 |V_m(t)|^2 \\
\leq & \mathrm{e}^{\mu_m} \frac{a(t)}{a(t_m)} \frac{\prod_{j=1}^{m-1}\left(1+|\theta_j(t)|\right)}{\prod_{j=1}^{m-1}\left(1-|\theta_j(t)|\right)} \left(\prod_{j=1}^{m-1}\left(1-|\theta_j(t_m)|^2\right)\right)\left(1+|\theta_m(t_m)|\right)^2 |V_{m+1}(t_m)|^2 \\
\leq & \exp\left(\sum_{k=m}^{m_1-1}\mu_k\right) \frac{a(t)}{a(t_{m_1-1})} \frac{\prod_{j=1}^{m-1}\left(1+|\theta_j(t)|\right)}{\prod_{j=1}^{m-1}\left(1-|\theta_j(t)|\right)} \frac{\prod_{k=m}^{m_1-1}\left(1+|\theta_k(t_k)|\right)}{\prod_{k=m}^{m_1-1}\left(1-|\theta_k(t_k)|\right)} \\
& \times \left(\prod_{j=1}^{m_1-1}\left(1-|\theta_j(t_{m_1-1})|^2\right)\right)|V_{m_1}(t_{m_1-1})|^2. \\
\leq & \exp\left(\sum_{k=m}^{m_1}\mu_k\right)\frac{a(t)}{a(0)}\left(\prod_{j=1}^{m-1}\frac{1+|\theta_j(t)|}{1-|\theta_j(t)|}\right)\left(\prod_{k=m}^{m_1-1}\frac{1+|\theta_k(t_k)|}{1-|\theta_k(t_k)|}\right) \\
& \times \left(\prod_{j=1}^{m_1-1}\frac{1+|\theta_j(0)|}{1-|\theta_j(0)|}\right)|V_1(0)|^2.
\end{aligned}$$

By Lemma 3.9 we have the following estimates:

$$\sum_{k=m}^{m_1}\mu_k \leq \mu\left(\langle\xi\rangle\right) \quad \text{and} \quad |\theta_j(\tau)| \leq \left(\frac{\nu\kappa}{N}\right)^j \leq 2^{-j}$$

for $\tau \in [t_j, t_{j-1}]$. It follows that

$$\left(\prod_{j=1}^{m-1}\frac{1+|\theta_j(t)|}{1-|\theta_j(t)|}\right)\left(\prod_{k=m}^{m_1-1}\frac{1+|\theta_k(t_k)|}{1-|\theta_k(t_k)|}\right)\left(\prod_{j=1}^{m_1-1}\frac{1+|\theta_j(0)|}{1-|\theta_j(0)|}\right) \leq \left(\prod_{j=1}^{\infty}\frac{1+2^{-j}}{1-2^{-j}}\right)^2 \leq \mathrm{e}^4.$$

Consequently, we obtain

$$|V_1(t,\xi)|^2 \leq \mathrm{e}^4 C_0^2 \exp\left(\mu\left(\langle\xi\rangle\right)\right)|V_1(0,\xi)|^2. \tag{3.28}$$

If $t_{m_0} < t \leq \tau_2$, then we also know the estimate (3.28) in the same way for the estimate in $Z_{H,m}$ by employing Lemma 3.10. In the sequel, there exists a positive constant C such that the following estimate is established:

$$|V_1(t,\xi)|^2 \leq \exp\left(C\mu\left(\langle\xi\rangle\right)\right)|V_1(0,\xi)|^2 \tag{3.29}$$

uniformly in Z_H.

3.7 Concluding of the proof

We recall that there exists a positive constant C such that the following estimate is established:

$$\mathcal{E}_j(t,\xi) \le \mathcal{E}_j(\tau_j,\xi)\exp\left(C\mu(\langle\xi\rangle)\right)$$

in $Z_{\Psi,k}$ for $k=0,1,\ldots$. From the definitions of $\mathcal{E}_j(t,\xi)$ $(j=0,1,2)$ and $V_1(t,\xi)$ we have

$$\mathcal{E}_0(\tau_0,\xi) = \frac{1}{2}\left(\frac{\Lambda(\tau_0)}{\Lambda(\tau_1)}\frac{\lambda(\tau_0)^2\Lambda(\tau_0)}{\Lambda(\tau_1)}|\xi|^2|v(\tau_0,\xi)|^2 + |v_t(\tau_0,\xi)|^2\right) \le \mathcal{E}_1(\tau_0,\xi),$$

$$\mathcal{E}_1(\tau_1,\xi) = \mathcal{E}_2(\tau_1,\xi)$$

and

$$|V_1(t,\xi)|^2 = a(t)^2|\xi|^2|v(t,\xi)|^2 + |v_t(t,\xi)|^2 \ge 2C_0^{-2}\mathcal{E}_2(t,\xi).$$

If $a_0>0$, then $|V_1(t,\xi)|^2 \simeq \mathcal{E}_j(t,\xi) \simeq \mathcal{E}(t,\xi)$ for $j=0,1,2$. Therefore, there exists a positive constant C such that the estimate (1.2) is established for any $(t,\xi)\in[0,T]\times\mathbb{R}^n$.

If $a_0=0$, then by (1.4) there exists a positive constant C such that $|\xi| \le \exp(C\mu(|\xi|))$. It follows that

$$\begin{aligned}|V_1(0,\xi)| &\le \left(C_0^2\lambda(0)^2|\xi|^2|v_0(\xi)|^2 + |v_1(\xi)|^2\right)\\ &\le \max\left\{1, C_0^2\lambda(0)^2\right\}\exp(2C\mu(|\xi|))\mathcal{E}(0,\xi).\end{aligned}$$

If $(t,\xi)\in Z_{\Psi,0}\cap\{(t,\xi)\in[0,T]\,;\,\langle\xi\rangle \ge R_{l_0}\}$, then noting the following estimates:

$$\begin{aligned}|v_t(t,\xi)|^2 \le& 2\mathcal{E}_0(t,\xi) \le 2\mathcal{E}_2(\tau_2,\xi)\exp\left(3C\mu(\langle\xi\rangle)\right) \le C_0^2|V_1(\tau_2,\xi)|^2\exp\left(3C\mu(\langle\xi\rangle)\right)\\ \le& C_0^2|V_1(0,\xi)|^2\exp\left(5C\mu(\langle\xi\rangle)\right)\end{aligned}$$

and

$$T|v_t(t,\xi)| \ge \left|\int_0^t v_t(s,\xi)ds\right| \ge |v(\xi,t)| - |v_1(\xi)|,$$

we have

$$|v(\xi,t)|^2 \le 2\left(T^2C_0^2\exp\left(5C\mu(\langle\xi\rangle)\right)+1\right)|V_1(0,\xi)|^2,$$

which follows that the estimate (1.2) is established. We have the estimate (1.2) in the other zones in the similar way. Thus we have completed the proof of Theorem 2.1.

4 Appendix

Lemma 4.1. *For any non-negative integers k, r_1 and r_2 the following estimate holds:*

$$\sum_{j=0}^{k}\left(\frac{r_1+r_2+k+j}{(r_1+j+1)(r_2+k-j+1)}\right)^2 \le \frac{4\pi^2}{3}. \tag{4.1}$$

Proof. We may suppose that $r_1 \le r_2$ without loss of generality. The proof is straightforward as follows:

$$\sum_{j=0}^{k}\left(\frac{r_1+r_2+k+j}{(r_1+j+1)(r_2+k-j+1)}\right)^2 \le 2\left(\frac{2r_2+k+1}{r_2+\left[\frac{k+1}{2}\right]+1}\right)^2\sum_{j=0}^{\infty}\frac{1}{(j+1)^2} \le \frac{4\pi^2}{3},$$

where $[\cdot]$ denotes the Gauss symbol. □

Lemma 4.2. *Let $\{M_k\}$ satisfy (1.16). For any non-negative integers j, k, q and r satisfying $0 \le j \le k$ the following estimates are established:*

$$\binom{k}{j}\frac{M_{r+j}M_{q+k-j}}{M_{q+r+k}} \le M_0, \tag{4.2}$$

and

$$\binom{k}{j}\frac{M_{r+j}M_{r+k-j}}{M_{r+k}} \le \frac{(r+k-1)!}{(k-1)!r!}M_r \tag{4.3}$$

for $k \ge 1$.

Proof. By (1.16) we have

$$\begin{aligned}\binom{k}{j}\frac{M_{r+j}M_{q+k-j}}{M_{q+r+k}} &= M_0\binom{k}{j}\prod_{l=0}^{q+k-j-1}\frac{1}{r+j+l+1}\frac{(r+j+l+1)M_{r+j+l}}{M_{r+j+l+1}}\frac{M_{l+1}}{M_l}\\ &\le M_0\binom{k}{j}\prod_{l=0}^{q+k-j-1}\frac{1}{r+j+l+1}\frac{(l+1)M_l}{M_{l+1}}\frac{M_{l+1}}{M_l}\\ &\le M_0\binom{k}{j}\prod_{l=0}^{k-j-1}\frac{l+1}{j+l+1} = M_0.\end{aligned}$$

Let us consider the case $r \ge 1$; otherwise the estimate (4.3) coincides with (4.2) as $q = r = 0$. By (1.16) we know

$$\begin{aligned}\binom{k}{j}\frac{M_{r+j}M_{r+k-j}}{M_{r+k}} &= M_r\binom{k}{j}\prod_{l=1}^{j}\frac{r+l}{r+k-j+l}\frac{\frac{M_{r+l}}{(r+l)M_{r+l-1}}}{\frac{M_{r+k-j+l}}{(r+k-j+l)M_{r+k-j+l-1}}}\\ &\le M_r\binom{k}{j}\prod_{l=1}^{j}\frac{r+l}{r+k-j+l}.\end{aligned}$$

Let us denote

$$L_j(k,r) = \frac{(k-j+r)!(j+r)!}{(k-j)!j!}$$

and

$$N_j(k,r) = \frac{\frac{(k+r-1)!}{(k-1)!r!}}{\binom{k}{j}\prod_{l=1}^{j}\frac{r+l}{r+k-j+l}} = \frac{(r+k-1)!}{(k-1)!}\frac{(k+r)!}{k!}\frac{1}{L_j(k,r)}.$$

Noting that $L_{j-1}(k,r) \le L_j(k,r)$ if and only if $2j-1 \le k$, we have

$$\max_{0\le j\le k}\{L_j(k,r)\} = L_{\left[\frac{k+1}{2}\right]}(k,r).$$

Consequently, if k is odd, then we obtain

$$\min_{0\le j\le k}\{N_j(k,r)\} = \frac{(k+r-1)\cdots k}{(\frac{k-1}{2}+r)\cdots(\frac{k-1}{2}+1)}\frac{(k+r)\cdots(k+1)}{(\frac{k+1}{2}+r)\cdots(\frac{k+1}{2}+1)} \ge 1.$$

On the other hand, if k is even, then we see

$$\min_{0\le j\le k}\{N_j(k,r)\} = \frac{(k+r-1)\cdots k}{(\frac{k}{2}+r)\cdots(\frac{k}{2}+1)}\frac{(k+r)\cdots(k+1)}{(\frac{k}{2}+r)\cdots(\frac{k}{2}+1)} \ge 1.$$

Therefore, we have $N_j(k,r) \ge 1$ for any $j \le k$, which completes the proof of (4.3). □

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