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A novel multi-component generalization of the short pulse equation and its multisoliton solutions

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We propose a novel multi-component system of nonlinear equations that generalizes the short pulse (SP) equation describing the propagation of ultra-short pulses in optical fibers. By means of the bilinear formalism combined with a hodograph transformation, we obtain its multisoliton solutions in the form of a parametric representation. Notably, unlike the determinantal solutions of the SP equation, the proposed system is found to exhibit solutions expressed in terms of pfaffians. The proof of the solutions is performed within the framework of an elementary theory of determinants. The reduced 2-component system deserves a special consideration. In particular, we show by establishing a Lax pair that the system is completely integrable. The properties of solutions such as loop solitons and breathers are investigated in detail, confirming their solitonic behavior. A variant of the 2-component system is also discussed with its multisoliton solutions.
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I. INTRODUCTION

The short pulse (SP) equation was derived as a model nonlinear equation describing the propagation of ultra-short pulses in isotropic optical fibers.[1] We write it in an appropriate dimensionless form as

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx}, \tag{1.1}$$

where $u = u(x, t)$ represents the magnitude of the electric field and subscripts x and t appended to u denote partial differentiations. The SP equation has appeared for the first time in an attempt to construct integrable differential equations associated with pseudospherical surfaces.[2] The integrability, soliton solutions, and other features of the SP equation common to the completely integrable partial differential equations (PDEs) have been studied from various points of view.[2–10] See also Ref. 11 for a recent review article on the SP equation which is mainly concerned with soliton and periodic solutions and their properties. It also provides a novel method for constructing multiperiodic solutions by means of the bilinear transformation method.

There exist a few generalizations of the SP equation to the 2-component systems that take into account the effects of polarization and nonisotropy. One is due to Pietrzyk *et al.* They proposed the following three integrable vector (or 2-component) SP equations:[12]

$$u_{xt} = u + \frac{1}{6}(u^3 + 3uv^2)_{xx}, \quad v_{xt} = v + \frac{1}{6}(v^3 + 3u^2v)_{xx}, \tag{1.2}$$

$$u_{xt} = u + \frac{1}{6}(u^3 - 3uv^2)_{xx}, \quad v_{xt} = v - \frac{1}{6}(v^3 - 3u^2v)_{xx}, \tag{1.3}$$

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx}, \quad v_{xt} = v + \frac{1}{2}(u^2v)_{xx}. \tag{1.4}$$

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Another one is given by Sakovich:[13]

$$u_{xt} = u + \frac{1}{6}(u^3 + uv^2)_{xx}, \quad v_{xt} = v + \frac{1}{6}(v^3 + u^2 v)_{xx}, \tag{1.5}$$

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx}, \quad v_{xt} = v + \frac{1}{6}(u^2 v)_{xx}. \tag{1.6}$$

As pointed out by Sakovich,[13] the two systems (1.2) and (1.3) can be reduced to the SP equation (1.1) by appropriate dependent variable transformations. Indeed, introducing the new variables p and q by $p = u + v$, $q = u - v$, the system of equations (1.2) can be decoupled and both p and q satisfy the SP equation (1.1), while for (1.3), the transformation $p = u + \mathrm{i}v$ and $q = u - \mathrm{i}v$ leads to the two decoupled SP equations as well. On the other hand, the integrability of the systems (1.5) and (1.6) was investigated by means of the Painlevé analysis. Sakovich showed that the above two systems pass the Painlevé test, providing a strong indication of their integrability. Nevertheless, their Lax representations, conservations laws, and soliton solutions have not been obtained as yet for the systems.

The purpose of this paper is to propose a novel multi-component analog of the SP equation and construct its multisoliton solutions. The system of equations presented here is composed of the following coupled nonlinear PDEs for the n variables $u_i (i = 1, 2, \ldots, n)$:

$$u_{i,xt} = u_i + \frac{1}{2}(F u_{i,x})_x, \quad (i = 1, 2, \ldots, n) \tag{1.7a}$$

with

$$F = \sum_{1 \le j < k \le n} c_{jk} u_j u_k. \tag{1.7b}$$

Here, c_{jk} are arbitrary constants with the symmetry $c_{jk} = c_{kj} (j, k = 1, 2, \ldots, n)$. For the special case of $n = 2$ with $c_{12} = 1$, this system becomes

$$u_{xt} = u + \frac{1}{2}(uvu_x)_x, \quad v_{xt} = v + \frac{1}{2}(uvv_x)_x, \tag{1.8}$$

where $u = u_1$ and $v = u_2$. Obviously, if we put $u = v$, then (1.8) reduces to the SP equation (1.1). A simple transformation recasts (1.8) to the system of equations

$$u_{xt} = u + \frac{1}{2}[(u^2 + v^2)u_x]_x, \quad v_{xt} = v + \frac{1}{2}[(u^2 + v^2)v_x]_x. \tag{1.9}$$

If $v = 0$, then this system reduces to the SP equation (1.1). The present paper is organized as follows. In Sec. II, we summarize an exact method of solution for the SP equation which will be suitable for application to the multi-component system. In Sec. III, we show by applying the standard procedure of the bilinear method that the system of equations (1.7) can be transformed to a coupled system of bilinear equations and obtain the multisoliton solution in the parametric form. Notably, the tau-functions constituting the solution are expressed in terms of pfaffians unlike the determinantal solutions of the SP equation.[9] The proof of the multisoliton solution is, however, performed with use of an elementary theory of determinants without recourse to the pfaffian theory. In Sec. IV, we consider system (1.8). In particular, we demonstrate that it is a completely integrable system by establishing a Lax pair. The multisoliton solution to the system is reduced from that of the n-component system. The properties of the 1- and 2-soliton solutions will be investigated in detail. Subsequently, we briefly discuss system (1.9). In Sec V, we conclude this study with a short summary and discuss some open problems associated with the multi-component SP equations.

II. SUMMARY OF THE EXACT METHOD OF SOLUTION

Here, we give a short summary of the exact method of solution for the SP equation. Although we have employed some nonlinear transformations to reduce the SP to the integrable sine-Gordon (sG) equations,[4,9,10] we provide a different approach which is more suitable for solving the system of equations (1.7).

A. Hodograph transformation

We first introduce the hodograph transformation $(x, t) \to (y, \tau)$ by

$$dy = r dx + \frac{1}{2}u^2 r dt, \quad d\tau = dt, \tag{2.1a}$$

where $r(>0)$ is a function of u to be determined later. Using (2.1a), the x and t derivatives are rewritten as

$$\frac{\partial}{\partial x} = r\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} + \frac{1}{2}u^2 r\frac{\partial}{\partial y}. \tag{2.1b}$$

It follows from (2.1b) that $x = x(y, \tau)$ satisfies the system of linear PDEs

$$x_y = \frac{1}{r}, \quad x_\tau = -\frac{u^2}{2}. \tag{2.2}$$

Equation (1.1) is then transformed into the form

$$u_{y\tau} = x_y u. \tag{2.3}$$

The form of r can be determined by the solvability condition of system (2.2), i.e., $x_{y\tau} = x_{\tau y}$. Indeed, this immediately gives $r_\tau = uu_y r^2$. On the other hand, it follows from (2.2) and (2.3) that $u = ru_{y\tau}$. Eliminating the variable u from both relations, one has $r_\tau = u_y u_{y\tau} r^3$. If we impose the boundary conditions $u(\pm\infty, \tau) = 0$, $r(\pm\infty, \tau) = 1$, then we obtain $r^2 = (1 - u_y^2)^{-1}$ after integrating this relation with respect to τ. Since $u_y = u_x/r$ by (2.1b), we can rewrite this expression into the form

$$r^2 = 1 + u_x^2. \tag{2.4}$$

The above relation has been used to transform the SP equation into the form of conservation law $r_t = (u^2 r/2)_x$. If one introduces a new variable ϕ by $u_y = \sin\phi$, then ϕ satisfies the sG equation $\phi_{y\tau} = \sin\phi$. This equation was the starting point in constructing multisoliton solutions of the SP equation.[9] Below, we develop an alternative method using (2.3) which will be relevant to application to the multi-component system.

B. Parametric representation of soliton solutions

The soliton solutions of Eq. (2.3) are constructed by a direct method using the bilinear formalism. To this end, we first introduce the following dependent variable transformations for u and x:

$$u = \frac{g}{f}, \tag{2.5}$$

$$x = y + \frac{h}{f}, \tag{2.6}$$

where f, g, and h are tau-functions. Note that we may add an arbitrary constant on the right-hand side of (2.6), if necessary. The second equation of (2.2) is then transformed to the bilinear equation

$$2D_\tau h \cdot f + g^2 = 0, \tag{2.7}$$

where the bilinear operators D_τ and D_y are defined by

$$D_\tau^m D_y^n f \cdot g = \left(\frac{\partial}{\partial \tau} - \frac{\partial}{\partial \tau'}\right)^m \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^n f(\tau, y) g(\tau', y')\Big|_{\tau'=\tau,\, y'=y}, \quad (m, n = 0, 1, 2, \ldots). \tag{2.8}$$

On the other hand, Eq. (2.3) becomes

$$\frac{gf_y}{f^3}(2f_\tau + h) - \frac{1}{f^2}(f_\tau g_y + f_y g_\tau + f_{y\tau} g + gh_y) + \frac{1}{f}(g_{y\tau} - g) = 0. \tag{2.9}$$

We can decouple Eq. (2.9) to a set of equations

$$2f_\tau + h = 0, \tag{2.10}$$

$$f_\tau g_y + f_y g_\tau + f_{y\tau} g + g h_y - f(g_{y\tau} - g) = 0. \tag{2.11}$$

Substituting h from (2.10) into Eqs. (2.11) and (2.7), we obtain the following system of bilinear equations for f and g:

$$D_y D_\tau f \cdot g = fg, \tag{2.12}$$

$$D_\tau^2 f \cdot f = \frac{1}{2} g^2. \tag{2.13}$$

It then follows from (2.6) and (2.10) that

$$x = y - 2\,\frac{f_\tau}{f}. \tag{2.14}$$

Thus, the soliton solutions of the SP equation are given by the parametric representations (2.5) and (2.14) in terms of the tau-functions f and g. In the simplest case of the 1-soliton solution, the solutions to Eqs. (2.12) and (2.13) are easily found to be as

$$f = 1 + \mathrm{e}^{2\xi}, \quad g = \frac{4}{p}\,\mathrm{e}^{\xi}, \quad \xi = py + \frac{1}{p}\tau + \xi_0, \tag{2.15}$$

where p and ξ_0 are constants related to the amplitude and phase of the soliton, respectively. The corresponding parametric representation of the 1-soliton solution is derived from (2.5), (2.14), and (2.15). It reads

$$u = \frac{2}{p}\,\mathrm{sech}\,\xi, \quad x = y - \frac{2}{p}\,\tanh\,\xi + x_0, \tag{2.16}$$

where $x_0 = -p/2$. For real p and ξ_0, the solution takes the form of a loop soliton.

C. Remark

We have already shown that the SP equation can be transformed into the sG equation and obtained the parametric representation of the N-soliton solution. Actually, it reads[9]

$$u = 2\mathrm{i}\left(\ln \frac{\tilde{f}'}{\tilde{f}}\right)_\tau, \quad x = y - 2\,(\ln\, \tilde{f}'\tilde{f})_\tau, \tag{2.17}$$

where $\tilde{f}$ and $\tilde{f}'$ are tau-functions for the sG equation $\phi_{y\tau} = \sin\,\phi,\ \phi = 2\mathrm{i}\ \ln(\tilde{f}'/\tilde{f})$ and they satisfy the bilinear equations

$$D_y D_\tau \tilde{f} \cdot \tilde{f} = \frac{1}{2}(\tilde{f}^2 - \tilde{f}'^2), \quad D_y D_\tau \tilde{f}' \cdot \tilde{f}' = \frac{1}{2}(\tilde{f}'^2 - \tilde{f}^2). \tag{2.18}$$

The explicit forms of the tau-functions are given by

$$\tilde{f} = \sum_{\mu=0,1} \exp\left[\sum_{j=1}^{N} \mu_j\left(\xi_j + \frac{\pi}{2}\mathrm{i}\right) + \sum_{1\le j<k\le N} \mu_j\mu_k\gamma_{jk}\right], \tag{2.19a}$$

$$\tilde{f}' = \sum_{\mu=0,1} \exp\left[\sum_{j=1}^{N} \mu_j\left(\xi_j - \frac{\pi}{2}\mathrm{i}\right) + \sum_{1\le j<k\le N} \mu_j\mu_k\gamma_{jk}\right], \tag{2.19b}$$

where

$$\xi_j = p_j y + \frac{1}{p_j}\tau + \xi_{j0}, \quad (j = 1, 2, \ldots, N), \tag{2.20a}$$

$$e^{\gamma_{jk}} = \left(\frac{p_j - p_k}{p_j + p_k}\right)^2, \quad (j, k = 1, 2, \ldots, N;\, j \ne k). \tag{2.20b}$$

Here, p_j and ξ_{j0} are arbitrary complex-valued parameters satisfying the conditions $p_j \neq \pm p_k$ for $j \neq k$ and N is an arbitrary positive integer. The notation $\sum_{\mu=0,1}$ implies the summation over all possible combinations of $\mu_1 = 0, 1, \mu_2 = 0, 1, \ldots, \mu_N = 0, 1$. Thus, we have two different expressions for the parametric soliton solutions of the SP equation, i.e., one is (2.5) with (2.14) and the other is (2.17). We can show that the tau-functions f and g are related to the tau-functions $\tilde{f}$ and $\tilde{f}'$ by the relations

$$f = \tilde{f}'\tilde{f}, \quad g = 2\mathrm{i}\, D_\tau \tilde{f}' \cdot \tilde{f}, \tag{2.21}$$

which will be inferred by comparing (2.5) and (2.14) with (2.17).

III. MULTI-COMPONENT SYSTEM

Let us now consider the multi-component system (1.7). The procedure for obtaining the parametric representation of soliton solutions parallels that developed in Sec. II for the SP equation. Hence, we omit the detail of the derivation and write down the final results. Specifically, we give the parametric representation of soliton solutions and associated system of bilinear equations corresponding to Eqs. (2.12) and (2.13). Then, we present the explicit form of the multisoliton solution of the bilinear equations. Last, the proof of the multisoliton solution is performed by using an elementary theory of determinants.

A. Parametric representation of soliton solutions

If we use the hodograph transformation (2.1a) with F given by (1.7b) in place of u^2

$$dy = r dx + \frac{1}{2} F r dt, \quad d\tau = dt, \tag{3.1}$$

we then obtain the equations corresponding to Eqs. (2.2) and (2.3) which are given, respectively, by

$$x_y = \frac{1}{r}, \quad x_\tau = -\frac{F}{2}. \tag{3.2}$$

$$u_{i,y\tau} = x_y u_i \quad (i = 1, 2, \ldots, n). \tag{3.3}$$

The solvability condition for Eq. (3.2) gives the explicit form of r^2 in terms of $u_{i,y}$ $(i = 1, 2, \ldots, n)$ as

$$r^2 = \frac{1}{1 - \sum_{1\leq j<k\leq n} c_{jk} u_{j,y} u_{k,y}}. \tag{3.4a}$$

If we use the relation $u_{j,y} = u_{j,x}/r$, then we can rewrite (3.4a) in terms of the original variable $u_{i,x}$ $(i = 1, 2, \ldots, n)$

$$r^2 = 1 + \sum_{1\leq j<k\leq n} c_{jk} u_{j,x} u_{k,x}. \tag{3.4b}$$

The parametric representation of the soliton solutions takes the form

$$u_i = \frac{g_i}{f}, \quad (i = 1, 2, \ldots, n), \quad x = y - 2\frac{f_\tau}{f}, \tag{3.5}$$

where the tau-functions f and $g_i (i = 1, 2, \ldots, n)$ satisfy the system of bilinear equations

$$D_y D_\tau f \cdot g_i = f g_i, \quad (i = 1, 2, \ldots, n), \tag{3.6}$$

$$D_\tau^2 f \cdot f = \frac{1}{2} \sum_{1\leq j<k\leq n} c_{jk} g_j g_k. \tag{3.7}$$

It follows from (3.2)–(3.4a) that $u_i = u_i(y, \tau)$ obey a closed system of PDEs

$$\frac{u_{i,y\tau}}{\sqrt{1 - \sum_{1\leq j<k\leq n} c_{jk} u_{j,y} u_{k,y}}} = u_i, \quad (i = 1, 2, \ldots, n). \tag{3.8}$$

Furthermore, if we introduce the new variables v_i by $v_i = u_{i,y}$ $(i = 1, 2, \ldots, n)$, then the above system can be recast to

$$\frac{\partial}{\partial y}\left[\frac{v_{i,\tau}}{\sqrt{1-\sum_{1\le j<k\le n} c_{jk}v_j v_k}}\right] = v_i, \quad (i = 1, 2, \ldots, n). \tag{3.9}$$

B. Multisoliton solution of bilinear equations

We first introduce vectors and matrices. Subsequently, we present the explicit multisoliton solution of the bilinear equations (3.6) and (3.7).

1. Definition

Let **a, b, c**, and **0** be row vectors having M components

$$\mathbf{a} = (a_1, a_2, \ldots, a_M), \quad \mathbf{b} = (b_1, b_2, \ldots, b_M), \quad \mathbf{c} = (c_1, c_2, \ldots, c_M),$$

$$\mathbf{d} = (d_1, d_2, \ldots, d_M), \quad \mathbf{0} = (0, 0, \ldots, 0), \tag{3.10a}$$

and $\mathbf{e}_i$ $(i = 1, 2, \ldots, n)$ be M-component row vectors defined below:

$$\mathbf{e}_1 = (\underbrace{1, 1, \ldots, 1}_{M_1}, \underbrace{0, 0, \ldots, 0}_{M-M_1}), \ldots, \mathbf{e}_i = (\underbrace{0, 0, \ldots, 0}_{M_1+\cdots+M_{i-1}}, \underbrace{1, 1, \ldots, 1}_{M_i}, \underbrace{0, 0, \ldots, 0}_{M-(M_1+\cdots+M_i)}),$$

$$\ldots, \mathbf{e}_n = (\underbrace{0, 0, \ldots, 0}_{M_1+\cdots+M_{n-1}}, \underbrace{1, 1, \ldots, 1}_{M_n}), \tag{3.10b}$$

where M and $M_i (i = 1, 2, \ldots, n)$ are positive integers satisfying the condition $\sum_{i=1}^{n} M_i = M$.

The following types of matrices appear in the process of proving the multisoliton solution:

$$D = (d_{ij})_{1\le i,j\le 2M} = \begin{pmatrix} A_M & I_M \\ -I_M & B_M \end{pmatrix}, \quad D(\mathbf{a};\mathbf{b}) = \begin{pmatrix} A_M & I_M & \mathbf{b}^T \\ -I_M & B_M & \mathbf{0}^T \\ \mathbf{a} & \mathbf{0} & 0 \end{pmatrix}, \tag{3.11a}$$

$$D(\mathbf{a},\mathbf{b};\mathbf{c},\mathbf{d}) = \begin{pmatrix} A_M & I_M & \mathbf{c}^T & \mathbf{d}^T \\ -I_M & B_M & \mathbf{0}^T & \mathbf{0}^T \\ \mathbf{a} & \mathbf{0} & 0 & 0 \\ \mathbf{b} & \mathbf{0} & 0 & 0 \end{pmatrix}, \quad D(\mathbf{a},\mathbf{e}_i;\mathbf{b},\mathbf{e}_j) = \begin{pmatrix} A_M & I_M & \mathbf{b}^T & \mathbf{0}^T \\ -I_M & B_M & \mathbf{0}^T & \mathbf{e}_j^T \\ \mathbf{a} & \mathbf{0} & 0 & 0 \\ \mathbf{0} & \mathbf{e}_i & 0 & 0 \end{pmatrix}, \tag{3.11b}$$

$$D(\mathbf{a},\mathbf{b},\mathbf{e}_i;\mathbf{c},\mathbf{d},\mathbf{e}_j) = \begin{pmatrix} A_M & I_M & \mathbf{c}^T & \mathbf{d}^T & \mathbf{0}^T \\ -I_M & B_M & \mathbf{0}^T & \mathbf{0}^T & \mathbf{e}_j^T \\ \mathbf{a} & \mathbf{0} & 0 & 0 & 0 \\ \mathbf{b} & \mathbf{0} & 0 & 0 & 0 \\ \mathbf{0} & \mathbf{e}_i & 0 & 0 & 0 \end{pmatrix}, \tag{3.11c}$$

where $A_M = (a_{ij})_{1\le i,j\le M}$ and $B_M = (b_{ij})_{1\le i,j\le M}$ are $M \times M$ skew-symmetric matrices, I_M is the $M \times M$ unit matrix, and the symbol T denotes the transpose.

The element a_{ij} of the matrix A_M is given by

$$a_{ij} = -\frac{p_i - p_j}{p_i + p_j}\,\mathrm{e}^{\xi_i+\xi_j} = -\frac{p_i - p_j}{p_i + p_j} z_i z_j, \ (i, j = 1, 2, \ldots, M), \tag{3.12}$$

where ξ_i is defined by (2.20a) and $z_i = \mathrm{e}^{\xi_i}$. To specify the matrix B_M, let $S_i\,(i = 1, 2, \ldots, n)$ be n disjoint sets consisting of positive integers

$$S_1 = \{1, \ldots, M_1\}, \ldots, S_i = \{M_1 + M_2 + \cdots + M_{i-1} + 1, \ldots, M_1 + \cdots + M_i\},$$
$$\ldots, S_n = \{M_1 + M_2 + \cdots + M_{n-1} + 1, \ldots, M_1 + \cdots + M_n\}. \tag{3.13}$$

Then

$$b_{\mu\nu} = \frac{1}{4} c_{ij} \frac{(p_\mu p_\nu)^2}{p_\mu^2 - p_\nu^2},\ \mu \in S_i, \nu \in S_j\ (\mu, \nu = 1, 2, \ldots, M\ (\mu \neq \nu); i, j = 1, 2, \ldots, n\ (i \neq j)), \tag{3.14}$$

$b_{\mu\nu} = 0$ if μ and ν belong to the same set and $b_{\mu\mu} = 0$ for all μ. Thus, B_M has the structure

$$B_M = \begin{pmatrix} O_{M_1\times M_1} & B_{M_1\times M_2} & \ldots & B_{M_1\times M_n} \\ -B^T_{M_1\times M_2} & O_{M_2\times M_2} & \ldots & B_{M_2\times M_n} \\ \vdots & \vdots & \ddots & \vdots \\ -B^T_{M_1\times M_n} & -B^T_{M_2\times M_n} & \ldots & O_{M_n\times M_n} \end{pmatrix}, \tag{3.15a}$$

$$B_{M_i\times M_j} = (b_{\mu\nu})_{\mu\in S_i, \nu\in S_j}\ (1 \leq i < j \leq n),$$
$$O_{M_i\times M_i} : M_i \times M_i \text{ null matrix } (i = 1, 2, \ldots, n). \tag{3.15b}$$

2. Multisoliton solution

Now, we state our main result.

Theorem 3.1: *The multisoliton solution of the system of bilinear equations (3.6) and (3.7) is given by the following form:*

$$f = \sqrt{F}, \quad F = |D|, \tag{3.16a}$$

$$g_i = \sqrt{G_i}, \quad G_i = |D(-\mathbf{z}, -\mathbf{e}_i; \mathbf{z}, \mathbf{e}_i)|,\ (i = 1, 2, \ldots, n), \tag{3.16b}$$

where $\mathbf{z}$ *is the M-component vector* $\mathbf{z} = (\mathrm{e}^{\xi_1}, \mathrm{e}^{\xi_2}, \ldots, \mathrm{e}^{\xi_M})$. *The parametric solution* u_i *(3.5) constructed from these tau-functions contains* M_i *solitons for each i.*

Note that f and g_i are pfaffians since each one of them is represented by the square root of the skew-symmetric determinant of even order. This fact is in striking contrast to the tau-functions of the N-soliton solution for the SP equation which can be represented by determinants.

C. PROOF OF MULTISOLITON SOLUTION

1. Basic formulas for determinants

Let $A = (a_{ij})_{1\,\leq\, i, j\,\leq\, M}$ be an $M \times M$ matrix and A_{ij} be the cofactor of the element a_{ij}. Then, the following three formulas for determinants are employed frequently in our analysis:[14]

$$\frac{\partial}{\partial x}|A| = \sum_{i,j=1}^{M} \frac{\partial a_{ij}}{\partial x} A_{ij}, \tag{3.17}$$

$$\begin{vmatrix} A & \mathbf{a}^T \\ \mathbf{b} & z \end{vmatrix} = |A| z - \sum_{i,j=1}^{M} A_{ij} a_i b_j, \tag{3.18}$$

$$|A(\mathbf{a}, \mathbf{b}; \mathbf{c}, \mathbf{d})||A| = |A(\mathbf{a}; \mathbf{c})||A(\mathbf{b}; \mathbf{d})| - |A(\mathbf{a}; \mathbf{d})||A(\mathbf{b}; \mathbf{c})|. \tag{3.19}$$

Formula (3.17) is the differential rule of the determinant and (3.18) is the expansion formula for a bordered determinant with respect to the last row and column. Formula (3.19) is Jacobi's identity and it will play a central role in the proof of the multisoliton solution.

2. Differential formulas

We give various differential formulas for the determinants F and G_i introduced in (3.16) which are necessary for the proof of the solution. The following formulas are derived easily with use of (3.17) and (3.18) as well as the relation $|D(-\mathbf{z};\mathbf{z})|=0$ which follows from the fact that the skew-symmetric determinant of odd order is identically zero. Hence, we quote only the results:

$$F_y=-2|D(-\mathbf{z};\mathbf{z}_y)|, \tag{3.20a}$$

$$F_\tau=-2|D(-\mathbf{z}_\tau;\mathbf{z})|, \tag{3.20b}$$

$$F_{y\tau}=-2|D(-\mathbf{z}_\tau;\mathbf{z}_y)|-2|D(-\mathbf{z},-\mathbf{z}_\tau;\mathbf{z},\mathbf{z}_y)|, \tag{3.20c}$$

$$F_{\tau\tau}=-2|D(-\mathbf{z}_{\tau\tau};\mathbf{z})|-2|D(-\mathbf{z},-\mathbf{z}_\tau;\mathbf{z},\mathbf{z}_\tau)|, \tag{3.20d}$$

$$G_{i,y}=2|D(-\mathbf{z},-\mathbf{e}_i;\mathbf{z}_y,\mathbf{e}_i)|, \tag{3.21a}$$

$$G_{i,\tau}=2|D(-\mathbf{z}_\tau,-\mathbf{e}_i;\mathbf{z},\mathbf{e}_i)|, \tag{3.21b}$$

$$G_{i,y\tau}=2|D(-\mathbf{z},-\mathbf{e}_i;\mathbf{z},\mathbf{e}_i)|+2|D(-\mathbf{z}_\tau,-\mathbf{e}_i;\mathbf{z}_y,\mathbf{e}_i)|+2|D(-\mathbf{z},-\mathbf{z}_\tau,-\mathbf{e}_i;\mathbf{z},\mathbf{z}_y,\mathbf{e}_i)|, \tag{3.21c}$$

where the M-component vectors $\mathbf{z}_y$, $\mathbf{z}_\tau$, and $\mathbf{z}_{\tau\tau}$ are given, respectively, by

$$\mathbf{z}_y=(p_1\mathrm{e}^{\xi_1},p_2\mathrm{e}^{\xi_2},\ldots,p_M\mathrm{e}^{\xi_M}),\quad \mathbf{z}_\tau=\left(\frac{\mathrm{e}^{\xi_1}}{p_1},\frac{\mathrm{e}^{\xi_2}}{p_2},\ldots,\frac{\mathrm{e}^{\xi_M}}{p_M}\right),\quad \mathbf{z}_{\tau\tau}=\left(\frac{\mathrm{e}^{\xi_1}}{p_1^2},\frac{\mathrm{e}^{\xi_2}}{p_2^2},\ldots,\frac{\mathrm{e}^{\xi_M}}{p_M^2}\right). \tag{3.22}$$

3. Proof of Eq. (3.6)

First, we show that the tau-functions (3.16) for the multisoliton solution satisfy the bilinear equation (3.6). To this end, we substitute f and g_i from (3.16) into Eq. (3.6) to obtain

$$\frac{G_i}{2F}\left(FF_{y\tau}-\frac{1}{2}F_yF_\tau\right)+\frac{F}{2G_i}\left(G_iG_{i,y\tau}-\frac{1}{2}G_{i,y}G_{i,\tau}\right)-\frac{1}{4}(F_yG_{i,\tau}+F_\tau G_{i,y})=FG_i. \tag{3.23}$$

We compute three terms on the left-hand side of (3.23) separately. Using (3.20a)-(3.20c) and the relation

$$|D(-\mathbf{z},-\mathbf{z}_\tau;\mathbf{z},\mathbf{z}_y)||D|=-|D(-\mathbf{z};\mathbf{z}_y)||D(-\mathbf{z}_\tau;\mathbf{z})|, \tag{3.24}$$

which follows from Jacobi's identity and the identity $|D(-\mathbf{z};\mathbf{z})|=0$, the first term on the left-hand side of (3.23) reduces to

$$\frac{G_i}{2F}\left(FF_{y\tau}-\frac{1}{2}F_yF_\tau\right)=-|D(-\mathbf{z}_\tau;\mathbf{z}_y)|G_i. \tag{3.25}$$

Next, it follows from (3.21a)–(3.21c) that

$$\begin{aligned}G_iG_{i,y\tau}-\frac{1}{2}G_{i,y}G_{i,\tau}&=2|D(-\mathbf{z},-\mathbf{e}_i;\mathbf{z},\mathbf{e}_i)|\Big\{|D(-\mathbf{z},-\mathbf{e}_i;\mathbf{z},\mathbf{e}_i)|+|D(-\mathbf{z}_\tau,-\mathbf{e}_i;\mathbf{z}_y,\mathbf{e}_i)|\\&+|D(-\mathbf{z},-\mathbf{z}_\tau,-\mathbf{e}_i;\mathbf{z},\mathbf{z}_y,\mathbf{e}_i)|\Big\}-2|D(-\mathbf{z},-\mathbf{e}_i;\mathbf{z}_y,\mathbf{e}_i)||D(-\mathbf{z}_\tau,-\mathbf{e}_i;\mathbf{z},\mathbf{e}_i)|.\end{aligned} \tag{3.26}$$

Referring again to Jacobi's identity and the identity $|D(-\mathbf{e}_i;\mathbf{e}_i)|=0$, one has

$$
\begin{aligned}
&|D(-\mathbf{z},-\mathbf{z}_\tau,-\mathbf{e}_i;\mathbf{z},\mathbf{z}_y,\mathbf{e}_i)||D(-\mathbf{e}_i;\mathbf{e}_i)|\\
&=|D(-\mathbf{z},-\mathbf{e}_i;\mathbf{z},\mathbf{e}_i)||D(-\mathbf{z}_\tau,-\mathbf{e}_i;\mathbf{z}_y,\mathbf{e}_i)|-|D(-\mathbf{z}_\tau,-\mathbf{e}_i;\mathbf{z},\mathbf{e}_i)||D(-\mathbf{z},-\mathbf{e}_i;\mathbf{z}_y,\mathbf{e}_i)|=0,
\end{aligned}
\tag{3.27}
$$

which, introduced into (3.26), simplifies the second term on the left-hand side of (3.23)

$$
\frac{F}{2G_i}\left(G_iG_{i,y\tau}-\frac{1}{2}G_{i,y}G_{i,\tau}\right)=F\Big\{|D(-\mathbf{z},-\mathbf{z}_\tau,-\mathbf{e}_i;\mathbf{z},\mathbf{z}_y,\mathbf{e}_i)|+G_i\Big\}. \tag{3.28}
$$

Last, formulas (3.20a), (3.20b), (3.21a), and (3.21b) give simply the third term on the left-hand side of (3.23):

$$
-\frac{1}{4}(F_yG_{i,\tau}+F_\tau G_{i,y})=|D(-\mathbf{z};\mathbf{z}_y)||D(-\mathbf{z}_\tau,-\mathbf{e}_i;\mathbf{z},\mathbf{e}_i)|+|D(-\mathbf{z}_\tau;\mathbf{z})||D(-\mathbf{z},-\mathbf{e}_i;\mathbf{z}_y,\mathbf{e}_i)|. \tag{3.29}
$$

Substituting (3.25), (3.28), and (3.29) into (3.23), the equation to be proved becomes

$$
\begin{aligned}
&|D||D(-\mathbf{z},-\mathbf{z}_\tau,-\mathbf{e}_i;\mathbf{z},\mathbf{z}_y,\mathbf{e}_i)|-|D(-\mathbf{z}_\tau;\mathbf{z}_y)||D(-\mathbf{z},-\mathbf{e}_i;\mathbf{z},\mathbf{e}_i)|\\
&+|D(-\mathbf{z};\mathbf{z}_y)||D(-\mathbf{z}_\tau,-\mathbf{e}_i;\mathbf{z},\mathbf{e}_i)|+|D(-\mathbf{z}_\tau;\mathbf{z})||D(-\mathbf{z},-\mathbf{e}_i;\mathbf{z}_y,\mathbf{e}_i)|=0.
\end{aligned}
\tag{3.30}
$$

The following formula can be verified by applying Jacobi's identity twice to the right-hand side of (3.31):

$$
\begin{vmatrix}
|D(\mathbf{a};\mathbf{a}')| & |D(\mathbf{a};\mathbf{b}')| & |D(\mathbf{a};\mathbf{c}')|\\
|D(\mathbf{b};\mathbf{a}')| & |D(\mathbf{b};\mathbf{b}')| & |D(\mathbf{b};\mathbf{c}')|\\
|D(\mathbf{c};\mathbf{a}')| & |D(\mathbf{c};\mathbf{b}')| & |D(\mathbf{c};\mathbf{c}')|
\end{vmatrix}
=|D|^2|D(\mathbf{a},\mathbf{b},\mathbf{c};\mathbf{a}',\mathbf{b}',\mathbf{c}')|. \tag{3.31}
$$

Assume that $|D|\neq 0$. Then, multiplying (3.30) by $|D|$ and using Jacobi's identity as well as the identities $|D(-\mathbf{e}_i;\mathbf{e}_i)|=|D(-\mathbf{z};\mathbf{z})|=0$, the resulting relation reduces to (3.31) with the identification $\mathbf{a}=-\mathbf{z}$, $\mathbf{b}=-\mathbf{z}_\tau$, $\mathbf{c}=-\mathbf{e}_i$, $\mathbf{a}'=\mathbf{z}$, $\mathbf{b}'=\mathbf{z}_y$, $\mathbf{c}'=\mathbf{e}_i$. This completes the proof of Eq. (3.6).

4. Proof of Eq. (3.7)

We proceed to the proof of Eq. (3.7). By using f and g_i from (3.16) and noting the symmetry $c_{ij}=c_{ji}$, we transform it to the form

$$
FF_{\tau\tau}-F_\tau^2=\frac{1}{4}\sum_{\substack{j,k=1\\(j\neq k)}}^{n}c_{jk}G_jG_k. \tag{3.32}
$$

If we substitute (3.16b), (3.20b), and (3.20d) into (3.32) and use the following relation with $j=k$

$$
|D(-\mathbf{e}_j;\mathbf{z})||D(-\mathbf{e}_k;\mathbf{z})|=|D||D(-\mathbf{z},-\mathbf{e}_j;\mathbf{z},\mathbf{e}_k)|,\ (j,k=1,2,\ldots,n), \tag{3.33}
$$

which comes from Jacobi's identity, we recast (3.32) in the form

$$
2|D|\Big\{|D(-\mathbf{z};\mathbf{z}_{\tau\tau})|-|D(-\mathbf{z},-\mathbf{z}_\tau;\mathbf{z},\mathbf{z}_\tau)|\Big\}=\frac{1}{4}\sum_{\substack{j,k=1\\(j\neq k)}}^{n}c_{jk}|D(-\mathbf{e}_j;\mathbf{z})||D(-\mathbf{e}_k;\mathbf{z})|. \tag{3.34}
$$

Last, replacing the right-hand side of (3.34) by the right-hand side of (3.33) and dividing the resultant equation by $2|D|$, the equation to be proved reduces to the following *linear* relation among determinants:

$$
|D(-\mathbf{z};\mathbf{z}_{\tau\tau})|-|D(-\mathbf{z},-\mathbf{z}_\tau;\mathbf{z},\mathbf{z}_\tau)|=\frac{1}{8}\sum_{\substack{j,k=1\\(j\neq k)}}^{n}c_{jk}|D(-\mathbf{z},-\mathbf{e}_j;\mathbf{z},\mathbf{e}_k)|. \tag{3.35}
$$

We now start the proof of (3.35). Define the $(2M+1)\times(2M+1)$ skew-symmetric matrix $D'=(d'_{ij})_{1\le i,j\le 2M+1}$ by

$$D' = D(-\mathbf{z};\mathbf{z}) = \begin{pmatrix} A_M & I_M & \mathbf{z}^T \\ -I_M & B_M & \mathbf{0}^T \\ -\mathbf{z} & \mathbf{0} & 0 \end{pmatrix}. \tag{3.36}$$

Let D_{ij} and D'_{ij} be the cofactors of the elements d_{ij} and d'_{ij}, respectively, and $D_{ij,kl}$ and $D'_{ij,kl}$ be second cofactors. Expanding the cofactor $D'_{M+j,M+i}$ with respect to the ith row, we obtain

$$D'_{M+j,M+i} = \sum_{k=1}^{M} D'_{i\,M+j,k\,M+i}a_{ik} + \sum_{k=1}^{M} D_{i\,M+j,k\,M+i}\,z_i z_k,\ (i,j=1,2,\ldots,M). \tag{3.37}$$

Similarly, referring to the structure of the matrix B_M defined by (3.15), the expansions of D_{ij} and D'_{ij} with respect to the $(M+i)$th column read

$$D_{ij} = \sum_{k=1}^{M} D_{i\,M+k,j\,M+i}b_{ki},\ (i,j=1,2,\ldots,M), \tag{3.38}$$

$$D'_{ij} = \sum_{k=1}^{M} D'_{i\,M+k,j\,M+i}b_{ki},\ (i,j=1,2,\ldots,M). \tag{3.39}$$

The proof of (3.35) can be performed on the basis of formulas (3.37)–(3.39). First, we multiply (3.37) by b_{ji}/p_i^2 and sum up with respect to i and j to obtain

$$\begin{aligned}\sum_{i,j=1}^{M} D'_{M+j,M+i}\frac{b_{ji}}{p_i^2} &= \sum_{i,j=1}^{M}\sum_{k=1}^{M} D'_{i\,M+j,k\,M+i}a_{ik}\frac{b_{ji}}{p_i^2} \\ &+ \sum_{i,j=1}^{M}\sum_{k=1}^{M} D_{i\,M+j,k\,M+i}\frac{b_{ji}}{p_i^2}z_i z_k,\ (i,j=1,2,\ldots,M).\end{aligned} \tag{3.40}$$

Note that for any function f_{ij}

$$\sum_{i,j=1}^{M} f_{ij} = \sum_{i,j=1}^{n}\sum_{\mu\in S_i}\sum_{\nu\in S_j} f_{\mu\nu}, \tag{3.41}$$

where the notation $\sum_{\mu\in S_i}$ implies that the dummy index μ runs over the set S_i. Applying this rule to the left-hand side of (3.40),

$$L \equiv \sum_{i,j=1}^{M} D'_{M+j,M+i}\frac{b_{ji}}{p_i^2} = \sum_{i,j=1}^{n}\sum_{\mu\in S_i}\sum_{\nu\in S_j} D'_{M+\nu,M+\mu}\frac{b_{\nu\mu}}{p_\mu^2}. \tag{3.42}$$

We modify L by taking into account the relations $b_{\nu\mu}=-b_{\mu\nu}$ and $D'_{M+\nu,M+\mu}=D'_{M+\mu,M+\nu}$ which follow from the skew-symmetry of the matrices D and D'. This leads to

$$\begin{aligned} L &= \frac{1}{2}\sum_{i,j=1}^{n}\sum_{\mu\in S_i}\sum_{\nu\in S_j} D'_{M+\nu,M+\mu}\left(-\frac{1}{p_\mu^2}+\frac{1}{p_\nu^2}\right)b_{\mu\nu} \\ &= \frac{1}{8}\sum_{\substack{i,j=1\\(i\neq j)}}^{n} c_{ij}\sum_{\mu\in S_i}\sum_{\nu\in S_j} D'_{M+\nu,M+\mu}, \end{aligned} \tag{3.43}$$

where in passing to the second line of (3.43), we used (3.14). It follows from (3.10b) and the formula (3.18) that

$$\sum_{\mu\in S_i}\sum_{\nu\in S_j} D'_{M+\nu,M+\mu} = |D'(-\mathbf{e}_i;\mathbf{e}_j)| = |D(-\mathbf{z},-\mathbf{e}_i;\mathbf{z},\mathbf{e}_j)|, \tag{3.44}$$

which, substituted in (3.43), gives

$$L = \frac{1}{8}\sum_{\substack{i,j=1\\(i\neq j)}}^{n} c_{ij}|D(-\mathbf{z},-\mathbf{e}_i;\mathbf{z},\mathbf{e}_j)|. \tag{3.45}$$

On the other hand, using (3.38) and (3.39), the right-hand side of (3.40) reduces to

$$R \equiv \sum_{i,k=1}^{M} D'_{ik}\frac{a_{ik}}{p_i^2} + \sum_{i,k=1}^{M} D_{ik}\frac{z_i z_k}{p_i^2}. \tag{3.46}$$

We substitute the explicit form of a_{ik} from (3.12) and take into account the symmetry $D'_{ik} = D'_{ki}$, the first term of R is modified as

$$\begin{aligned}\sum_{i,k=1}^{M} D'_{ik}\frac{a_{ik}}{p_i^2} &= -\frac{1}{2}\sum_{i,k=1}^{M} D'_{ik}\left(\frac{1}{p_i^2}-\frac{1}{p_k^2}\right)\frac{p_i-p_k}{p_i+p_k}z_i z_k\\ &= \frac{1}{2}\sum_{i,k=1}^{M} D'_{ik}\left(\frac{1}{p_i^2}-\frac{2}{p_i p_k}+\frac{1}{p_k^2}\right)z_i z_k.\end{aligned} \tag{3.47}$$

It turns out by applying the formula (3.18) to (3.47) that

$$\begin{aligned}\sum_{i,k=1}^{M} D'_{ik}\frac{a_{ik}}{p_i^2} &= \frac{1}{2}|D'(-\mathbf{z};\mathbf{z}_{\tau\tau})| - |D'(-\mathbf{z}_\tau;\mathbf{z}_\tau)| + \frac{1}{2}|D'(-\mathbf{z}_{\tau\tau};\mathbf{z})|\\ &= \frac{1}{2}|D(-\mathbf{z},-\mathbf{z};\mathbf{z},\mathbf{z}_{\tau\tau})| - |D(-\mathbf{z},-\mathbf{z}_\tau;\mathbf{z},\mathbf{z}_\tau)| + \frac{1}{2}|D(-\mathbf{z},-\mathbf{z}_{\tau\tau};\mathbf{z},\mathbf{z})\\ &= -|D(-\mathbf{z},-\mathbf{z}_\tau;\mathbf{z},\mathbf{z}_\tau)|,\end{aligned} \tag{3.48}$$

where in passing to the last line, we used the fact that any determinant which contains two identical rows (or columns) is zero. The similar procedure applied to the second term of R yields

$$\sum_{i,k=1}^{M} D_{ik}\frac{z_i z_k}{p_i^2} = |D(-\mathbf{z};\mathbf{z}_{\tau\tau})|. \tag{3.49}$$

Adding (3.48) and (3.49), we finally obtain

$$R = |D(-\mathbf{z};\mathbf{z}_{\tau\tau})| - |D(-\mathbf{z},-\mathbf{z}_\tau;\mathbf{z},\mathbf{z}_\tau)|. \tag{3.50}$$

The desired relation (3.35) follows immediately from (3.40), (3.45), and (3.50), completing the proof of Eq. (3.7).

D. Remarks

1. Let $C = (c_{ij})_{1\le i,j\le n}$ be a real symmetric matrix whose diagonal elements are zero, i.e., $c_{ii} = 0(i = 1, 2, \dots, n)$, and $P = (p_{ij})_{1\le i,j\le n}$ is a regular matrix. Then, under appropriate orthogonal transformation $u_i = \sum_{j=1}^{n} p_{ij}u'_j$, the quadratic form (1.7b) can be recast to a canonical form

$$F = \sum_{i=1}^{p} u_i'^2 - \sum_{i=1}^{q} u_{p+i}'^2,\ (p+q\le n), \tag{3.51}$$

where $p(q)$ is the number of positive (negative) eigenvalues of C, and p and q are determined uniquely by C.[15] Note that since $\mathrm{Tr}\, C = 0$, $p \neq 0$ and $q \neq 0$. Under the same transformation, the system of bilinear equations (3.6) and (3.7) can be converted into the system

$$D_y D_\tau f \cdot g_i' = f g_i', \ (i = 1, 2, \ldots, p+q), \tag{3.52}$$

$$D_\tau^2 f \cdot f = \frac{1}{2}\left(\sum_{i=1}^{p} g_i'^2 - \sum_{i=1}^{q} g_{p+i}'^2\right), \tag{3.53}$$

where $u_i' = g_i'/f (i = 1, 2, \ldots, p+q)$. For example, if $c_{ij} = 1$ $(i \neq j)$, $c_{ii} = 0$, then $p = 1$ and $q = n - 1$ since the eigenvalues of C are $n - 1$ (simple root) and -1 (($n - 1$)-ple root).

2. When F is a positive definite quadratic form of $u_i\, (i = 1, 2, \ldots, n)$, we can put $p = n$ and $q = 0$ in (3.52) and (3.53) provided that C has n distinct positive eigenvalues. The system corresponding to (1.7) becomes

$$u_{i,xt} = u_i + \frac{1}{2}\left[\left(\sum_{j=1}^{n} u_j^2\right) u_{i,x}\right]_x, \ (i = 1, 2, \ldots, n). \tag{3.54}$$

If we consider the continuum limit $n \to \infty$ for (3.54), then we have a $(2 + 1)$-dimensional nonlocal PDE of the form

$$u_{xt} = u + \frac{1}{2}\left(u_x \int_{-\infty}^{\infty} u^2 dz\right)_x, \quad u = u(x, z, t). \tag{3.55}$$

This equation is an analog of the $(2 + 1)$-dimensional nonlocal nonlinear Schrödinger equation

$$\mathrm{i}u_t = u_{xx} + 2u \int_{-\infty}^{\infty} |u|^2 dz, \quad u = u(x, z, t), \tag{3.56}$$

arising from a continuum limit of the multi-component nonlinear Schrödinger equation.[16,17] By means of the hodograph transformation

$$dy = r dx + \left(\int_{-\infty}^{\infty} u^2 dz\right) r d\tau, \quad dt = d\tau, \tag{3.57}$$

we obtain the parametric representation of the solution for Eq. (3.55)

$$u = \frac{g}{f}, \quad x = y - 2\frac{f_\tau}{f}, \tag{3.58}$$

where $f = f(y, \tau)$ and $g = g(y, z, \tau)$ satisfy the system of bilinear equations

$$D_y D_\tau f \cdot g = f g, \quad D_\tau^2 f \cdot f = \frac{1}{2}\int_{-\infty}^{\infty} g^2 dz. \tag{3.59}$$

We will discuss the integrability of Eq. (3.55) in a separate paper.

3. The bilinear equation (3.7) takes the same form as that of a coupled modified Koreweg-de Vries equations proposed in Ref. 18 where the proof of the multisoliton solution has been performed by lengthy calculations using various formulas of pfaffians. Here, we have provided a novel proof relying only on an elementary theory of determinants.

4. The coupled PDEs proposed recently in Ref. 19

$$u_{i,xt} = u_i - \sum_{1\le j<k\le n} c_{jk}(u_{j,x}u_k - u_j u_{k,x})u_i, \ (i = 1, 2, \ldots, n), \tag{3.60}$$

where the coupling constants c_{jk} are skew-symmetric, are transformed to the following system of bilinear equations through the dependent variable transformations $u_i = g_i/f\, (i = 1, 2, \ldots, n)$:

$$D_x D_t f \cdot g_i = f g_i, \ (i = 1, 2, \ldots, n), \tag{3.61}$$

$$D_x D_t f \cdot f = \sum_{1\le j<k\le n} c_{jk} D_x g_j \cdot g_k. \tag{3.62}$$

Recall that the bilinear equation (3.61) coincides with (3.6) if we replace the variables x and t by y and τ, respectively. We conjecture that the multisoliton solution of Eqs. (3.61) and (3.62) will be given by (3.16) where the matrix B_M has the form

$$b_{\mu\nu} = -c_{ij}\,\frac{p_\mu p_\nu}{p_\mu + p_\nu},\ \ \mu \in S_i, \nu \in S_j\ (\mu, \nu = 1, 2, \ldots, M\ (\mu \neq \nu); i, j = 1, 2, \ldots, n\ (i \neq j)), \tag{3.63}$$

in place of (3.14). Obviously, the corresponding tau-functions f and g_i satisfy Eq. (3.61) since its proof does not depend on the explicit form of B_M except that it is a skew-symmetric matrix with the constant elements. For the 2-component system, we have checked that Eqs. (3.61) and (3.62) exhibit the 2- and 3-soliton solutions, i.e., $M_1 = M_2 = 2$, $M_1 = M_2 = 3$. The proof of the general multisoliton solution will be reported elsewhere.

IV. TWO-COMPONENT SYSTEM

Here, we consider the 2-component system (1.8) in detail. We first show the integrability of the system by constructing a Lax pair and then present the multisoliton solution. We also discuss an integrable system (1.9) which is closely related to system (1.8).

A. Integrability

For system (1.8), Eqs. (3.2) and (3.3) corresponding to Eqs. (1.8) read

$$x_{y\tau} = -\frac{1}{2}(uv)_y, \quad u_{y\tau} = x_y u, \quad v_{y\tau} = x_y v, \tag{4.1}$$

where the first of these equations comes from the y-derivative of the second equation of (3.2) with $F = uv$. The system of equations (4.1) can be derived from the compatibility condition of the system of linear PDEs

$$\Psi_y = U\Psi,\ \ \Psi_\tau = V\Psi \tag{4.2a}$$

with

$$U = \lambda\begin{pmatrix} x_y & u_y \\ v_y & -x_y \end{pmatrix}, \quad V = \frac{1}{2}\begin{pmatrix} 0 & -u \\ v & 0 \end{pmatrix} + \frac{1}{4\lambda}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4.2b}$$

where λ is a spectral parameter. Note in this expression that $x_y = \sqrt{1 - u_y v_y}$. Indeed, it follows from the condition $\Psi_{y\tau} = \Psi_{\tau y}$ that

$$U_\tau - V_y + UV - VU = O, \tag{4.3}$$

which yields Eqs. (4.1). Using (2.1b), we can rewrite (4.2) in terms of the original variables x and t

$$\Psi_x = \tilde{U}\Psi,\ \ \Psi_t = \tilde{V}\Psi \tag{4.4a}$$

with

$$\tilde{U} = \lambda\begin{pmatrix} 1 & u_x \\ v_x & -1 \end{pmatrix}, \quad \tilde{V} = \frac{1}{2}\begin{pmatrix} 0 & -u \\ v & 0 \end{pmatrix} + \frac{1}{4\lambda}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{\lambda}{2}\begin{pmatrix} uv & uvu_x \\ uvv_x & -uv \end{pmatrix}. \tag{4.4b}$$

This is a Lax pair for the system of equations (1.8). Note that when $u = v$, (4.4) reduces to the Lax pair for the SP equation.[4] One can apply the inverse scattering transform (IST) method to establish the complete integrability of the system (1.8).

B. Multisoliton solution

1. N-soliton solution

The parametric representation of the multisoliton solution of Eq. (1.8) is given by (3.5) with $n = 2$

$$u = \frac{g_1}{f}, \quad v = \frac{g_2}{f}, \quad x = y - 2\,\frac{f_\tau}{f}. \tag{4.5}$$

Here, we consider the case where both u and v contain N solitons. Correspondingly, we set $M_1 = M_2 = N$ and $M = 2N$ in (4.5). The tau-functions f and $g_i (i = 1, 2)$ from (3.16) are represented by the following formulas:

$$f = \sqrt{F}, \quad g_i = \sqrt{G_i}\ (i = 1, 2) \tag{4.6a}$$

with

$$F = |D| = \begin{vmatrix} A_{2N} & I_{2N} \\ -I_{2N} & B_{2N} \end{vmatrix}, \tag{4.6b}$$

$$G_i = |D(-\mathbf{z}, -\mathbf{e}_i; \mathbf{z}, \mathbf{e}_i)| = \begin{vmatrix} A_{2N} & I_{2N} & \mathbf{z}^T & \mathbf{0}^T \\ -I_{2N} & B_{2N} & \mathbf{0}^T & \mathbf{e}_i^T \\ -\mathbf{z} & \mathbf{0} & 0 & 0 \\ \mathbf{0} & -\mathbf{e}_i & 0 & 0 \end{vmatrix}, \ (i = 1, 2). \tag{4.6c}$$

Here, the $2N \times 2N$ skew-symmetric matrices A_{2N} and B_{2N} have the elements

$$A_{2N} = (a_{ij})_{1\le i,j\le 2N}, \quad a_{ij} = -\frac{p_i - p_j}{p_i + p_j}\,\mathrm{e}^{\xi_i+\xi_j} \equiv -\frac{p_i - p_j}{p_i + p_j} z_i z_j,$$

$$\xi_i = p_i y + \frac{1}{p_i}\tau + \xi_{i0}, \ (i = 1, 2, \ldots, 2N), \tag{4.6d}$$

$$B_{2N} = \begin{pmatrix} O_{N\times N} & B_{N\times N} \\ -B_{N\times N}^T & O_{N\times N} \end{pmatrix},$$

$$B_{N\times N} = (b_{i\,N+j})_{1\le i,j\le N}, \quad b_{i\,N+j} = \tfrac{1}{4}\tfrac{(p_i p_{N+j})^2}{p_i^2 - p_{N+j}^2}, \ (i, j = 1, 2, \ldots, N), \tag{4.6e}$$

and the $2N$-component vectors $\mathbf{z}$ and $\mathbf{e}_i\,(i = 1, 2)$ are given, respectively, by

$$\mathbf{z} = (\mathrm{e}^{\xi_1}, \mathrm{e}^{\xi_2}, \ldots, \mathrm{e}^{\xi_{2N}}), \quad \mathbf{e}_1 = (\underbrace{1, 1, \ldots, 1}_{N}, \underbrace{0, 0, \ldots, 0}_{N}), \quad \mathbf{e}_2 = (\underbrace{0, 0, \ldots, 0}_{N}, \underbrace{1, 1, \ldots, 1}_{N}). \tag{4.6f}$$

Note that the N-soliton solution contains $4N$ complex-valued parameters $p_i, \xi_{i0}\,(i = 1, 2, \ldots, 2N)$. An alternative parametrization with the same number of the parameters is possible if one puts $p_{N+i} = p_i\,(i = 1, 2, \ldots, N)$ and replaces ξ_{i0} and ξ_{N+i0} by $\xi_{i0} + \ln a_i$ and $\xi_{i0} + \ln b_i\,(i = 1, 2, \ldots, N)$, respectively, where a_i and b_i are new parameters. In the following, we present a few examples of solutions and investigate their properties.

2. One-loop soliton solution

We give two types of tau-functions described above:

$$f = 1 + \frac{1}{4}\frac{(p_1 p_2)^2}{(p_1 + p_2)^2} z_1 z_2, \quad g_1 = z_1, \quad g_2 = z_2, \tag{4.7a}$$

$$f = 1 + \frac{a_1 b_1 p_1^2}{16} z_1^2, \quad g_1 = a_1 z_1, \quad g_2 = b_1 z_1, \ (a_1, a_2, p_1 > 0), \tag{4.7b}$$

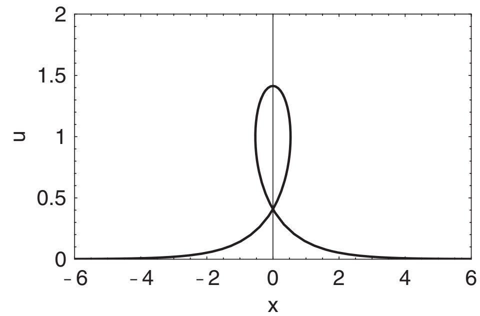

FIG. 1. The profile of the 1-loop soliton solution u with the parameters $p_1 = 1.0$, $a_1 = 0.5$, and $b_1 = 1.0$.

The solution corresponding to (4.7b) is calculated from (3.5) to give

$$u = \frac{2}{p_1}\sqrt{\frac{a_1}{b_1}}\,\mathrm{sech}(\xi_1+\delta_1), \quad v = \frac{2}{p_1}\sqrt{\frac{b_1}{a_1}}\,\mathrm{sech}(\xi_1+\delta_1), \tag{4.8a}$$

$$x = y - \frac{2}{p_1}\tanh(\xi_1+\delta_1), \quad \delta_1 = \ln\left(\frac{\sqrt{a_1b_1}\,p_1}{4}\right),\ (a_1, b_1, p_1 > 0). \tag{4.8b}$$

A profile of u is depicted in Fig. 1. It represents a loop soliton with the amplitude $\frac{2}{p_1}\sqrt{\frac{a_1}{b_1}}$ and the velocity $c_1 = 1/p_1^2$. Note that the amplitude of the loop soliton is defined by the maximum value of u which is attained at $\xi_1 = -\delta_1$ in the present example. The property of v is the same as that of u except the amplitude given by $\frac{2}{p_1}\sqrt{\frac{b_1}{a_1}}$. By comparing (2.16) and (4.8), we see that the loop soliton has the same structure as that of the loop soliton of the SP equation.

3. Two-loop soliton solution

As in the case of the 1-soliton solution, we write down two types of tau-functions for the 2-soliton solution:

$$f = 1 + \frac{1}{4}\frac{(p_1p_3)^2}{(p_1+p_3)^2}z_1z_3 + \frac{1}{4}\frac{(p_1p_4)^2}{(p_1+p_4)^2}z_1z_4 + \frac{1}{4}\frac{(p_2p_3)^2}{(p_2+p_3)^2}z_2z_3 + \frac{1}{4}\frac{(p_2p_4)^2}{(p_2+p_4)^2}z_2z_4$$
$$+\frac{1}{16}\frac{(p_1p_2p_3p_4)^2(p_1-p_2)^2(p_3-p_4)^2}{(p_1+p_3)^2(p_2+p_3)^2(p_1+p_4)^2(p_2+p_4)^2}z_1z_2z_3z_4, \tag{4.9a}$$

$$g_1 = z_1 + z_2 + \frac{1}{4}\frac{p_3^2(p_1-p_2)^2}{(p_1+p_3)^2(p_2+p_3)^2}z_1z_2z_3 + \frac{1}{4}\frac{p_4^4(p_1-p_2)^2}{(p_1+p_4)^2(p_2+p_4)^2}z_1z_2z_4, \tag{4.9b}$$

$$g_2 = z_3 + z_4 + \frac{1}{4}\frac{p_1^4(p_3-p_4)^2}{(p_1+p_3)^2(p_1+p_4)^2}z_1z_3z_4 + \frac{1}{4}\frac{p_2^4(p_3-p_4)^2}{(p_2+p_3)^2(p_2+p_4)^2}z_2z_3z_4, \tag{4.9c}$$

$$f = 1 + \frac{1}{16}a_1b_1p_1^2z_1^2 + \frac{1}{4}(a_1b_2+a_2b_1)\frac{(p_1p_2)^2}{(p_1+p_2)^2}z_1z_2 + \frac{1}{16}a_2b_2p_2^2z_2^2$$
$$+\frac{1}{256}a_1a_2b_1b_2\frac{(p_1p_2)^2(p_1-p_2)^4}{(p_1+p_2)^4}(z_1z_2)^2, \tag{4.10a}$$

$$g_1 = a_1z_1 + a_2z_2 + \frac{1}{16}a_1a_2b_1\frac{p_1^2(p_1-p_2)^2}{(p_1+p_2)^2}z_1^2z_2 + \frac{1}{16}a_1a_2b_2\frac{p_2^2(p_1-p_2)^2}{(p_1+p_2)^2}z_1z_2^2, \tag{4.10b}$$

$$g_2 = b_1z_1 + b_2z_2 + \frac{1}{16}a_1b_1b_2\frac{p_1^2(p_1-p_2)^2}{(p_1+p_2)^2}z_1^2z_2 + \frac{1}{16}a_2b_1b_2\frac{p_2^2(p_1-p_2)^2}{(p_1+p_2)^2}z_1z_2^2. \tag{4.10c}$$

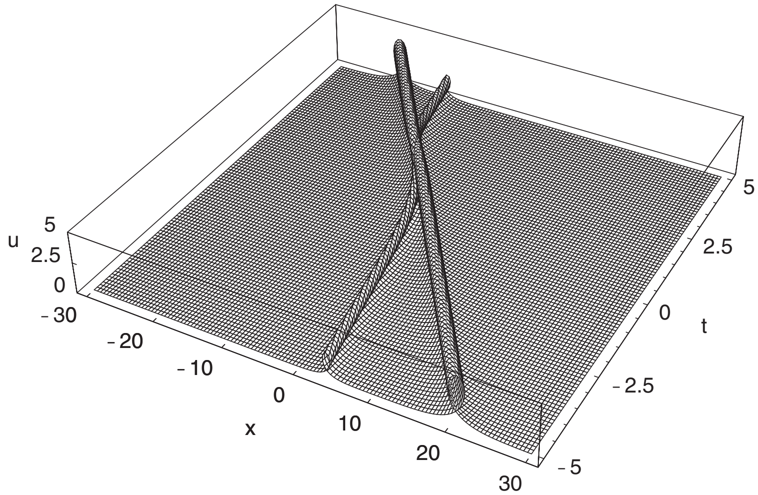

FIG. 2. The time evolution of the 2-loop soliton solution u with the parameters $p_1 = 0.5$, $p_2 = 1.0$, $a_1 = 1.0$, $a_2 = 2.0$, $b_1 = 1.0$, $b_2 = 2.0$, and $x_{10} = x_{20} = 0$.

We consider the 2-soliton solution corresponding to the tau-functions (4.10). Figure 2 shows the time evolution of the 2-soliton solution u. It represents the interaction of two loop solitons, each takes the form of the 1-loop soliton given by (4.8), as we demonstrate now.

We investigate the asymptotic behavior of the solution u. To this end, we assume $0 < p_1 < p_2$ and $a_i > 0$, $b_i > 0\,(i = 1, 2)$. Then, an asymptotic analysis similar to that developed for the 2-loop soliton solution of the SP equation shows that as $t \to -\infty$, u behaves like[9]

$$u = u_1 + u_2 \sim \frac{2}{p_1}\sqrt{\frac{a_1}{b_1}}\,\mathrm{sech}(\xi_1 + \delta_1') + \frac{2}{p_2}\sqrt{\frac{a_2}{b_2}}\,\mathrm{sech}(\xi_2 + \delta_2), \tag{4.11a}$$

$$x + c_1 t - x_{10} \sim \frac{\xi_1}{p_1} - \frac{2}{p_1}\tanh(\xi_1 + \delta_1') - \frac{2}{p_1} - \frac{4}{p_2}, \ \text{for } u_1, \tag{4.11b}$$

$$x + c_2 t - x_{20} \sim \frac{\xi_1}{p_2} - \frac{2}{p_2}\tanh(\xi_2 + \delta_2) - \frac{2}{p_2}, \ \text{for } u_2, \tag{4.11c}$$

where

$$c_i = \frac{1}{p_i^2}, \quad \delta_i = \ln\left(\frac{\sqrt{a_i b_i}}{4}p_i\right), \quad \delta_i' = \ln\left[\frac{\sqrt{a_i b_i}}{4}p_i\left(\frac{p_1 - p_2}{p_1 + p_2}\right)^2\right], \ (i = 1, 2). \tag{4.11d}$$

As $t \to +\infty$, on the other hand

$$u = u_1 + u_2 \sim \frac{2}{p_1}\sqrt{\frac{a_1}{b_1}}\,\mathrm{sech}(\xi_1 + \delta_1) + \frac{2}{p_2}\sqrt{\frac{a_2}{b_2}}\,\mathrm{sech}(\xi_2 + \delta_2'), \tag{4.12a}$$

$$x + c_1 t - x_{10} \sim \frac{\xi_1}{p_1} - \frac{2}{p_1}\tanh(\xi_1 + \delta_1) - \frac{2}{p_1}, \ \text{for } u_1, \tag{4.12b}$$

$$x + c_2 t - x_{20} \sim \frac{\xi_1}{p_2} - \frac{2}{p_2}\tanh(\xi_2 + \delta_2') - \frac{2}{p_2} - \frac{4}{p_1}, \ \text{for } u_2. \tag{4.12c}$$

We observe that the solution u splits into two loop solitons as time evolves, each of which has the form of a single loop soliton. The only effect due to the interaction of two loop solitons is the phase shift. To see this, let x_{ic} be the center position of the ith soliton. Then, it follows from the asymptotic forms (4.11) and (4.12) that

$$x_{1c}+c_1t-x_{10}\sim-\frac{\delta_1'}{p_1}-\frac{2}{p_1}-\frac{4}{p_2},\quad x_{2c}+c_2t-x_{20}\sim-\frac{\delta_2}{p_2}-\frac{2}{p_2},\ (t\to\ -\infty), \tag{4.13a}$$

$$x_{1c}+c_1t-x_{10}\sim-\frac{\delta_1}{p_1}-\frac{2}{p_1},\quad x_{2c}+c_2t-x_{20}\sim-\frac{\delta_2'}{p_2}-\frac{2}{p_2}-\frac{4}{p_1}.\ (t\to\ +\infty). \tag{4.13b}$$

Since two solitons propagate to the left, the phase shift of the ith soliton can be defined by the relation

$$\Delta_i=x_{ic}(t\to-\infty)-x_{ic}(t\to+\infty),\ \ (i=1,2). \tag{4.14}$$

Thus, from (4.13) and (4.14) one has

$$\Delta_1=-\frac{1}{p_1}\ln\left(\frac{p_1-p_2}{p_1+p_2}\right)^2-\frac{4}{p_2}, \tag{4.15a}$$

$$\Delta_2=\frac{1}{p_2}\ln\left(\frac{p_1-p_2}{p_1+p_2}\right)^2+\frac{4}{p_1}. \tag{4.15b}$$

The same calculation can be applied to v as well. The corresponding asymptotic formulas are obtained simply by interchanging a_i and b_i $(i=1,2)$ in the above expressions. It should be remarked that the above formulas for the phase shift do not depend on a_i and b_i $(i=1,2)$ and are determined only by the amplitude parameters p_1 and p_2. They coincide with those of the 2-loop soliton solution of the SP equation.[9] A novel feature of the solution in the present 2-component system is that the large soliton propagates slower than the small soliton if the inequality $\frac{1}{p_1}\sqrt{\frac{a_1}{b_1}}<\frac{1}{p_2}\sqrt{\frac{a_2}{b_2}}$ holds. This fact is seen from the asymptotic forms (4.11a) and (4.12a) of the solution with the velocities of u_1 and u_2 being given, respectively, by $c_1=1/p_1^2$ and $c_2=1/p_2^2$ $(c_2<c_1)$.

4. Breather solutions

The breather solutions are constructed from the soliton solutions by following the same manipulation as that used for the soliton solutions of the SP equation.[9] Here, we present the 1-breather solution. In this case, we put

$$p_1=a+\mathrm{i}b=p_2^*,\ \xi_{10}=\lambda+\mathrm{i}\mu=\xi_{20}^*,\ a_1=\alpha_1\mathrm{e}^{\mathrm{i}\phi_1}=a_2^*,\ b_1=\beta_1\mathrm{e}^{\mathrm{i}\psi_1}=b_2^* \tag{4.16}$$

in (4.10) to obtain the tau-functions f, g_1, and g_2. Here, a, b, α_1, and β_1 are positive constants, λ, μ, ϕ_1, and ψ_1 are real constants, and the asterisk denotes complex conjugate. After a few calculations, we find the following expressions:

$$f=\frac{4}{b^2}\mathrm{e}^{2(\theta+\theta_0)}\hat{f},\quad g_1=\frac{16\frac{a}{b}\sqrt{\frac{\alpha_1}{\beta_1}}}{\sqrt{a^2+b^2}}\mathrm{e}^{2(\theta+\theta_0)}\hat{g}_1,\quad g_2=\frac{16\frac{a}{b}\sqrt{\frac{\beta_1}{\alpha_1}}}{\sqrt{a^2+b^2}}\mathrm{e}^{2(\theta+\theta_0)}\hat{g}_2 \tag{4.17a}$$

with

$$\hat{f}=b^2\cosh^2(\theta+\theta_0)+a^2\cos^2(\chi+\chi_0+\delta')-(a^2+b^2)\sin^2\delta, \tag{4.17b}$$

$$\hat{g}_1=\sin(\chi_0-\delta)\sin(\chi+\chi_0+\delta')\cosh(\theta+\theta_0)-\cos(\chi_0-\delta)\cos(\chi+\chi_0+\delta')\sinh(\theta+\theta_0), \tag{4.17c}$$

$$\hat{g}_2=\sin(\chi_0+\delta)\sin(\chi+\chi_0+\delta')\cosh(\theta+\theta_0)-\cos(\chi_0+\delta)\cos(\chi+\chi_0+\delta')\sinh(\theta+\theta_0), \tag{4.17d}$$

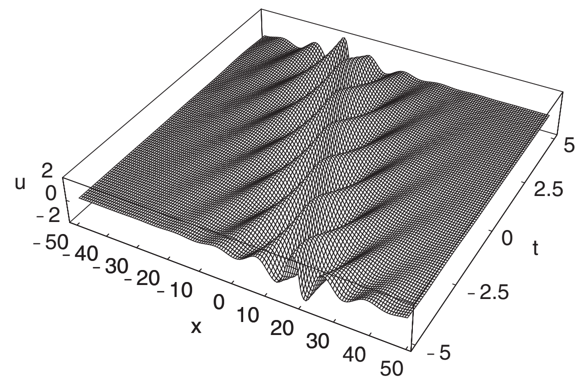

FIG. 3. The time evolution of the 1-breather solution u with the parameters $a = 0.1$, $b = 0.5$, $\alpha_1 = 1.0$, $\beta_1 = 2.0$, $\phi_1 = 0$, $\psi_1 = \pi/2$, and $\lambda = \mu = 0$.

where

$$\theta = a\left(y + \frac{1}{a^2+b^2}\,\tau\right) + \lambda, \quad \chi = b\left(y - \frac{1}{a^2+b^2}\,\tau\right) + \mu, \tag{4.18a}$$

$$\mathrm{e}^{\theta_0} = \frac{b}{4a}\sqrt{\alpha_1\beta_1(a^2+b^2)}, \quad \tan\,\chi_0 = \frac{b}{a}, \quad \delta = \frac{1}{2}(\phi_1 - \psi_1), \quad \delta' = \frac{1}{2}(\phi_1 + \psi_1). \tag{4.18b}$$

Substituting (4.17) into (3.5), we obtain the parametric representation of the 1-breather solution:

$$u = \frac{4ab\sqrt{\frac{\alpha_1}{\beta_1}}}{\sqrt{a^2+b^2}}\,\frac{\hat{g}_1}{\hat{f}}, \quad v = \frac{4ab\sqrt{\frac{\beta_1}{\alpha_1}}}{\sqrt{a^2+b^2}}\,\frac{\hat{g}_2}{\hat{f}}, \tag{4.19a}$$

$$x = y - \frac{2ab}{a^2+b^2}\,\frac{b\,\sinh\,2(\theta+\theta_0) + a\,\sin\,2(\chi+\chi_0+\delta')}{\hat{f}} - \frac{4a}{a^2+b^2}. \tag{4.19b}$$

Both u and v include two different phases θ and χ. The former characterizes the envelope of the breather, whereas the latter governs the internal oscillation. Figure 3 shows the time evolution of the 1-breather solution u. It represents an oscillating localized pulse moving to the left. Contrary to the single loop soliton, the profile of the pulse is nonstationary in the comoving coordinate system. An inspection shows that solution (4.19) exhibits singularities as encountered in the case of the breather solution of the SP equation. Therefore, certain condition must be imposed on the parameters a, b, and δ to produce the regular breather. However, we do not pursue the detail here.

C. Related integrable system

1. The 2-component system

The 2-component system (1.8) can be transformed to another integrable system (1.9) by a simple transformation. To show this, we put

$$u = \tilde{u} + \mathrm{i}\tilde{v}, \quad v = \tilde{u} - \mathrm{i}\tilde{v}, \tag{4.20}$$

and substitute this into Eqs. (1.8), we obtain a system of equations for $\tilde{u}$ and $\tilde{v}$:

$$\tilde{u}_{xt} = \tilde{u} + \frac{1}{2}[(\tilde{u}^2 + \tilde{v}^2)\tilde{u}_x]_x, \tag{4.21a}$$

$$\tilde{v}_{xt} = \tilde{v} + \frac{1}{2}[(\tilde{u}^2 + \tilde{v}^2)\tilde{v}_x]_x. \tag{4.21b}$$

This system is a special case of the n-component system (3.54) with $n = 2$.

2. N-soliton solution

The parametric representation of the N-soliton solution for Eqs. (4.21) can be expressed in the form

$$\tilde{u} = \frac{\tilde{g}_1}{\tilde{f}}, \quad \tilde{v} = \frac{\tilde{g}_2}{\tilde{f}}, \quad x = y - 2\frac{\tilde{f}_\tau}{\tilde{f}}, \tag{4.22}$$

where the tau-functions $\tilde{g}_1$, $\tilde{g}_2$, and $\tilde{f}$ satisfy the system of bilinear equations:

$$D_y D_\tau \tilde{f} \cdot \tilde{g}_i = \tilde{f}\tilde{g}_i, \ (i = 1, 2), \tag{4.23a}$$

$$D_\tau^2 \tilde{f} \cdot \tilde{f} = \frac{1}{2}(\tilde{g}_1^2 + \tilde{g}_2^2). \tag{4.23b}$$

Here, we consider real-valued solutions $\tilde{u}$ and $\tilde{v}$ for system (4.21). The tau-functions representing the N-soliton solution are obtained from (4.6) by putting $p_{i+N} = p_i^*, \ \xi_{i+N\,0} = \xi_{i0}^*\,(i = 1, 2, \ldots, N)$. Then, the expressions corresponding to (4.6d), (4.6e), and (4.6f) become

$$A_{2N} = \begin{pmatrix} A_1 & A_2 \\ A_2^* & A_1^* \end{pmatrix}, \quad A_1 = \left(-\frac{p_i - p_j}{p_i + p_j} z_i z_j\right)_{1\le i,j\le N}, \quad A_2 = \left(-\frac{p_i - p_j^*}{p_i + p_j^*} z_i z_j^*\right)_{1\le i,j\le N}, \tag{4.24a}$$

$$B_{2N} = \begin{pmatrix} O_{N\times N} & B_1 \\ B_1^* & O_{N\times N} \end{pmatrix}, \quad B_1 = \left(\frac{1}{4}\frac{(p_i p_j^*)^2}{p_i^2 - p_j^{*2}}\right)_{1\le i,j\le N}, \tag{4.24b}$$

$$\mathbf{z} = (\mathrm{e}^{\xi_1}, \mathrm{e}^{\xi_2}, \ldots, \mathrm{e}^{\xi_N}, \mathrm{e}^{\xi_1^*}, \mathrm{e}^{\xi_2^*}, \ldots, \mathrm{e}^{\xi_N^*}). \tag{4.24c}$$

We can see that the tau-functions f, g_1, and g_2 (4.6) with (4.24) satisfy the conditions $f = f^*$ and $g_2 = g_1^*$, which, combined with (4.5) and (4.20), give

$$\tilde{f} = f, \quad \tilde{g}_1 = \frac{1}{2}(g_1 + g_1^*), \quad \tilde{g}_2 = \frac{1}{2\mathrm{i}}(g_1 - g_1^*). \tag{4.25}$$

As in the case of the N-soliton solution of the 2-component system (1.8) (see Sec. IV B), we have an alternative parametrization of the N-soliton solution. Namely, we replace ξ_{j0} by $\xi_{j0} + \ln(a_j + \mathrm{i}b_j)\,(j = 1, 2, \ldots, N)$ where a_j and b_j are real parameters, and then take p_j and $\xi_{j0}\,(j = 1, 2, \ldots, N)$ being real in the expressions of the tau-functions. This procedure yields the tau-functions corresponding to (4.7b) and (4.10), for example. Actually, for the 1-soliton solution, the corresponding tau-functions are given by

$$\tilde{f} = 1 + \frac{1}{16}(a_1^2 + b_1^2)p_1^2 z_1^2, \quad \tilde{g}_1 = a_1 z_1, \quad \tilde{g}_2 = b_1 z_1, \tag{4.26}$$

and for the 2-soliton solution, they read

$$\tilde{f} = 1 + \frac{1}{16}(a_1^2 + b_1^2)p_1^2 z_1^2 + \frac{1}{2}(a_1 a_2 + b_1 b_2)\frac{(p_1 p_2)^2}{(p_1 + p_2)^2} z_1 z_2 + \frac{1}{16}(a_2^2 + b_2^2)p_2^2 z_2^2 + \frac{1}{256}(a_1^2 + b_1^2)(a_2^2 + b_2^2)\frac{(p_1 p_2)^2 (p_1 - p_2)^4}{(p_1 + p_2)^4}(z_1 z_2)^2, \tag{4.27a}$$

$$\tilde{g}_1 = a_1 z_1 + a_2 z_2 + \frac{1}{16}a_2(a_1^2 + b_1^2)\frac{p_1^2 (p_1 - p_2)^2}{(p_1 + p_2)^2} z_1^2 z_2 + \frac{1}{16}a_1(a_2^2 + b_2^2)\frac{p_2^2(p_1 - p_2)^2}{(p_1 + p_2)^2} z_1 z_2^2, \tag{4.27b}$$

$$\tilde{g}_2 = b_1 z_1 + b_2 z_2 + \frac{1}{16}b_2(a_1^2 + b_1^2)\frac{p_1^2 (p_1 - p_2)^2}{(p_1 + p_2)^2} z_1^2 z_2 + \frac{1}{16}b_1(a_2^2 + b_2^2)\frac{p_2^2(p_1 - p_2)^2}{(p_1 + p_2)^2} z_1 z_2^2. \tag{4.27c}$$

It can be seen that substitution of (4.26) and (4.27) into (4.22) produces the 1- and 2-loop soliton solutions, respectively.

3. Breather solutions

One can confirm by direct calculation that the tau-functions (4.27) satisfy the bilinear equations (4.23). In the process, the reality of the parameters has not been used. This fact enables us to extend the range of the parameters to complex values. Thus, the breather solutions are constructed from the soliton solutions by applying the procedure developed in Sec. IV B. In practice, according to parametrization (4.16), one has

$$\tilde{f} = \frac{4}{b^2}\mathrm{e}^{2(\theta+\theta_0)}\bar{f}, \quad \tilde{g}_1 = \frac{16\alpha_1 a}{\gamma b\sqrt{a^2+b^2}}\,\mathrm{e}^{2(\theta+\theta_0)}\,\bar{g}_1, \quad \tilde{g}_2 = \frac{16\beta_1 a}{\gamma b\sqrt{a^2+b^2}}\,\mathrm{e}^{2(\theta+\theta_0)}\,\bar{g}_2 \tag{4.28a}$$

with

$$\bar{f} = b^2\cosh^2(\theta+\theta_1) + a^2\cos^2(\chi+\chi_0+\kappa) + \frac{1}{2}(a^2+b^2)\left(\frac{\alpha_1^2+\beta_1^2}{\gamma^2}-1\right), \tag{4.28b}$$

$$\begin{aligned}\bar{g}_1 = {}& \sin(\chi_0-\phi_1+\kappa)\sin(\chi+\chi_0+\kappa)\,\cosh(\theta+\theta_1)\\ &-\cos(\chi_0-\phi_1+\kappa)\cos(\chi+\chi_0+\kappa)\sinh(\theta+\theta_1),\end{aligned} \tag{4.28c}$$

$$\begin{aligned}\bar{g}_2 = {}& \sin(\chi_0-\psi_1+\kappa)\sin(\chi+\chi_0+\kappa)\,\cosh(\theta+\theta_1)\\ &-\cos(\chi_0-\psi_1+\kappa)\cos(\chi+\chi_0+\kappa)\sinh(\theta+\theta_1),\end{aligned} \tag{4.28d}$$

where the parameters θ_1, γ, and κ are defined by

$$\mathrm{e}^{\theta_1} = \frac{b}{4a}\sqrt{a^2+b^2}\,\gamma, \quad \gamma = [\alpha_1^4 + 2\alpha_1^2\beta_1^2\,\cos\,2(\phi_1-\psi_1)+\beta_1^4]^{\frac{1}{4}}, \tag{4.28e}$$

$$\tan\,2\kappa = \frac{\alpha_1^2\,\sin\,2\phi_1 + \beta_1^2\,\sin\,2\psi_1}{\alpha_1^2\,\cos\,2\phi_1+\beta_1^2\,\cos\,2\phi_1}, \tag{4.28f}$$

and θ, χ, and θ_0 are already given, respectively, by (4.18a) and (4.18b). Substituting (4.28) into (4.22), we obtain the parametric representation of the 1-breather solution:

$$\tilde{u} = \frac{4\alpha_1 ab}{\gamma\sqrt{a^2+b^2}}\,\frac{\bar{g}_1}{\bar{f}}, \quad \tilde{v} = \frac{4\beta_1 ab}{\gamma\sqrt{a^2+b^2}}\,\frac{\bar{g}_2}{\bar{f}}, \tag{4.29a}$$

$$x = y - \frac{2ab}{a^2+b^2}\,\frac{b\,\sinh\,2(\theta+\theta_1)+a\,\sin\,2(\chi+\chi_0+\kappa)}{\bar{f}} - \frac{4a}{a^2+b^2}. \tag{4.29b}$$

Of particular interest is a circularly polarized wave for which the solution exhibits a simple structure, as we shall now demonstrate. In this case, we put $\alpha_1 = \beta_1$, $\chi_0-\phi_1+\kappa = \frac{\pi}{2}$ and $\phi_1-\psi_1 = \frac{\pi}{2}$ to obtain the tau-functions

$$\tilde{f} = 1 + \frac{\alpha_1^2(a^2+b^2)^2}{4a^2}\,\mathrm{e}^{2\theta}, \quad \tilde{g}_1 = 2\alpha_1\,\mathrm{e}^{\theta}\cos(\chi+\phi_1), \quad \tilde{g}_2 = 2\alpha_1\,\mathrm{e}^{\theta}\sin(\chi+\phi_1). \tag{4.30}$$

Then, the solution takes the form

$$\tilde{u} = \frac{2a}{a^2+b^2}\frac{\cos(\chi+\phi_1)}{\cosh(\theta+\theta_0')}, \quad \tilde{v} = \frac{2a}{a^2+b^2}\frac{\sin(\chi+\phi_1)}{\cosh(\theta+\theta_0')}, \quad \left(\mathrm{e}^{\theta_0'} = \frac{\alpha_1(a^2+b^2)}{2a}\right), \tag{4.31a}$$

$$x = y - \frac{2a}{a^2+b^2}\,\tanh(\theta+\theta_0') - \frac{2a}{a^2+b^2}. \tag{4.31b}$$

The parametric solution (4.31) represents a nonsingular breather if the inequality $0 < a/b < 1$ holds. Figure 4 shows the time evolution of $u(\equiv\tilde{u})$ given by (4.31).

One can show by a direct calculation that the solution (4.31) satisfies the integral relations

$$\int_{-\infty}^{\infty}\tilde{u}\,dx = 0, \quad \int_{-\infty}^{\infty}\tilde{v}\,dx = 0, \tag{4.32}$$

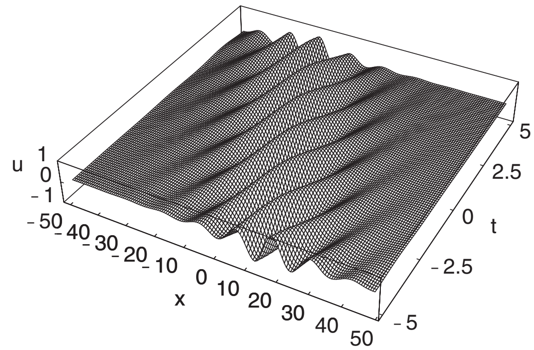

FIG. 4. The time evolution of the 1-breather solution $u(\equiv \tilde{u})$ with the parameters $a = 0.1$, $b = 0.5$, $\alpha_1 = 1.0$, $\phi_1 = 0$, and $\lambda = \mu = 0$.

implying that both $\tilde{u}$ and $\tilde{v}$ are zero mean fields. This fact indicates clearly an oscillating character of the solution. Note that the above relations represent the conservation laws derived from the system of equations (4.21) for localized waves. It is interesting to observe that in the small amplitude limit $a/b \to 0$, the profile of $\tilde{u}$ bears resemblance to that of the soliton solution of the nonlinear Schrödinger equation.

D. Remarks

1. The system of equations (4.1) is equivalent to a coupled dispersionless system for the variables $r = r(x, t)$, $s = s(x, t)$, and $q = q(x, t)$:

$$q_{xt} + (rs)_x = 0, \quad r_{xt} - 2q_x r = 0, \quad s_{xt} - 2q_x s = 0. \tag{4.33}$$

The Lax pair associated with system (4.33) has been obtained and the IST has been applied to it to construct soliton solutions.[20] In particular, 1- and 2-soliton solutions have been presented for r, s, and q_x. Here, we present the formula for the general multisoliton solution for the first time.

2. The system of bilinear equations (4.23) can be derived from system (4.33) with a reduction $s = r^*$ through appropriate dependent variable transformations.[21] See also an analysis by means of the IST.[22]

V. CONCLUSION

In this paper, we proposed a novel multi-component system associated with the SP equation and constructed its multisoliton solutions in terms of pfaffians. We also considered the equations reduced from our system. In particular, the 2-component system (1.8) was found to be completely integrable for which the explicit Lax pair was presented. We also provided the loop soliton and breather solutions for the system and investigated their properties. We also addressed system (1.9) which stems from system (1.8) by a simple transformation. In conclusion, we shall discuss some open problems associated with the multi-component system under consideration.

1. One interesting issue to be resolved in a future work is the proof of the complete integrability of the n-component system (1.7) by using the IST. To construct the Lax pair for the system, one way will be to start from the system of bilinear equations (3.6) and (3.7) to obtain the Bäcklund transformation among the tau-functions and then derive the scheme of the IST following the standard procedure in the bilinear formalism.

2. Other issues to be reserved for detailed study have already been described in Sec. III D. Of particular importance is the construction of the multisoliton solution of the n-component system (3.54) with $n \geq 3$. Unlike the 2-component system, the linear transformation such as (4.20) does not exist to convert system (1.7) to system (3.54). Hence, one must solve the system of equations (3.52) and (3.53) with $p = n$ and $q = 0$. It will be a relatively simple task to obtain the 1- and

2-soliton solutions analogous to (4.26) and (4.27). Nevertheless, a systematic approach is necessary to construct general multisoliton solutions.

3. The system of equations (1.5) has been derived as a unidirectional model describing the propagation of circularly polarized ultra-short pulses in a Kerr medium.[23] The solution of breather type has been obtained by means of an analysis as well as numerical computations.[24,25] However, in view of the extremely complicated structure of the breather solution,[24] it seems to be unlikely that the system admits multibreather solutions as well. Thus, we suspect the complete integrability of the system even if it has passed the Painlevé test. On the other hand, although the difference between (1.5) and (1.9) is the location of the x derivative on the right-hand side, the latter shares many common features to the integrable systems such as the complete integrability and the existence of multisoliton solutions. At present, however, the relevance of the system to the description of the dynamics of ultra-short pulses in optical fibers is not clear. Nevertheless, it would be of interest to examine the possibility of the system as a physical model for the 2-component generalization of the SP equation.

Note added in proof: After the acceptance of the paper for publication, the author was informed by Professor Müller-Hoissen that he and his coworker proposed the multi-component system (3.54) and obtained the N-soliton solution of the 2-component system by means of their bidifferential calculus approach.[26] However, the construction of the N-soliton solution of the n-component system with $n \geq 3$ still remains open.

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