Schrödinger uncertainty relation, Wigner-Yanase-Dyson skew information and metric adjusted correlation measure

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Abstract. In this paper, we give Schrödinger-type uncertainty relation using the Wigner-Yanase-Dyson skew information. In addition, we give Schrödinger-type uncertainty relation by use of a two-parameter extended correlation measure. We finally show the further generalization of Schrödinger-type uncertainty relation by use of the metric adjusted correlation measure. These results generalize our previous result in [Phys. Rev. A, Vol.82(2010), 034101].

Keywords : Trace inequality, Wigner-Yanase-Dyson skew information, Schrödinger uncertainty relation and metric adjusted correlation measure

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1 Introduction

In quantum information theory, one of the most important results is the strong subadditivity of von Neumann entropy [22]. This important property of von Neumann entropy can be proven by the use of Lieb's theorem [16] which gave a complete solution for the conjecture of the convexity of Wigner-Yanase-Dyson skew information. In addition, the uncertainty relation has been widely studied in quantum information theory [21, 31, 29]. In particular, the relations between skew information and uncertainty relation have been studied in [17, 4, 8, 9, 7]. Quantum Fisher information is also called monotone metric which was introduced by Petz [23] and the Wigner-Yanase-Dyson metric is connected to quantum Fisher information (monotone metric) as a special case. Recently, Hansen gave a further development of the notion of monotone metric, so-called metric adjusted skew information [12]. The Wigner-Yanase-Dyson skew information is also connected to the metric adjusted skew information as a special case. That is, the metric adjusted skew information gave a class including the Wigner-Yanase-Dyson skew information, while the monotone metric gave a class including the Wigner-Yanase-Dyson metric. In the paper [12], the metric adjusted correlation measure was also introduced as a generalization of the quantum covariance and correlation measure defined in [17]. Therefore there is a significance to give the relation among the Wigner-Yanase-Dyson skew information, metric adjusted correlation measure and uncertainty relation for the fundamental studies on quantum information theory.

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We start from the Heisenberg uncertainty relation [13]:

$$V_\rho(A)V_\rho(B) \geq \frac{1}{4}|Tr[\rho[A,B]]|^2 \quad (1)$$

for a quantum state (density operator) ρ and two observables (self-adjoint operators) A and B. The further stronger result was given by Schrödinger in [27, 28]:

$$V_\rho(A)V_\rho(B) - |Re\left\{Cov_\rho(A,B)\right\}|^2 \geq \frac{1}{4}|Tr[\rho[A,B]]|^2, \quad (2)$$

where the covariance is defined by $Cov_\rho(A,B) \equiv Tr[\rho\,(A - Tr[\rho A]I)\,(B - Tr[\rho B]I)]$.

The Wigner-Yanase skew information represents a measure for non-commutativity between a quantum state ρ and an observable H. Luo introduced the quantity $U_\rho(H)$ representing a quantum uncertainty excluding the classical mixture [18]:

$$U_\rho(H) \equiv \sqrt{V_\rho(H)^2 - (V_\rho(H) - I_\rho(H))^2}, \quad (3)$$

with the Wigner-Yanase skew information [32]:

$$I_\rho(H) \equiv \frac{1}{2}Tr\left[(i[\rho^{1/2}, H_0])^2\right] = Tr[\rho H_0^2] - Tr[\rho^{1/2}H_0\rho^{1/2}H_0], \quad H_0 \equiv H - Tr[\rho H]I$$

and then he successfully showed a new Heisenberg-type uncertainty relation on $U_\rho(H)$ in [18]:

$$U_\rho(A)U_\rho(B) \geq \frac{1}{4}|Tr[\rho[A,B]]|^2. \quad (4)$$

As stated in [18], the physical meaning of the quantity $U_\rho(H)$ can be interpreted as follows. For a mixed state ρ, the variance $V_\rho(H)$ has both classical mixture and quantum uncertainty. Also, the Wigner-Yanase skew information $I_\rho(H)$ represents a kind of quantum uncertainty [19, 20]. Thus, the difference $V_\rho(H) - I_\rho(H)$ has a classical mixture so that we can regard that the quantity $U_\rho(H)$ has a quantum uncertainty excluding a classical mixture. Therefore it is meaningful and suitable to study an uncertainty relation for a mixed state by the use of the quantity $U_\rho(H)$.

Recently, a one-parameter extension of the inequality (4) was given in [33]:

$$U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) \geq \alpha(1-\alpha)|Tr[\rho[A,B]]|^2, \quad (5)$$

where

$$U_{\rho,\alpha}(H) \equiv \sqrt{V_\rho(H)^2 - (V_\rho(H) - I_{\rho,\alpha}(H))^2},$$

with the Wigner-Yanase-Dyson skew information $I_{\rho,\alpha}(H)$ is defined by

$$I_{\rho,\alpha}(H) \equiv \frac{1}{2}Tr\left[(i[\rho^\alpha, H_0])(i[\rho^{1-\alpha}, H_0])\right] = Tr[\rho H_0^2] - Tr[\rho^\alpha H_0 \rho^{1-\alpha} H_0],$$

It is notable that the convexity of $I_{\rho,\alpha}(H)$ with respect to ρ was successfully proven by Lieb in [16]. The further generalization of the Heisenberg-type uncertainty relation on $U_\rho(H)$ has been given in [34] using the generalized Wigner-Yanase-Dyson skew information introduced in [3]. See also [1, 5, 7, 8] for the recent studies on skew informations and uncertainty relations.

Motivated by the fact that the Schrödinger uncertainty relation is a stronger result than the Heisenberg uncertainty relation, a new Schrödinger-type uncertainty relation for mixed states using Wigner-Yanase skew information was shown in [4]. That is, for a quantum state ρ and two observables A and B, we have

$$U_\rho(A)U_\rho(B) - |Re\left\{Corr_\rho(A,B)\right\}|^2 \geq \frac{1}{4}|Tr[\rho[A,B]]|^2, \quad (6)$$

where the correlation measure [17] is defined by

$$Corr_\rho(X,Y) \equiv Tr[\rho X^* Y] - Tr[\rho^{1/2} X^* \rho^{1/2} Y]$$

for any operators X and Y. This result refined the Heisenberg-type uncertainty relation (4) shown in [18] for mixed states (general states). We easily find that the inequality (6) is equivalent to the following inequality:

$$U_\rho(A)U_\rho(B) \geq |Corr_\rho(A,B)|^2. \tag{7}$$

The main purpose of this paper is to give some extensions of the inequality (7) by using the Wigner-Yanase-Dyson skew information $I_{\rho,\alpha}(H)$ and the metric adjusted correlation measure introduced in [12].

2 Schrödinger uncertainty relation with Wigner-Yanase-Dyson skew information

In this section, we give a generalization of the Schrödinger type uncertainty relation (7) by the use of the quantity $U_{\rho,\alpha}(H)$ defined by the Wigner-Yanase-Dyson skew information $I_{\rho,\alpha}(H)$.

Theorem 2.1 *For $\alpha \in [1/2, 1]$, a quantum state ρ and two observables A and B, we have*

$$U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) \geq 4\alpha(1-\alpha)|Corr_{\rho,\alpha}(A,B)|^2. \tag{8}$$

where the generalized correlation measure [14, 36] is defined by

$$Corr_{\rho,\alpha}(X,Y) \equiv Tr[\rho X^* Y] - Tr[\rho^\alpha X^* \rho^{1-\alpha} Y]$$

for any operators X and Y.

To prove Theorem 2.1, we need the following lemmas.

Lemma 2.2 ([33]) *For a spectral decomposition of $\rho = \sum_{j=1}^{\infty} \lambda_j |\phi_j\rangle\langle\phi_j|$, putting $h_{ij} \equiv \langle\phi_i|H_0|\phi_j\rangle$, we have the following relations.*

(i) For the Wigner-Yanase-Dyson skew information, we have

$$I_{\rho,\alpha}(H) = \sum_{i<j} \left(\lambda_i^\alpha - \lambda_j^\alpha\right)\left(\lambda_i^{1-\alpha} - \lambda_j^{1-\alpha}\right)|h_{ij}|^2.$$

(ii) For the quantity associated to the Wigner-Yanase-Dyson skew information:

$$J_{\rho,\alpha}(H) \equiv \frac{1}{2} Tr\left[\left(\{\rho^\alpha, H_0\}\right)\left(\{\rho^{1-\alpha}, H_0\}\right)\right] = Tr[\rho H_0^2] + Tr[\rho^\alpha H_0 \rho^{1-\alpha} H_0],$$

where $\{X,Y\} \equiv XY + YX$ is an anti-commutator, we have

$$J_{\rho,\alpha}(H) \geq \sum_{i<j} \left(\lambda_i^\alpha + \lambda_j^\alpha\right)\left(\lambda_i^{1-\alpha} + \lambda_j^{1-\alpha}\right)|h_{ij}|^2.$$

Lemma 2.3 ([2, 33]) *For any $t > 0$ and $\alpha \in [0,1]$, we have*

$$(1-2\alpha)^2(t-1)^2 \geq (t^\alpha - t^{1-\alpha})^2.$$

Proof of Theorem 2.1: We take a spectral decomposition $\rho = \sum_{j=1}^{\infty} \lambda_j |\phi_j\rangle\langle\phi_j|$. If we put $a_{ij} = \langle\phi_i|A_0|\phi_j\rangle$ and $b_{ji} = \langle\phi_j|B_0|\phi_i\rangle$, where $A_0 = A - Tr[\rho A]I$ and $B_0 = B - Tr[\rho B]I$, then we have

$$\begin{aligned} Corr_{\rho,\alpha}(A,B) &= Tr[\rho AB] - Tr[\rho^\alpha A \rho^{1-\alpha} B] \\ &= Tr[\rho A_0 B_0] - Tr[\rho^\alpha A_0 \rho^{1-\alpha} B_0] \\ &= \sum_{i,j=1}^{\infty} (\lambda_i - \lambda_i^\alpha \lambda_j^{1-\alpha}) a_{ij} b_{ji} \\ &= \sum_{i \neq j} (\lambda_i - \lambda_i^\alpha \lambda_j^{1-\alpha}) a_{ij} b_{ji} \\ &= \sum_{i<j} \left\{ (\lambda_i - \lambda_i^\alpha \lambda_j^{1-\alpha}) a_{ij} b_{ji} + (\lambda_j - \lambda_j^\alpha \lambda_i^{1-\alpha}) a_{ji} b_{ij} \right\}. \end{aligned} \tag{9}$$

Thus we have

$$|Corr_{\rho,\alpha}(A,B)| \leq \sum_{i<j} \left\{ |\lambda_i - \lambda_i^\alpha \lambda_j^{1-\alpha}||a_{ij}||b_{ji}| + |\lambda_j - \lambda_j^\alpha \lambda_i^{1-\alpha}||a_{ji}||b_{ij}| \right\}.$$

Since $|a_{ij}| = |a_{ji}|$ and $|b_{ij}| = |b_{ji}|$, taking a square of both sides and then using Schwarz inequality and Lemma 2.2, we have

$$\begin{aligned} & 4\alpha(1-\alpha)|Corr_{\rho,\alpha}(A,B)|^2 \\ & \leq 4\alpha(1-\alpha) \left\{ \sum_{i<j} \left\{ |\lambda_i - \lambda_i^\alpha \lambda_j^{1-\alpha}| + |\lambda_j - \lambda_j^\alpha \lambda_i^{1-\alpha}| \right\} |a_{ij}||b_{ji}| \right\}^2 \\ & = \left\{ \sum_{i<j} 2\sqrt{\alpha(1-\alpha)} \left(\lambda_i^\alpha + \lambda_j^\alpha \right) |\lambda_i^{1-\alpha} - \lambda_j^{1-\alpha}||a_{ij}||b_{ji}| \right\}^2 \\ & \leq \left\{ \sum_{i<j} 2\sqrt{\alpha(1-\alpha)} |\lambda_i - \lambda_j||a_{ij}||b_{ji}| \right\}^2 \\ & \leq \left\{ \sum_{i<j} \left\{ \left(\lambda_i^\alpha - \lambda_j^\alpha \right) \left(\lambda_i^{1-\alpha} - \lambda_j^{1-\alpha} \right) \left(\lambda_i^\alpha + \lambda_j^\alpha \right) \left(\lambda_i^{1-\alpha} + \lambda_j^{1-\alpha} \right) \right\}^{1/2} |a_{ij}||b_{ji}| \right\}^2 \\ & \leq \left\{ \sum_{i<j} \left(\lambda_i^\alpha - \lambda_j^\alpha \right) \left(\lambda_i^{1-\alpha} - \lambda_j^{1-\alpha} \right) |a_{ij}|^2 \right\} \left\{ \sum_{i<j} \left(\lambda_i^\alpha + \lambda_j^\alpha \right) \left(\lambda_i^{1-\alpha} + \lambda_j^{1-\alpha} \right) |b_{ij}|^2 \right\} \\ & \leq I_{\rho,\alpha}(A) J_{\rho,\alpha}(B) \end{aligned}$$

In the above process, the inequality $(x^\alpha + y^\alpha)|x^{1-\alpha} - y^{1-\alpha}| \leq |x - y|$ for $x, y \geq 0$ and $\alpha \in [\frac{1}{2}, 1]$ and the inequality $4\alpha(1-\alpha)(x-y)^2 \leq (x^\alpha - y^\alpha)\left(x^{1-\alpha} - y^{1-\alpha}\right)(x^\alpha + y^\alpha)\left(x^{1-\alpha} + y^{1-\alpha}\right)$ for $x, y \geq 0$ and $\alpha \in [0, 1]$, which can be proven by Lemma 2.3, were used. By the similar way, we also have

$$4\alpha(1-\alpha)|Corr_{\rho,\alpha}(A,B)|^2 \leq I_{\rho,\alpha}(B) J_{\rho,\alpha}(A).$$

Thus for $\alpha \geq \frac{1}{2}$ we have

$$4\alpha(1-\alpha)|Corr_{\rho,\alpha}(A,B)|^2 \leq U_{\rho,\alpha}(A) U_{\rho,\alpha}(B). \tag{10}$$

■

Note that Theorem 2.1 recovers the inequality (7), if we take $\alpha = \frac{1}{2}$.

Remark 2.4 *We take* $\alpha = 0.1$ *and*

$$\rho = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, A = \begin{pmatrix} 2 & 2-i \\ 2+i & 1 \end{pmatrix}, B = \begin{pmatrix} 2 & i \\ -i & 1 \end{pmatrix},$$

then we have

$$U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) - 4\alpha(1-\alpha)|Corr_{\rho,\alpha}(A,B)|^2 \simeq -0.28332.$$

Therefore the inequality (8) does not hold for $\alpha \in [0, 1/2)$ *in general.*

Corollary 2.5 *Under the same assumptions with Theorem 2.1, we have the following inequality:*

$$U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) - 4\alpha(1-\alpha)\left(|Re\left\{Corr_{\rho,\alpha}(A,B)\right\}|^2 - |Im\left\{Tr[\rho^\alpha A \rho^{1-\alpha} B]\right\}|^2\right) \geq \alpha(1-\alpha)|Tr[\rho[A,B]]|^2. \tag{11}$$

Proof: From

$$Im\left\{Corr_{\rho,\alpha}(A,B)\right\} = \frac{1}{2i}Tr\left[\rho[A,B]\right] - Im\left\{Tr[\rho^\alpha A\rho^{1-\alpha}B]\right\},$$

we have

$$\frac{1}{4}|Tr\left[\rho[A,B]\right]|^2 \leq |Im\left\{Corr_{\rho,\alpha}(A,B)\right\}|^2 + |Im\left\{Tr[\rho^\alpha A\rho^{1-\alpha}B]\right\}|^2.$$

Thus we have

$$\begin{aligned} |Corr_{\rho,\alpha}(A,B)|^2 &= |Re\left\{Corr_{\rho,\alpha}(A,B)\right\}|^2 + |Im\left\{Corr_{\rho,\alpha}(A,B)\right\}|^2 \\ &\geq |Re\left\{Corr_{\rho,\alpha}(A,B)\right\}|^2 + \frac{1}{4}|Tr\left[\rho[A,B]\right]|^2 - |Im\left\{Tr[\rho^\alpha A\rho^{1-\alpha}B]\right\}|^2, \end{aligned}$$

which proves the corollary.

■

Remark 2.6 *The following inequality does not hold in general for* $\alpha \in [\frac{1}{2}, 1]$*:*

$$|Re\left\{Corr_{\rho,\alpha}(A,B)\right\}|^2 \geq |Im\left\{Tr[\rho^\alpha A\rho^{1-\alpha}B]\right\}|^2. \tag{12}$$

Because we have a counter-example as follows. We take $\alpha = \frac{2}{3}$ *and*

$$\rho = \frac{1}{7}\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}, A = \begin{pmatrix} 2 & 2-i \\ 2+i & 1 \end{pmatrix}, B = \begin{pmatrix} 2 & i \\ -i & 1 \end{pmatrix},$$

then we have

$$|Re\left\{Corr_{\rho,\alpha}(A,B)\right\}|^2 - |Im\left\{Tr[\rho^\alpha A\rho^{1-\alpha}B]\right\}|^2 \simeq -0.0548142.$$

This shows Theorem 2.1 does not refine the inequality (5) in general.

3 Two-parameter extensions

In this section, we introduce the parametric extended correlation measure $Corr_{\rho,\alpha,\gamma}(X,Y)$ by the convex combination between $Corr_{\rho,\alpha}(X,Y)$ and $Corr_{\rho,1-\alpha}(X,Y)$. Then we establish the parametric extended Schrödinger-type uncertainty relation applying the parametric extended correlation measure $Corr_{\rho,\alpha,\gamma}(X,Y)$. In addition, introducing the symmetric extended correlation measure $Corr^{(sym)}_{\rho,\alpha,\gamma}(X,Y)$ by the convex combination between $Corr_{\rho,\alpha}(X,Y)$ and $Corr_{\rho,\alpha}(Y,X)$, we show its Schrödinger-type uncertainty relation.

Definition 3.1 *We define the parametric extended correlation measure* $Corr_{\rho,\alpha,\gamma}(X,Y)$ *for two parameters* $\alpha,\gamma\in[0,1]$ *by*

$$Corr_{\rho,\alpha,\gamma}(X,Y)\equiv\gamma Corr_{\rho,\alpha}(X,Y)+(1-\gamma)Corr_{\rho,1-\alpha}(X,Y) \tag{13}$$

for any operators X *and* Y*.*

Note that we have $Corr_{\rho,\alpha,\gamma}(H,H)=I_{\rho,\alpha}(H)$ for any observable H. Then we can prove the following inequality.

Theorem 3.2 *If* $0\le\alpha,\gamma\le\frac{1}{2}$ *or* $\frac{1}{2}\le\alpha,\gamma\le 1$*, then we have*

$$U_{\rho,\alpha}(A)U_{\rho,\alpha}(B)\ge 4\alpha(1-\alpha)|Corr_{\rho,\alpha,\gamma}(A,B)|^2$$

for two observables A*,* B *and a quantum state* ρ*.*

Proof: By the similar way of the proof of Theorem 2.1, we have Eq.(9) and we also have

$$\begin{aligned}Corr_{\rho,1-\alpha}(A,B) &= Tr[\rho AB]-Tr[\rho^{1-\alpha}A\rho^{\alpha}B]\\ &= \sum_{i<j}\left\{(\lambda_i-\lambda_i^{1-\alpha}\lambda_j^{\alpha})a_{ij}b_{ji}+(\lambda_j-\lambda_j^{1-\alpha}\lambda_i^{\alpha})a_{ji}b_{ij}\right\}.\end{aligned} \tag{14}$$

Thus we have

$$\begin{aligned}Corr_{\rho,\alpha,\gamma}(A,B) &= \gamma Corr_{\rho,\alpha}(A,B)+(1-\gamma)Corr_{\rho,\alpha}(A,B)\\ &= \sum_{i<j}\left\{\gamma\lambda_i^{\alpha}(\lambda_i^{1-\alpha}-\lambda_j^{1-\alpha})+(1-\gamma)\lambda_i^{1-\alpha}(\lambda_i^{\alpha}-\lambda_j^{\alpha})\right\}a_{ij}b_{ji}\\ &\quad+\sum_{i<j}\left\{\gamma\lambda_j^{\alpha}(\lambda_j^{1-\alpha}-\lambda_i^{1-\alpha})+(1-\gamma)\lambda_j^{1-\alpha}(\lambda_j^{\alpha}-\lambda_i^{\alpha})\right\}a_{ji}b_{ij}.\end{aligned}$$

Since we have $|a_{ij}|=|a_{ji}|$ and $|b_{ij}|=|b_{ji}|$, we then have

$$\begin{aligned}|Corr_{\rho,\alpha,\gamma}(A,B)| &\le \sum_{i<j}\left\{\gamma(\lambda_i^{\alpha}+\lambda_j^{\alpha})|\lambda_i^{1-\alpha}-\lambda_j^{1-\alpha}|+(1-\gamma)(\lambda_i^{1-\alpha}+\lambda_j^{1-\alpha})|\lambda_i^{\alpha}-\lambda_j^{\alpha}|\right\}|a_{ij}||b_{ji}|\\ &\le \sum_{i<j}|\lambda_i-\lambda_j||a_{ij}||b_{ji}|,\end{aligned}$$

thanks to the inequality

$$\gamma(x^{\alpha}+y^{\alpha})|x^{1-\alpha}-y^{1-\alpha}|+(1-\gamma)(x^{1-\alpha}+y^{1-\alpha})|x^{\alpha}-y^{\alpha}|\le|x-y| \tag{15}$$

for $0\le\alpha,\gamma\le\frac{1}{2}$ or $\frac{1}{2}\le\alpha,\gamma\le 1$, and $x,y\ge 0$. The rest of the proof goes similar way to that of Theorem 2.1.

■

Corollary 3.3 *For any* $\alpha\in[0,1]$*, two observables* A*,* B *and a quantum state* ρ*, we have*

$$U_{\rho,\alpha}(A)U_{\rho,\alpha}(B)\ge 4\alpha(1-\alpha)|Corr_{\rho,\alpha,\frac{1}{2}}(A,B)|^2.$$

Proof: If $\gamma=\frac{1}{2}$, then the equality of the inequality (15) holds for any $\alpha\in[0,1]$ and $x,y\ge 0$. Therefore we have the present corollary from Theorem 3.2. ■

We may define the following correlation measure instead of Definition 3.1.

Definition 3.4 *We define a symmetric extended correlation measure* $Corr^{(sym)}_{\rho,\alpha,\gamma}(X,Y)$ *for two parameters* $\alpha,\gamma\in[0,1]$ *by*

$$Corr^{(sym)}_{\rho,\alpha,\gamma}(X,Y)\equiv\gamma Corr_{\rho,\alpha}(X,Y)+(1-\gamma)Corr_{\rho,\alpha}(Y,X) \tag{16}$$

for any operators X *and* Y*.*

Note that we have $Corr^{(sym)}_{\rho,\alpha,\gamma}(A,B)=Corr^{(sym)}_{\rho,\alpha,\gamma}(B,A)$ for self-adjoint operators A and B. Then we have the following therem by the similar proof of the above using the inequality

$$(x^{\alpha}+y^{\alpha})|x^{1-\alpha}-y^{1-\alpha}|\leq|x-y|$$

for $x,y\geq 0$ and $\alpha\geq\frac{1}{2}$.

Theorem 3.5 *For* $\alpha\in[\frac{1}{2},1]$ *and* $\gamma\in[0,1]$*, we have*

$$U_{\rho,\alpha}(A)U_{\rho,\alpha}(B)\geq 4\alpha(1-\alpha)|Corr^{(sym)}_{\rho,\alpha,\gamma}(A,B)|^2$$

for two observables A*,* B *and a quantum state* ρ*.*

4 A further generalization by metric adjusted correlation measure

Inspired by the recent results in [10] and the concept of metric adjusted skew information introduced by Hansen in [12], we here give a further generalization for Schrödinger-type uncertainty relation applying metric adjusted correlation measure introduced in [12]. We firstly give some notations according to those in [10]. Let $M_n(\mathbb{C})$ and $M_{n,sa}(\mathbb{C})$ be the set of all $n\times n$ complex matrices and all $n\times n$ self-adjoint matrices, equipped with the Hilbert-Schmidt scalar product $\langle A,B\rangle=Tr[A^*B]$, respectively. Let $M_{n,+}(\mathbb{C})$ be the set of all positive definite matrices of $M_{n,sa}(\mathbb{C})$ and $M_{n,+,1}(\mathbb{C})$ be the set of all density matrices, that is

$$M_{n,+,1}(\mathbb{C})\equiv\{\rho\in M_{n,sa}(\mathbb{C})|Tr\rho=1,\rho>0\}\subset M_{n,+}(\mathbb{C}).$$

Here $X\in M_{n,+}(\mathbb{C})$ means we have $\langle\phi|X|\phi\rangle\geq 0$ for any vector $|\phi\rangle\in\mathbb{C}^n$. In the study of quantum physics, we usually use a positive semidefinite matrix with a unit trace as a density operator ρ. In this section, we assume the invertibility of ρ.

A function $f:(0,+\infty)\to\mathbb{R}$ is said operator monotone if the inequalities $0\leq f(A)\leq f(B)$ hold for any $A,B\in M_{n,sa}(\mathbb{C})$ such that $0\leq A\leq B$. An operator monotone function $f:(0,+\infty)\to(0,+\infty)$ is said symmetric if $f(x)=xf(x^{-1})$ and normalized if $f(1)=1$. We represents the set of all symmetric normalized operator monotone functions by $\mathcal{F}_{op}$. We have the following examples as elements of $\mathcal{F}_{op}$:

Example 4.1 ([12, 10, 6, 25])

$$f_{RLD}(x)=\frac{2x}{x+1},\quad f_{SLD}(x)=\frac{x+1}{2},\quad f_{BKM}(x)=\frac{x-1}{\log x},$$

$$f_{WY}(x)=\left(\frac{\sqrt{x}+1}{2}\right)^2,\quad f_{WYD}(x)=\alpha(1-\alpha)\frac{(x-1)^2}{(x^{\alpha}-1)(x^{1-\alpha}-1)},\quad\alpha\in(0,1).$$

The functions $f_{BKM}(x)$ and $f_{WYD}(x)$ are normalized in the sense that $\lim_{x\to 1} f_{BKM}(x) = 1$ and $\lim_{x\to 1} f_{WYD}(x) = 1$. Note that a simple proof of the operator monotonicity of $f_{WYD}(x)$ was given in [6]. See also [30] for the proof of the operator monotonicity of $f_{WYD}(x)$ by use of majorization.

Remark 4.2 ([10, 15, 24, 25]) *For any $f \in \mathcal{F}_{op}$, we have the following inequalities:*

$$\frac{2x}{x+1} \le f(x) \le \frac{x+1}{2}, \quad x > 0.$$

That is, all $f \in \mathcal{F}_{op}$ lies in between the harmonic mean and the arithmetic mean.

For $f \in \mathcal{F}_{op}$ we define $f(0) = \lim_{x\to 0} f(x)$. We also denote the sets of regular and non-regular functions by

$$\mathcal{F}_{op}^r = \{f \in \mathcal{F}_{op} | f(0) \neq 0\} \ \text{and} \ \mathcal{F}_{op}^n = \{f \in \mathcal{F}_{op} | f(0) = 0\}.$$

Definition 4.3 ([8, 10]) *For $f \in \mathcal{F}_{op}^r$, we define the function $\tilde{f}$ by*

$$\tilde{f}(x) = \frac{1}{2}\left\{(x+1) - (x-1)^2 \frac{f(0)}{f(x)}\right\}, \quad (x > 0).$$

Then we have the following theorem.

Theorem 4.4 ([8, 6, 26]) *The correspondence $f \to \tilde{f}$ is a bijection between $\mathcal{F}_{op}^r$ and $\mathcal{F}_{op}^n$.*

We can use matrix mean theory introduced by Kubo-Ando in [15]. Then a mean m_f corresponds to each operator monotone function $f \in \mathcal{F}_{op}$ by the following formula

$$m_f(A,B) = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2},$$

for $A, B \in M_{n,+}(\mathbb{C})$. By the notion of matrix mean, we may define the set of the monotone metrics [23] by the following formula

$$\langle A, B\rangle_{\rho,f} = Tr[A m_f(L_\rho, R_\rho)^{-1}(B)],$$

where $L_\rho(A) = \rho A$ and $R_\rho(A) = A\rho$.

Definition 4.5 ([12, 8]) *For $A, B \in M_{n,sa}(\mathbb{C})$, $\rho \in M_{n,+,1}(\mathbb{C})$ and $f \in \mathcal{F}_{op}^r$, we define the following quantities:*

$$Corr_\rho^f(A,B) \equiv \frac{f(0)}{2}\langle i[\rho, A], i[\rho, B]\rangle_{\rho,f}, \ I_\rho^f(A) \equiv Corr_\rho^f(A,A),$$

$$C_\rho^f(A,B) \equiv Tr[m_f(L_\rho, R_\rho)(A)B], \ C_\rho^f(A) \equiv C_\rho^f(A,A),$$

$$U_\rho^f(A) \equiv \sqrt{V_\rho(A)^2 - (V_\rho(A) - I_\rho^f(A))^2}.$$

The quantity $I_\rho^f(A)$ is known as metric adjusted skew information [12]. It is notable that the metric adjusted correlation measure $Corr_\rho^c(A,B)$ was firstly introduced in [12] for a regular Morozova-Chentsov function c. Recently the notation $I_\rho^c(A,B)$ in [1] and the notation $I_\rho^f(A,B)$ in [11] were used. In addition, it is useful for the readers to be noted that the correlation $I_\rho^f(A,B)$ can be expressed as a difference of covariances [11]. Throughout the present paper, we use the notation $Corr_\rho^f(A,B)$ as the metric adjusted correlation measure, to avoid the confusion of the readers. (In the previous sections, we have already used $Corr_\rho(A,B)$, $Corr_{\rho,\alpha}(A,B)$ and $Corr_{\rho,\alpha,\gamma}(A,B)$ as correlation measures and done $I_\rho(H)$ and $I_{\rho,\alpha}(H)$ as skew informations.) Then we have the following proposition.

Proposition 4.6 ([8, 10]) *For $A, B \in M_{n,sa}(\mathbb{C})$, $\rho \in M_{n,+,1}(\mathbb{C})$ and $f \in \mathcal{F}_{op}^r$, we have the following relations, where we put $A_0 \equiv A - Tr[\rho A]I$ and $B_0 \equiv B - Tr[\rho B]I$.*

(1) $I_\rho^f(A) = Tr[\rho A_0^2] - Tr[m_{\tilde{f}}(L_\rho, R_\rho)(A_0)A_0] = V_\rho(A) - C_\rho^{\tilde{f}}(A_0)$.

(2) $J_\rho^f(A) = Tr[\rho A_0^2] + Tr[m_{\tilde{f}}(L_\rho, R_\rho)(A_0)A_0] = V_\rho(A) + C_\rho^{\tilde{f}}(A_0)$.

(3) $0 \leq I_\rho^f(A) \leq U_\rho^f(A) \leq V_\rho(A)$.

(4) $U_\rho^f(A) = \sqrt{I_\rho^f(A) J_\rho^f(A)}$.

(5) $Corr_\rho^f(A, B) = \frac{1}{2}Tr[\rho A_0 B_0] + \frac{1}{2}Tr[\rho B_0 A_0] - Tr[m_{\tilde{f}}(L_\rho, R_\rho)(A_0)B_0] = \frac{1}{2}Tr[\rho A_0 B_0] + \frac{1}{2}Tr[\rho B_0 A_0] - C_\rho^{\tilde{f}}(A_0, B_0)$.

The following inequality is the further generalization of Corollary 3.3 by the use of the metric adjusted correlation measure.

Theorem 4.7 *For $f \in \mathcal{F}_{op}^r$, if we have*

$$\frac{x+1}{2} + \tilde{f}(x) \geq 2f(x), \tag{17}$$

then we have

$$U_\rho^f(A) U_\rho^f(B) \geq 4f(0)|Corr_\rho^f(A, B)|^2, \tag{18}$$

for $A, B \in M_{n,sa}(\mathbb{C})$ and $\rho \in M_{n,+,1}(\mathbb{C})$.

In order to prove Theorem 4.7, we use the following two lemmas.

Lemma 4.8 ([35]) *If Eq.(17) is satisfied, then we have the following inequality:*

$$\left(\frac{x+y}{2}\right)^2 - m_{\tilde{f}}(x, y)^2 \geq f(0)(x-y)^2.$$

Proof: By Eq.(17), we have

$$\frac{x+y}{2} + m_{\tilde{f}}(x, y) \geq 2m_f(x, y).$$

We also have

$$\begin{aligned} m_{\tilde{f}}(x, y) &= y\tilde{f}\left(\frac{x}{y}\right) \\ &= \frac{y}{2}\left\{\frac{x}{y} + 1 - \left(\frac{x}{y} - 1\right)^2 \frac{f(0)}{f(x/y)}\right\} \\ &= \frac{x+y}{2} - \frac{f(0)(x-y)^2}{2m_f(x, y)}. \end{aligned}$$

Therefore

$$\begin{aligned} \left(\frac{x+y}{2}\right)^2 - m_{\tilde{f}}(x, y)^2 &= \left\{\frac{x+y}{2} - m_{\tilde{f}}(x, y)\right\}\left\{\frac{x+y}{2} + m_{\tilde{f}}(x, y)\right\} \\ &\geq \frac{f(0)(x-y)^2}{2m_f(x, y)} 2m_f(x, y) \\ &= f(0)(x-y)^2. \end{aligned}$$

■

We have the following expressions for the quantities $I_\rho^f(A)$, $J_\rho^f(A)$, $U_\rho^f(A)$ and $Corr_\rho^f(A,B)$ by using Proposition 4.6 and a mean $m_{\tilde{f}}$.

Lemma 4.9 ([10]) *Let $\{|\phi_1\rangle, |\phi_2\rangle, \cdots, |\phi_n\rangle\}$ be a basis of eigenvectors of ρ, corresponding to the eigenvalues $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$. We put $a_{jk} = \langle\phi_j|A_0|\phi_k\rangle, b_{jk} = \langle\phi_j|B_0|\phi_k\rangle$, where $A_0 \equiv A - Tr[\rho A]I$ and $B_0 \equiv B - Tr[\rho B]I$ for $A, B \in M_{n,sa}(\mathbb{C})$ and $\rho \in M_{n,+,1}(\mathbb{C})$. Then we have*

$$\begin{aligned} I_\rho^f(A) &= \frac{1}{2}\sum_{j,k}(\lambda_j+\lambda_k)a_{jk}a_{kj} - \sum_{j,k} m_{\tilde{f}}(\lambda_j,\lambda_k)a_{jk}a_{kj} \\ &= 2\sum_{j<k}\left\{\frac{\lambda_j+\lambda_k}{2} - m_{\tilde{f}}(\lambda_j,\lambda_k)\right\}|a_{jk}|^2, \end{aligned}$$

$$\begin{aligned} J_\rho^f(A) &= \frac{1}{2}\sum_{j,k}(\lambda_j+\lambda_k)a_{jk}a_{kj} + \sum_{j,k} m_{\tilde{f}}(\lambda_j,\lambda_k)a_{jk}a_{kj} \\ &\geq 2\sum_{j<k}\left\{\frac{\lambda_j+\lambda_k}{2} + m_{\tilde{f}}(\lambda_j,\lambda_k)\right\}|a_{jk}|^2, \end{aligned}$$

$$U_\rho^f(A)^2 = \frac{1}{4}\left(\sum_{j,k}(\lambda_j+\lambda_k)|a_{jk}|^2\right)^2 - \left(\sum_{j,k} m_{\tilde{f}}(\lambda_j,\lambda_k)|a_{jk}|^2\right)^2$$

and

$$\begin{aligned} Corr_\rho^f(A,B) &= \frac{1}{2}\sum_{j,k}\lambda_j a_{jk}b_{kj} + \frac{1}{2}\sum_{j,k}\lambda_k a_{jk}b_{kj} - \sum_{j,k} m_{\tilde{f}}(\lambda_j,\lambda_k)a_{jk}b_{kj} \\ &= \sum_{j<k}\left(\frac{\lambda_j+\lambda_k}{2} - m_{\tilde{f}}(\lambda_j,\lambda_k)\right)a_{jk}b_{kj} + \sum_{j<k}\left(\frac{\lambda_k+\lambda_j}{2} - m_{\tilde{f}}(\lambda_k,\lambda_j)\right)a_{kj}b_{jk}. \end{aligned} \tag{19}$$

We are now in a position to prove Theorem 4.7.

Proof of Theorem 4.7: From Eq.(19), we have

$$\begin{aligned} |Corr_\rho^f(A,B)| &\leq \sum_{j<k}\left|\left(\frac{\lambda_j+\lambda_k}{2} - m_{\tilde{f}}(\lambda_j,\lambda_k)\right)a_{jk}b_{kj}\right| + \sum_{j<k}\left|\left(\frac{\lambda_j+\lambda_k}{2} - m_{\tilde{f}}(\lambda_k,\lambda_j)\right)a_{kj}b_{jk}\right| \\ &\leq \sum_{j<k}\left|\frac{\lambda_j+\lambda_k}{2} - m_{\tilde{f}}(\lambda_j,\lambda_k)\right||a_{jk}||b_{kj}| + \sum_{j<k}\left|\frac{\lambda_j+\lambda_k}{2} - m_{\tilde{f}}(\lambda_k,\lambda_j)\right||a_{kj}||b_{jk}| \\ &= 2\sum_{j<k}\left|\frac{\lambda_j+\lambda_k}{2} - m_{\tilde{f}}(\lambda_j,\lambda_k)\right||a_{jk}||b_{kj}| \\ &\leq \sum_{j<k}|\lambda_j-\lambda_k||a_{jk}||b_{kj}|. \end{aligned}$$

Then we have

$$f(0)|Corr_\rho^f(A,B)|^2 \leq \left(\sum_{j<k} f(0)^{1/2}|\lambda_j-\lambda_k||a_{jk}||b_{kj}|\right)^2$$

$$
\begin{aligned}
&\le \left(\sum_{j<k}\left\{\left(\frac{\lambda_j+\lambda_k}{2}\right)^2 - m_{\tilde{f}}(\lambda_j,\lambda_k)^2\right\}^{1/2}|a_{jk}||b_{kj}|\right)^2 \\
&\le \left(\sum_{j<k}\left\{\frac{\lambda_j+\lambda_k}{2} - m_{\tilde{f}}(\lambda_j,\lambda_k)\right\}|a_{jk}|^2\right) \\
&\quad \times \left(\sum_{j<k}\left\{\frac{\lambda_j+\lambda_k}{2} + m_{\tilde{f}}(\lambda_j,\lambda_k)\right\}|b_{kj}|^2\right) \\
&\le \frac{1}{4} I_\rho^f(A) J_\rho^f(B).
\end{aligned}
$$

By the similar way, we also have

$$
I_\rho^f(B) J_\rho^f(A) \ge 4f(0)|Corr_\rho^f(A,B)|^2.
$$

Hence we have the desired inequality (18).

■

Remark 4.10 *Under the same assumptions with Theorem 4.7, we have the following Heisenberg-type uncertainty relation [35]:*

$$
U_\rho^f(A) U_\rho^f(B) \ge f(0) |Tr\,[\rho[A,B]]\,|^2 \tag{20}
$$

by the similar way to the proof of Theorem 4.7, since we have

$$
|Tr\,[\rho[A,B]]\,| \le 2\sum_{j<k} |\lambda_j - \lambda_k||a_{jk}||b_{kj}|.
$$

As stated in Remark 2.6, there is no ordering between the right hand side of the inequality (18) and that of the inequality (20), in general.

If we use the function

$$
f_{WYD}(x) = \alpha(1-\alpha)\frac{(x-1)^2}{(x^\alpha - 1)(x^{1-\alpha}-1)}, \quad \alpha \in (0,1),
$$

then we obtain the following uncertainty relation.

Corollary 4.11 *For $A, B \in M_{n,sa}(\mathbb{C})$ and $\rho \in M_{n,+,1}(\mathbb{C})$, we have*

$$
U_\rho^{f_{WYD}}(A) U_\rho^{f_{WYD}}(B) \ge 4\alpha(1-\alpha)|Corr_\rho^{f_{WYD}}(A,B)|^2.
$$

Proof: From the definition

$$
f_{WYD}(x) = \alpha(1-\alpha)\frac{(x-1)^2}{(x^\alpha - 1)(x^{1-\alpha}-1)},
$$

it is clear that

$$
\tilde{f}_{WYD}(x) = \frac{1}{2}\{x+1-(x^\alpha-1)(x^{1-\alpha}-1)\}.
$$

By Lemma 2.3, we have for $0 \le \alpha \le 1$ and $x > 0$,

$$
(1-2\alpha)^2(x-1)^2 - (x^\alpha - x^{1-\alpha})^2 \ge 0.
$$

This inequality can be rewritten by

$$(x^{2\alpha}-1)(x^{2(1-\alpha)}-1) \geq 4\alpha(1-\alpha)(x-1)^2.$$

Thus we have

$$\begin{aligned}\frac{x+1}{2}+\tilde{f}_{WYD}(x) &= x+1-\frac{1}{2}(x^{\alpha}-1)(x^{1-\alpha}-1)\\ &= \frac{1}{2}(x^{\alpha}+1)(x^{1-\alpha}+1)\\ &\geq 2\alpha(1-\alpha)\frac{(x-1)^2}{(x^{\alpha}-1)(x^{1-\alpha}-1)}\\ &= 2f_{WYD}(x).\end{aligned}$$

Thus we obtain the aimed result from Theorem 4.7.

∎

Note that Corollary 3.3 coincides with Corollary 4.11, since we have $U_{\rho,\alpha}(A) = U_{\rho}^{f_{WYD}}(A)$ which is obtained by the fact the function $f_{WYD}(x)$ corresponds to the Wigner-Yanase-Dyson skew information. We also note that we have $Corr_{\rho}^{f_{WYD}}(A,B) = Corr_{\rho,\alpha,\frac{1}{2}}^{(sym)}(A,B)$ and $Corr_{\rho}^{f_{WYD}}(A,B) \neq Corr_{\rho,\alpha,\frac{1}{2}}(A,B)$ in general.

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