VALUE DISTRIBUTION OF THE GAUSS MAP OF IMPROPER AFFINE SPHERES

YU KAWAKAMI AND DAISUKE NAKAJO

ABSTRACT. We give the best possible upper bound for the number of exceptional values of the Lagrangian Gauss map of complete improper affine fronts in the affine three-space. We also obtain the sharp estimate for weakly complete case. As an application of this result, we provide a new and simple proof of the parametric affine Bernstein problem for improper affine spheres. Moreover, we get the same estimate for the ratio of canonical forms of weakly complete flat fronts in hyperbolic three-space.

INTRODUCTION

The study of improper affine spheres has been related to various subjects in geometry and analysis. In fact, improper affine spheres in the affine three-space $\mathbf{R}^3$ are locally graphs of solutions of the Monge-Ampère equation $\det{(\nabla^2 f)} = 1$, and Calabi [3] proved that there exists a local correspondence between solutions of the equation of improper affine spheres in $\mathbf{R}^3$ and solutions of the equation of minimal surfaces in Euclidean three-space. Recently, Martínez [29] discovered the correspondence between improper affine spheres and smooth special Lagrangian immersions in the complex two-space $\mathbf{C}^2$. Moreover, from the viewpoint of this correspondence, he introduced the notion of improper affine maps, that is, a class of (locally strongly convex) improper affine spheres with some admissible singularities and gave a holomorphic representation formula for them. Later, the second author [31], Umehara and Yamada [45] showed that an improper affine map is a front in $\mathbf{R}^3$, and hence we call this class improper affine fronts in this paper. Martínez [29] also defined the Lagrangian Gauss map of improper affine fronts in $\mathbf{R}^3$ and obtained the characterization of a complete (in the sense of [27, 29], see also Section 1 of this paper) improper affine front whose Lagrangian Gauss map is constant. We note that the second author [31] constructed a representation formula for indefinite improper affine spheres with some admissible singularities.

On the other hand, the study of value distribution property of the Gauss map of complete minimal surfaces in Euclidean three-space has accomplished many significant results.

This study is a generalization of the classical Bernstein theorem [1] and initiated by Osserman [35, 36, 37]. Fujimoto [8, 9, 10] showed that the best possible upper bound for the number D_g of exceptional values of the Gauss map g of complete nonflat minimal surfaces in Euclidean three-space is "four". Ros [39] gave a different proof of this result. Moreover, Osserman [36, 37] proved that the Gauss map of a nonflat algebraic minimal surface can omit at most three values (by an algebraic minimal surface, we mean a complete minimal surface with finite total curvature). Recently, the first author, Kobayashi and Miyaoka [21] gave an effective ramification estimate for the Gauss map of a wider class of complete minimal surfaces that includes algebraic minimal surfaces (this class is called the pseudo-algebraic minimal surfaces). It also provided new proofs of the Fujimoto and the Osserman theorems in this class and revealed the geometric meaning behind them. The first author obtained the same estimate for the hyperbolic Gauss map of pseudo-algebraic constant mean curvature one surfaces in hyperbolic three-space $\mathbf{H}^3$ [19]. These estimates correspond to the defect relation in Nevanlinna theory ([17], [22], [33] and [40]).

The purpose of this paper is to study value distribution property of the Lagrangian Gauss map of improper affine fronts in $\mathbf{R}^3$. The organization of this paper is as follows: In Section 1, we recall some definitions and basic facts about improper affine fronts in $\mathbf{R}^3$ which are used throughout this paper. We review, in particular, the definitions of completeness in the sense of [27, 29] and weakly completeness in the sense of [45]. In Section 2, we give the upper bound for the totally ramified value number δ_ν of the Lagrangian Gauss map ν of complete improper affine fronts in $\mathbf{R}^3$ (Theorem 2.2). This estimate is effective in the sense that the upper bound which we obtained is described in terms of geometric invariants and sharp for some topological cases. Moreover, as a corollary of this estimate, we also obtain the best possible upper bound for the number D_ν of exceptional values of the Lagrangian Gauss map in this class (Corollary 2.4). We note that this class corresponds to that of algebraic minimal surfaces in Euclidean three-space. In Section 3, by applying the Fujimoto argument, we give the optimal estimate for D_ν of weakly complete improper affine fronts in $\mathbf{R}^3$ (Theorem 3.2). We note that the best possible upper bound for D_g of complete minimal surfaces obtained by Fujimoto is "four", but the best possible upper bound for D_ν of this class is "three". As an application of this estimate, from the viewpoint of the value distribution property, we provide a new and simple proof of the well-known result ([2], [15]) that any affine complete improper affine sphere must be an elliptic paraboloid (Corollary 3.6). This result is the special case of the parametric affine Bernstein problem of affine maximal surfaces, which states that any affine complete affine maximal surface must be an elliptic paraboloid ([4], [28], and [44]). In Section 4, after reviewing some definitions and fundamental properties on flat fronts in $\mathbf{H}^3$, we study the value distribution of the ratio of canonical forms of weakly complete flat fronts in $\mathbf{H}^3$. Flat surfaces (resp. fronts) in $\mathbf{H}^3$ are closely related to improper affine spheres (resp. fronts) in $\mathbf{R}^3$ (See [30] and also [14]). Indeed, we show that the ratio of

canonical forms of weakly complete flat fronts in $\mathbf{H}^3$ have some properties similar to the Lagrangian Gauss map of weakly complete improper affine fronts in $\mathbf{R}^3$ (Propositions 3.1 and 4.4, Theorems 3.2 and 4.5, Corollaries 3.6 and 4.6). By Corollaries 3.6 and 4.6, we can prove that the uniqueness theorems of complete surfaces (these results are called the parametric Bernstein type theorems) for these classes follow from the Liouville property, that is, the boundedness of their Gauss maps.

Finally, the authors would like to particularly thank to Professors Wayne Rossman, Masaaki Umehara and Kotaro Yamada for their useful advice. The authors also thank to Professors Ryoichi Kobayashi, Masatoshi Kokubu, Reiko Miyaoka, Junjiro Noguchi and Yoshihiro Ohnita for their encouragement of our study.

1. PRELIMINARIES

We first briefly recall some definitions and basic facts about affine differential geometry. Details can be found, for instance, in [28] and [34]. Let Σ be an oriented two-manifold, and (ψ, ξ) a pair of an immersion $\psi\colon \Sigma \to \mathbf{R}^3$ into the affine three-space $\mathbf{R}^3$ and a vector field ξ on Σ along ψ which is transversal to $\psi_*(T\Sigma)$. Then the Gauss-Weingarten equations of (ψ, ξ) are as follows:

$$(1.1) \qquad \begin{cases} D_X \psi_* Y = \psi_*(\nabla_X Y) + g(X, Y)\xi\,, \\ D_X \xi = -\psi_*(SX) + \tau(X)\xi\,, \end{cases}$$

where D is the standard flat connection on $\mathbf{R}^3$. Here, g is called the *affine metric* (or *Blaschke metric*) of the pair (f, ξ). Indeed, we can easily show that the rank of g is invariant under the change of the transversal vector field ξ. When g is positive definite, we call ψ a *locally strongly convex immersion*. From now on, we only consider the locally strongly convex case. Given an immersion $\psi\colon \Sigma \to \mathbf{R}^3$, we can uniquely choose the transversal vector field ξ which satisfies the following conditions:

(i) $\tau \equiv 0$ (or equivalently $D_X \xi \in \psi_*(T\Sigma)$ for all $X \in \mathfrak{X}(\Sigma)$) ,

(ii) $\mathrm{vol}_g(X_1, X_2) = \det(\psi_* X_1, \psi_* X_2, \xi)$ for all $X_1, X_2 \in \mathfrak{X}(\Sigma)$,

where vol_g is the volume form of the Riemannian metric g and $\det$ is the standard volume element of $\mathbf{R}^3$. The transversal vector field ξ which satisfies the two conditions above is called an *affine normal* (or *Blaschke normal*), and a pair (ψ, ξ) of an immersion and its affine normal is called a *Blaschke immersion*. A Blaschke immersion (f, ξ) with $S = 0$ in (1.1) is called an *improper affine sphere*. In this case, the transversal vector field ξ is constant because $\tau \equiv 0$. Thus a transversal vector field ξ of an improper affine sphere is given by $\xi = (0, 0, 1)$ after a suitable affine transformation of $\mathbf{R}^3$. The *conormal map* $N\colon \Sigma \to (\mathbf{R}^3)^*$ into the dual space of the affine three-space $(\mathbf{R}^3)^*$ for a given Blaschke immersion (f, ξ) is defined as the immersion which satisfy the following conditions:

(i) $N(f_* X) = 0$ for all $X \in \mathfrak{X}(\Sigma)$,

(ii) $N(\xi) = 1$.

For an improper affine sphere with affine normal $(0, 0, 1)$, we can write $N = (n, 1)$ with a smooth map $n \colon \Sigma \to \mathbf{R}^2$.

Let $\mathbf{C}^2$ denotes the complex two-space with the complex coordinates $\zeta = (\zeta_1, \zeta_2)$, where $\zeta = x + \sqrt{-1}y$ $(x, y \in \mathbf{R}^2)$. We consider the standard metric g', the symplectic form ω', and the complex two-form Ω' given by

$$\begin{aligned}
g' &= |d\zeta_1|^2 + |d\zeta_2|^2, \\
\omega' &= \frac{\sqrt{-1}}{2}(d\zeta_1 \wedge d\bar{\zeta}_1 + d\zeta_2 \wedge d\bar{\zeta}_2), \\
\Omega' &= d\zeta_1 \wedge d\zeta_2.
\end{aligned}$$

Let $L \colon \Sigma \to \mathbf{C}^2$ be an special Lagrangian immersion with respect to the calibration $\Re(\sqrt{-1}\Omega')$. As in [12], L can be characterized as an immersion in $\mathbf{C}^2$ satisfying

$$\omega'|_{L(\Sigma)} \equiv 0, \quad \Im(\sqrt{-1}\Omega'|_{L(\Sigma)}) \equiv 0,$$

where $\Re$ and $\Im$ represent real and imaginary part, respectively.

Then there exists the following correspondence between improper affine spheres in $\mathbf{R}^3$ and some nondegenerate special Lagrangian immersions in $\mathbf{C}^2$.

Fact 1.1. [29, Theorem 1] *Let $\psi = (x, \varphi) \colon \Sigma \to \mathbf{R}^3 = \mathbf{R}^2 \times \mathbf{R}$ be an improper affine sphere with the conormal map $N = (n, 1)$. The map $L_\psi \colon \Sigma \to \mathbf{C}^2$ given by*

$$L_\psi := x + \sqrt{-1}n$$

is an special Lagrangian immersion such that

(i) *The induced metric $d\tau^2 := \langle dx, dx \rangle + \langle dn, dn \rangle$ is conformal to the affine metric g of ψ,*

(ii) *The metric $ds^2 := \langle dx, dx \rangle$ is a nondegenerate flat metric,*

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbf{R}^2$.

Fact 1.2. [29, Theorem 2] *Let $L_\psi = x + \sqrt{-1}n \colon \Sigma \to \mathbf{C}^2$ be a special Lagrangian immersion such that $ds^2 := \langle dx, dx \rangle$ is nondegenarate. Then*

$$\psi = \left(x, -\int \langle n, dx \rangle \right)$$

is an improper affine sphere which is well-defined if and only if $\int_c \langle n, dx \rangle = 0$ for any loop c on Σ.

Next, using the notations defined as above, we introduce the notion of improper affine fronts, which is a generalization of improper affine spheres with some admissible singularities.

Definition 1.3 ([29, Definition 1]). A map $\psi = (x, \varphi) \colon \Sigma \to \mathbf{R}^3 = \mathbf{R}^2 \times \mathbf{R}$ is called an *improper affine front* if ψ is expressed as

$$\psi = \left(x, - \int \langle n, dx \rangle \right)$$

by a special Lagrangian immersion $L_\psi = x + \sqrt{-1}n \colon \Sigma \to \mathbf{C}^2$, where $\langle \cdot , \cdot \rangle$ denotes the standard inner product in $\mathbf{R}^2$. Nonregular points of ψ correspond with degenerate points of $ds^2 := \langle dx, dx \rangle$. We call ds^2 the *flat fundamental form* of ψ.

From Facts 1.1 and 1.2, at the nondegenerate points of ds^2, the induced metric $d\tau^2 := \langle dx, dx \rangle + \langle dn, dn \rangle$ is conformal to the affine metric $g := -\langle dx, dn \rangle$.

For any improper affine front $\psi \colon \Sigma \to \mathbf{R}^3$, considering the conformal structure given by the induced metric $d\tau^2$ of its associated special Lagrangian immersion L_ψ, we regard Σ as a Riemann surface.

Since every special Lagrangian immersion in $\mathbf{C}^2$ is a holomorphic curve with respect to the complex coordinates $\zeta = (\zeta_1, \zeta_2)$ (see [16]), we see that there exists a holomorphic regular curve $\alpha \colon \Sigma \to \mathbf{C}^2$, $\alpha := (F, G)$, such that if we identify vectors of $\mathbf{R}^2$ with complex numbers in the standard way:

$$(r, s) = r + \sqrt{-1}s, \quad r, s \in \mathbf{R}$$

then we can write

$$(1.2) \qquad\qquad x = G + \bar{F}, \quad n = \bar{F} - G$$

and since the inner product of two vectors $\zeta_i = r_i + \sqrt{-1}s_i$ ($i = 1, 2$) is given by $\langle \zeta_1, \zeta_2 \rangle = \Re(\zeta_1 \bar{\zeta_2})$, then the flat fundamental form ds^2, the induced metric $d\tau^2$ and the affine metric g are given, respectively, by

$$
\begin{aligned}
(1.3) \qquad ds^2 &= |dF + dG|^2 = |dF|^2 + |dG|^2 + dG dF + \overline{dF dG}, \\
d\tau^2 &= 2(|dF|^2 + |dG|^2), \\
g &= |dG|^2 - |dF|^2.
\end{aligned}
$$

The nontrivial part of the Gauss map of $L_\psi \colon \Sigma \to \mathbf{C}^2 \cong \mathbf{R}^4$ (see [5]) is the meromorphic function $\nu \colon \Sigma \to \mathbf{C} \cup \{\infty\}$ given by

$$(1.4) \qquad\qquad \nu := \frac{dF}{dG}$$

which is called the *Lagrangian Gauss map* of ψ.

Martínez [29] gave the following representation formula for improper affine fronts in terms of two holomorphic functions. This generalized a formula in [7].

Fact 1.4 ([29, Theorem 3]). *Let $\psi = (x, \varphi)\colon \Sigma \to \mathbf{R}^3 = \mathbf{C} \times \mathbf{R}$ be an improper affine front. Then there exists a holomorphic regular curve $\alpha := (F, G)\colon \Sigma \to \mathbf{C}^2$ such that*

$$(1.5) \qquad \psi := \left(G + \bar{F}, \frac{|G|^2 - |F|^2}{2} + \Re\left(GF - \int F dG \right) \right).$$

Here, the conormal map of ψ becomes

$$N = (\bar{F} - G, 1).$$

Conversely, given a Riemann surface Σ and a holomorphic regular curve $\alpha := (F, G)\colon \Sigma \to \mathbf{C}^2$, then (1.5) gives an improper affine front which is well-defined if and only if $\Re \int_c F dG = 0$ for any loop c in Σ.

We call the pair (F, G) the *Weierstrass data* of ψ. Note that the singular points of ψ correspond with the points where $|dF| = |dG|$, that is, $|\nu| = 1$ ([29], see also [31]).

An improper affine front $\psi\colon \Sigma \to \mathbf{R}^3$ is said to be *complete* if there exists a symmetric two-tensor T such that $T = 0$ outside a compact set $C \subset \Sigma$ and $ds^2 + T$ is a complete Riemannian metric on Σ, where ds^2 is the flat fundamental form of ψ. This definition is similar to the definition of completeness for fronts [27].

Fact 1.5. *A complete improper affine front $\psi\colon \Sigma \to \mathbf{R}^3$ satisfies the following two conditions:*

(i) *Σ is biholomorphic to $\overline{\Sigma}_\gamma \backslash \{p_1, \ldots, p_k\}$, where $\overline{\Sigma}_\gamma$ is a closed Riemann surface of genus γ and $p_j \in \overline{\Sigma}_\gamma$ $(j = 1, \ldots, k)$ [13].*

(ii) *The Weierstrass data (F, G) of ψ can be extended meromorphically to $\overline{\Sigma}_\gamma$. In particular, its Lagrangian Gauss map can also be a meromorphic function on $\overline{\Sigma}_\gamma$ [29].*

Each puncture point p_j $(j = 1, \ldots, k)$ is called an *end* of ψ. On the other hand, an improper affine front is said to be *weakly complete* if the induced metric $d\tau^2$ as in (1.3) is complete. Note that the universal cover of a weakly complete improper affine front is also weakly complete, but completeness is not preserved when lifting to the universal cover. The relationship between completeness and weakly completeness in this class is as follows:

Fact 1.6 ([45, Remark 4]). *An improper affine front in $\mathbf{R}^3$ is complete if and only if it is weakly complete, the singular set is compact and all ends are biholomorphic to a punctured disk.*

Finally, we give two examples in [29, Section 4] which play important roles in the following sections.

Example 1.7 (Elliptic paraboloids). An elliptic paraboloid can be obtained by taking $\Sigma = \mathbf{C}$ and Weierstrass data (cz, z), where c is constant. It is complete, and its Lagrangian

Gauss map is constant. Note that, if $|c| = 1$, then an improper affine front constructed from this data is a line in $\mathbf{R}^2$.

Example 1.8 (Rotational improper affine fronts)**.** A rotational improper affine front is obtained by considering $\Sigma = \mathbf{C}\backslash\{0\}$ and Weierstrass data $(z, \pm r^2/z)$, where $r \in \mathbf{R}\backslash\{0\}$. It is complete and its Lagrangian Gauss map $\nu = \mp z^2/r^2$. In particular, ν omits two values, $0, \infty$.

2. A RAMIFICATION ESTIMATE FOR THE LAGRANGIAN GAUSS MAP OF COMPLETE IMPROPER AFFINE FRONTS

We first recall the definion of the totally ramified value number of a meromorphic function.

Definition 2.1 ([32])**.** Let Σ be a Riemann surface and h a meromorphic function on Σ. We call $b \in \mathbf{C} \cup \{\infty\}$ a *totally ramified value* of h when h branches at any inverse image of b. We regard exceptional values also as totally ramified values, here we call a point of $\mathbf{C} \cup \{\infty\}\backslash h(\Sigma)$ an exceptional value of h. Let $\{a_1, \ldots, a_{r_0}, b_1, \ldots, b_{l_0}\} \subset \mathbf{C} \cup \{\infty\}$ be the set of totally ramified values of h, where a_j's are exceptional values. For each a_j, set $m_j = \infty$, and for each b_j, define m_j to be the minimum of the multiplicities of h at points $h^{-1}(b_j)$. Then we have $m_j \geq 2$. We call

$$\delta_h = \sum_{a_j, b_j}\left(1 - \frac{1}{m_j}\right) = r_0 + \sum_{j=1}^{l_0}\left(1 - \frac{1}{m_j}\right)$$

the *totally ramified value number* of h.

Because the Lagrangian Gauss map ν of an improper affine front $\psi\colon \Sigma \to \mathbf{R}^3$ is a meromorphic function on Σ, we can consider the totally ramified value number δ_ν of ν. By virtue of Fact 1.5, we regard Σ as a punctured Riemann surface $\overline{\Sigma}_\gamma\backslash\{p_1, \ldots, p_k\}$, where $\overline{\Sigma}_\gamma$ is a closed Riemann surface of genus γ and $p_j \in \overline{\Sigma}_\gamma$ $(j = 1, \ldots, k)$. Then we give the upper bound for δ_ν of complete improper affine fronts in $\mathbf{R}^3$. Here, we denote by D_ν the number of exceptional values of ν. By definition, it follows immediately that $D_\nu \leq \delta_\nu$.

Theorem 2.2. *Let* $\psi\colon \Sigma = \overline{\Sigma}_\gamma\backslash\{p_1, \ldots, p_k\} \to \mathbf{R}^3$ *be a complete improper affine front and* $\nu\colon \Sigma \to \mathbf{C} \cup \{\infty\}$ *the Lagrangian Gauss map of* ψ*. Suppose that* ν *is nonconstant and* d *is the degree of* ν *considered as a map on* $\overline{\Sigma}_\gamma$*. Then we have*

$$(2.1) \qquad D_\nu \leq \delta_\nu \leq 2 + \frac{2}{R}, \qquad \frac{1}{R} := \frac{\gamma - 1 + k/2}{d} < \frac{1}{2}.$$

In particular, $D_\nu \leq \delta_\nu < 3$*.*

Remark 2.3. The geometric meaning of "2" in the upper bound of (2.1) is the Euler number of the Riemann sphere. The geometric meaning of the ratio R is given in [21, Section 6].

Proof. By Fact 1.6, if ds^2 is complete, then $d\tau^2$ is a complete Riemannian metric. Then the metric $d\tau^2$ is represented as

$$(2.2) \qquad d\tau^2 = 2(|dF|^2 + |dG|^2) = 2\left(1 + \left|\frac{dF}{dG}\right|^2\right)|dG|^2 = 2(1 + |\nu|^2)|dG|^2\,.$$

Since $d\tau^2$ is nondegenerate on Σ, the poles of ν of order k coincide exactly with the zeros of dG of order k. By the completeness of $d\tau^2$, dG has a pole of order $\mu_j \geq 1$ at p_j [37]. Moreover we show that $\mu_j \geq 2$ for each p_j because G is single-valued on $\overline{\Sigma}_\gamma$. Applying the Riemann-Roch theorem to dG on $\overline{\Sigma}_\gamma$, we have

$$(2.3) \qquad d - \sum_{j=1}^{k} \mu_j = 2\gamma - 2\,.$$

Thus we get

$$(2.4) \qquad d = 2\gamma - 2 + \sum_{j=1}^{k} \mu_j \geq 2(\gamma - 1 + k) > 2\left(\gamma - 1 + \frac{k}{2}\right),$$

and

$$(2.5) \qquad \frac{1}{R} < \frac{1}{2}\,.$$

Assume that ν omits $r_0 = D_\nu$ values. Let n_0 be the sum of the branching orders at the image of these exceptional values of ν. Then we have

$$(2.6) \qquad k \geq dr_0 - n_0\,.$$

Let $b_1, \ldots, b_{l_0}$ be the totally ramified values which are not exceptional values and n_r the sum of branching order at the inverse image of b_i $(i = 1, \ldots, l_0)$ of ν. For each b_i, we denote

$$m_i = \min_{\nu^{-1}(b_i)}\{\text{multiplicity of } \nu(z) = b_i\}\,,$$

then the number of points in the inverse image $\nu^{-1}(b_i)$ is less than or equal to d/m_i. Thus we get

$$(2.7) \qquad dl_0 - n_r \leq \sum_{i=1}^{l_0} \frac{d}{m_i}\,.$$

This implies

$$(2.8) \qquad l_0 - \sum_{i=1}^{l_0} \frac{1}{m_i} \leq \frac{n_r}{d}\,.$$

Let n_ν be the total branching order of ν on $\overline{\Sigma}_\gamma$. Then applying the Riemann-Hurwitz formula to the meromorphic function ν on $\overline{\Sigma}_\gamma$, we have

$$(2.9) \qquad n_\nu = 2(d + \gamma - 1)\,.$$

Hence we obtain

$$\delta_\nu = r_0 + \sum_{j=1}^{l_0} \left(1 - \frac{1}{m_j}\right) \leq \frac{n_0 + k}{d} + \frac{n_r}{d} \leq \frac{n_\nu + k}{d} = 2 + \frac{2}{R}\,.$$

$\square$

The system of inequalities (2.1) is sharp in the following cases:
(i) When $(\gamma, k, d) = (0, 1, n)$ $(n \in \mathbf{N})$, we have

$$\delta_\nu \leq 2 - \frac{1}{n}\,.$$

In this case, we can set $\Sigma = \mathbf{C}$. Since Σ is simply connected, we have no period condition. We define a Weierstrass data on Σ, by

(2.10) $$(F, G) = \left(\frac{z^{n+1}}{n+1}, z\right).$$

By Fact 1.4, we can construct a complete improper affine front $\psi\colon \Sigma \to \mathbf{R}^3$ whose Weierstrass data is (2.10). In particular, its Lagrangian Gauss map ν has $\delta_\nu = 2 - (1/n)$. In fact, $\nu = z^n$, and it has one exceptional value and another totally ramified value of multiplicity n at $z = 0$. Thus (2.1) is sharp in this case.
(ii) When $(\gamma, k, d) = (0, 2, 2)$, we have

$$D_\nu \leq \delta_\nu \leq 2\,.$$

In this case, we can set $\Sigma = \mathbf{C}\backslash\{0\}$. On the other hand, a rotational improper affine front (Example 1.8) has $D_\nu = \delta_\nu = 2$. Thus (2.1) is also sharp in this case.

As a corollary of Theorem 2.2, we obtain the maximal number of exceptional values of the Lagrangian Gauss map of complete improper affine fronts in $\mathbf{R}^3$.

Corollary 2.4. *Let ψ be a complete improper affine front in $\mathbf{R}^3$. If its Lagrangian Gauss map ν is nonconstant, then ν can omit at most two values.*

The number "two" is sharp, because the Lagrangian Gauss map of a rotational improper affine front (Example 1.8) omits two values. Hence we provide the best possible upper bound for D_ν in complete case.

3. The maximal number of exceptional values of the Lagrangian Gauss map of weakly complete improper affine fronts

In this section, we study the value distribution of the Lagrangian Gauss map of weakly complete improper affine fronts in $\mathbf{R}^3$. We begin to consider the case where the Lagrangian Gauss map is constant.

Proposition 3.1. *Let $\psi\colon \Sigma \to \mathbf{R}^3$ be a weakly complete improper affine front. If its Lagrangian Gauss map ν is constant, then ψ is an elliptic paraboloid.*

Proof. Since the metric $d\tau^2$ is represented as (2.2), if ν is constant, then the Gaussian curvature $K_{d\tau^2}$ of $d\tau^2$ vanishes identically on Σ. By the Huber theorem, Σ is a closed Riemann surface of genus γ with k points removed, that is, $\Sigma = \overline{\Sigma}_\gamma \backslash \{p_1, \ldots, p_k\}$. Moreover we obtain the formula (see [6, Corollary 1] or [42])

$$\frac{1}{2\pi} \int_\Sigma (-K_{d\tau^2}) dA = -\chi(\overline{\Sigma}_\gamma) - \sum_{j=1}^k \mathrm{ord}_{p_j}(d\tau^2),$$

where dA denotes the area element of $d\tau^2$ and $\chi(\overline{\Sigma}_\gamma)$ the Euler number of $\overline{\Sigma}_\gamma$. Since the metric $d\tau^2$ is complete, $\mathrm{ord}_{p_j} d\tau^2 \leq -1$ holds for each end p_j. Thus if ν is constant, then we get $\gamma = 0$ and

$$(3.1) \qquad\qquad \sum_{j=1}^k \mathrm{ord}_{p_j}(d\tau^2) = -2 \,.$$

Since $d\tau^2$ is well-defined on Σ, we need to consider the following two cases:

(a) The improper affine front ψ has two ends p and q, and $\mathrm{ord}_p d\tau^2 = \mathrm{ord}_q d\tau^2 = -1$,
(b) The improper affine front ψ has one end p, and $\mathrm{ord}_p d\tau^2 = -2 \,.$

In this class, the case (a) cannot occur because F and G are single-valued on Σ. Thus we have only to consider the case (b). Then we may assume that $p = \infty$ after a suitable Möbius transformation of the Riemann sphere $\overline{\Sigma}_0$. Since ν is constant, dF and dG are well-defined on $\overline{\Sigma}_0$, and it holds that

$$\mathrm{ord}_\infty dF = \mathrm{ord}_\infty dG = -2 \,.$$

We thus have $dF = cdz$ and $dG = dz$, that is, $F(z) = cz$ and $G(z) = z$ for some constant c. Therefore the result follows from Example 1.7. $\square$

We next give the best possible upper bound for the number of exceptional values of the Lagrangian Gauss map of weakly complete improper affine fronts in $\mathbf{R}^3$.

Theorem 3.2. *Let ψ be a weakly complete improper affine front in $\mathbf{R}^3$. If its Lagrangian Gauss map ν is nonconstant, then ν can omit at most three values.*

The number "three" is sharp because there exist the following examples.

Example 3.3 (Improper affine fronts of Voss type)**.** We consider the Lagrangian Gauss map ν and the holomorphic one-form dG on $\Sigma = \mathbf{C} \backslash \{a_1, a_2\}$ for distinct points $a_1, a_2 \in \mathbf{C}$, by

$$(3.2) \qquad\qquad \nu = z, \quad dG = \frac{dz}{\prod_j (z - a_j)} \,.$$

Since F and G are not well-defined on Σ, we obtain an improper affine front $\psi \colon \mathbf{D} \to \mathbf{R}^3$ on the universal covering disk $\mathbf{D}$ of Σ. Since the metric $d\tau^2$ is complete, we can get a

weakly complete improper affine front whose Lagrangian Gauss map omits three values, a_1, a_2 and ∞.

Before proceeding to the proof of Theorem 3.2, we recall two function-theoretical lemmas. For two distinct values α, $\beta \in \mathbf{C} \cup \{\infty\}$, we set

$$|\alpha, \beta| := \frac{|\alpha - \beta|}{\sqrt{1 + |\alpha|^2}\sqrt{1 + |\beta|^2}}$$

if $\alpha \neq \infty$ and $\beta \neq 0$, and $|\alpha, \infty| = |\infty, \alpha| := 1/\sqrt{1 + |\alpha|^2}$. Note that, if we take v_1, $v_2 \in \mathbf{S}^2$ with $\alpha = \varpi(v_1)$ and $\beta = \varpi(v_2)$, we have that $|\alpha, \beta|$ is a half of the chordal distance between v_1 and v_2, where ϖ denotes the stereographic projection of $\mathbf{S}^2$ onto $\mathbf{C} \cup \{\infty\}$.

Lemma 3.4 ([10, (8.12) in page 136]). *Let ν be a nonconstant meromorphic function on $\Delta_R = \{z \in \mathbf{C} \,;\, |z| < R\}$ $(0 < R \leq +\infty)$ which omits q values $\alpha_1, \ldots, \alpha_q$. If $q > 2$, then for each positive η with $\eta < (q-2)/q$, there exists some positive constant $C > 0$ such that*

$$(3.3) \qquad \frac{|\nu'|}{(1 + |\nu|^2)\prod_{j=1}^{q}|\nu, \alpha_j|^{1-\eta}} \leq C\frac{R}{R^2 - |z|^2}\,.$$

Lemma 3.5 ([9, Lemma 1.6.7]). *Let $d\sigma^2$ be a conformal flat metric on an open Riemann surface Σ. Then, for each point $p \in \Sigma$, there exists a local diffeomorphism Ψ of a disk $\Delta_{R_0} = \{z \in \mathbf{C} \,;\, |z| < R_0\}$ $(0 < R_0 \leq +\infty)$ onto an open neighborhood of p with $\Psi(0) = p$ such that Ψ is a local isometry, namely, the pull-back $\Psi^*(d\sigma^2)$ is equal to the standard Euclidean metric ds_{Euc}^2 on Δ_{R_0} and, for a point a_0 with $|a_0| = 1$, the Ψ-image Γ_{a_0} of the curve $L_{a_0} = \{w := a_0 s \,;\, 0 < s < R_0\}$ is divergent in Σ.*

Proof of Theorem 3.2. This is proved by contradiction. Suppose that ν omits four distinct values $\alpha_1, \ldots, \alpha_4$. For our purpose, we may assume $\alpha_4 = \infty$ and that Σ is biholomorphic to the unit disk because Σ can be replaced by its universal covering surface and Theorem 3.2 is obvious in the case where $\Sigma = \mathbf{C}$ by the little Picard theorem. We choose some η with $0 < \eta < 1/4$ and set $\lambda := 1/(2 - 4\eta)$. Then $1/2 < \lambda < 1$ holds. Now we define a new metric

$$(3.4) \qquad d\sigma^2 = |G_z'|^{\frac{2}{1-\lambda}}\left(\frac{1}{|\nu_z'|}\prod_{j=1}^{3}\left(\frac{|\nu - \alpha_j|}{\sqrt{1 + |\alpha_j|^2}}\right)^{1-\eta}\right)^{\frac{2\lambda}{1-\lambda}}|dz|^2$$

on the set $\Sigma' := \{z \in \Sigma \,;\, \nu_z'(z) \neq 0\}$, where $dG = G_z' dz$ and $\nu_z' = d\nu/dz$. Take a point $p \in \Sigma'$. Since the metric $d\sigma^2$ is flat on Σ', by Lemma 3.5, there exists a local isometry Ψ satisfying $\Psi(0) = p$ from a disk $\Delta_R = \{z \in \mathbf{C} \,;\, |z| < R\}$ $(0 < R \leq +\infty)$ with the standard Euclidean metric onto an open neighborhood of p in Σ' with the metric $d\sigma^2$, such that, for a point a_0 with $|a_0| = 1$, the Ψ-image Γ_{a_0} of the curve $L_{a_0} = \{w := a_0 s \,;\, 0 \leq s < R\}$

is divergent in Σ'. For brevity, we denote the function $\nu \circ \Psi$ on Δ_R by ν in the followings. By Lemma 3.4, we get

$$(3.5) \qquad R \leq C \frac{1 + |\nu(0)|^2}{|\nu_z'(0)|} \prod_{j=1}^{4} |\nu(0), \alpha_j|^{1-\eta} < +\infty.$$

Hence

$$(3.6) \qquad L_{d\sigma}(\Gamma_{a_0}) = \int_{\Gamma_{a_0}} d\sigma = \int_{L_{a_0}} ds_{Euc}^2 = R < +\infty,$$

where $L_{d\sigma}(\Gamma_{a_0})$ denotes the length of Γ_{a_0} with respect to the metric $d\sigma^2$.

We assume that Ψ-image Γ_{a_0} tends to a point $p_0 \in \Sigma \backslash \Sigma'$ as $s \to R$. Taking a local complex coordinate ζ in a neighborhood of p_0 with $\zeta(p_0) = 0$, we can write $d\sigma^2 = |\zeta|^{-2\lambda/(1-\lambda)} w |d\zeta|^2$ with some positive smooth function w. Since $\lambda/(1-\lambda) > 1$, we have

$$R = \int_{\Gamma_{a_0}} d\sigma \geq C' \int_{\Gamma_{a_0}} \frac{|d\zeta|}{|\zeta|^{\lambda/(1-\lambda)}} = +\infty$$

which contradicts (3.5). Thus Γ_{a_0} diverges outside any compact subset of Σ as $s \to R$.

On the other hand, since $d\sigma^2 = |dz|^2$, we obtain by (3.4)

$$(3.7) \qquad |G_z'| = \left(|\nu_z'| \prod_{j=1}^{3} \left(\frac{\sqrt{1 + |\alpha_j|^2}}{|\nu - \alpha_j|} \right)^{1-\eta} \right)^{\lambda}.$$

By Lemma 3.4, we have

$$\begin{aligned}
\Psi^* d\tau &= \sqrt{2} |G_z'| \sqrt{1 + |\nu|^2} |dz| \\
&= \sqrt{2} \left(|\nu_z'| (1 + |\nu|^2)^{1/2\lambda} \prod_{j=1}^{3} \left(\frac{\sqrt{1 + |\alpha_j|^2}}{|\nu - \alpha_j|} \right)^{1-\eta} \right)^{\lambda} |dz| \\
&= \sqrt{2} \left(\frac{|\nu_z'|}{(1 + |\nu|^2) \prod_{j=1}^{4} |\nu, \alpha_j|^{1-\eta}} \right)^{\lambda} |dz| \\
&\leq \sqrt{2} C^{\lambda} \left(\frac{R}{R^2 - |z|^2} \right)^{\lambda} |dz|
\end{aligned}$$

Thus, if we denote the distance $d(p)$ from a point $p \in \Sigma$ to the boundary of Σ as the greatest lower bound of the lengths with respect to the metric $d\tau^2$ of all divergent paths in Σ, then we have

$$d(p) \leq \int_{\Gamma_{a_0}} d\tau = \int_{L_{a_0}} \Psi^* d\tau = \sqrt{2} C^{\lambda} \int_{L_{a_0}} \left(\frac{R}{R^2 - |z|^2} \right)^{\lambda} |dz| \leq \sqrt{2} C^{\lambda} \frac{R^{1-\lambda}}{1 - \lambda} < +\infty$$

because $1/2 < \lambda < 1$. However, it contradicts the assumption that $d\tau^2$ is complete. $\qquad \square$

As a corollary of Theorem 3.2, we provide a new and simple proof of the uniqueness theorem for affine complete improper affine spheres from the viewpoint of the value distribution property.

Corollary 3.6. *Any affine complete improper affine sphere must be an elliptic paraboloid.*

Proof. Because an improper affine sphere has no singularities, the complement of the image of its Lagrangian Gauss map ν contains at least the circle $\{|\nu| = 1\} \subset \mathbf{C} \cup \{\infty\}$. Thus, by exchanging roles of dF and dG if necessarily, it holds that $|\nu| < 1$, that is, $|dF| < |dG|$. On the other hand, we have

$$g = |dG|^2 - |dF|^2 < 2(|dF|^2 + |dG|^2) = d\tau^2.$$

Thus if an improper affine sphere is affine complete, then it is also weakly complete. Hence, by Propotion 3.1 and Theorem 3.2, it is an elliptic paraboloid. $\square$

4. VALUE DISTRIBUTION OF THE RATIO OF CANONICAL FORMS FOR WEAKLY COMPLETE FLAT FRONTS IN HYPERBOLIC THREE-SPACE

We first summarize here definitions and basic facts on weakly complete flat fronts in $\mathbf{H}^3$ which we shall need. For more details, we refer the reader to [11], [24], [25], [27] and [43].

Let $\mathbf{L}^4$ be the Lorentz-Minkowski four-space with inner product of signature $(-, +, +, +)$. Then the hyperbolic three-space is given by

$$(4.1) \qquad \mathbf{H}^3 = \{(x_0, x_1, x_2, x_3) \in \mathbf{L}^4 \,|\, -(x_0)^2 + (x_1)^2 + (x_2)^2 + (x_3)^2 = -1, x_0 > 0\}$$

with the induced metric from $\mathbf{L}^4$, which is a simply connected Riemannian three-manifold with constant sectional curvature -1. Identifying $\mathbf{L}^4$ with the set of 2×2 Hermitian matrices Herm(2)= $\{X^* = X\}$ $(X^* := {}^t\overline{X})$ by

$$(4.2) \qquad (x_0, x_1, x_2, x_3) \longleftrightarrow \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix}$$

where $i = \sqrt{-1}$, we can write

$$(4.3) \qquad \begin{aligned} \mathbf{H}^3 &= \{X \in \text{Herm}(2)\,;\, \det X = 1, \text{trace}\, X > 0\} \\ &= \{aa^*\,;\, a \in SL(2, \mathbf{C})\} \end{aligned}$$

with the metric

$$\langle X, Y \rangle = -\frac{1}{2}\,\text{trace}\,(X\widetilde{Y}), \quad \langle X, X \rangle = -\det(X),$$

where $\widetilde{Y}$ is the cofactor matrix of Y. The complex Lie group $PSL(2, \mathbf{C}) := SL(2, \mathbf{C})/\{\pm\text{id}\}$ acts isometrically on $\mathbf{H}^3$ by

$$(4.4) \qquad \mathbf{H}^3 \ni X \longmapsto aXa^*,$$

where $a \in PSL(2, \mathbf{C})$.

Let Σ be an oriented two-manifold. A smooth map $f\colon \Sigma \to \mathbf{H}^3$ is called a *front* if there exists a Legendrian immersion

$$L_f\colon \Sigma \to T_1^*\mathbf{H}^3$$

into the unit cotangent bundle of $\mathbf{H}^3$ whose projection is f. Identifying $T_1^*\mathbf{H}^3$ with the unit tangent bundle $T_1\mathbf{H}^3$, we can write $L_f = (f, n)$, where $n(p)$ is a unit vector in $T_{f(p)}\mathbf{H}^3$ such that $\langle df(p), n(p)\rangle = 0$ for each $p \in M$. We call n a *unit normal vector field* of the front f. A point $p \in \Sigma$ where $\operatorname{rank}(df)_p < 2$ is called a *singularity* or *singular point*. A point which is not singular is called *regular point*, where the first fundamental form is positive definite.

The *parallel front* f_t of a front f at distance t is given by $f_t(p) = \operatorname{Exp}_{f(p)}(tn(p))$, where "Exp" denotes the exponential map of $\mathbf{H}^3$. In the model for $\mathbf{H}^3$ as in (4.1), we can write

$$(4.5) \qquad f_t = (\cosh t)f + (\sinh t)n, \quad n_t = (\cosh t)n + (\sinh t)f,$$

where n_t is the unit normal vector field of f_t.

Based on the fact that any parallel surface of a flat surface is also flat at regular points, we define flat fronts as follows: A front $f\colon \Sigma \to \mathbf{H}^3$ is called a *flat front* if, for each $p \in M$, there exists a real number $t \in \mathbf{R}$ such that the parallel front f_t is a flat immersion at p. By definition, $\{f_t\}$ forms a family of flat fronts. We note that an equivalent definition of flat fronts is that the Gaussian curvature of f vanishes at all regular points. However, there exists a case where this definition is not suitable. For details, see [27, Remark 2.2].

We assume that f is flat. Then there exists a (unique) complex structure on Σ and a holomorphic Legendrian immersion

$$(4.6) \qquad \mathcal{E}_f\colon \widetilde{\Sigma} \to SL(2, \mathbf{C})$$

such that f and L_f are projections of $\mathcal{E}_f$, where $\widetilde{\Sigma}$ is the universal covering surface of Σ. Here, $\mathcal{E}_f$ being a holomorphic Legendrian map means that $\mathcal{E}_f^{-1}d\mathcal{E}_f$ is off-diagonal (see [11], [26], [27]). We call $\mathcal{E}_f$ the *holomorphic Legendrian lift* of f. The map f and its unit normal vector field n are

$$(4.7) \qquad f = \mathcal{E}_f \mathcal{E}_f^*, \quad n = \mathcal{E}_f e_3 \mathcal{E}_f^*, \quad e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

If we set

$$(4.8) \qquad \mathcal{E}_f^{-1}d\mathcal{E}_f = \begin{pmatrix} 0 & \theta \\ \omega & 0 \end{pmatrix},$$

the first and second fundamental forms $ds^2 = \langle df, df\rangle$ and $dh^2 = -\langle df, dn\rangle$ are given by

$$(4.9) \qquad \begin{aligned} ds^2 &= |\omega + \bar{\theta}|^2 = Q + \bar{Q} + (|\omega|^2 + |\theta|^2), \quad Q = \omega\theta \\ dh^2 &= |\theta|^2 - |\omega|^2 \end{aligned}$$

for holomorphic one-forms ω and θ defined on $\widetilde{\Sigma}$, with $|\omega|^2$ and $|\theta|^2$ well-defined on Σ itself. We call ω and θ the *canonical forms* of f. The holomorphic two-differential Q appearing in the $(2,0)$-part of ds^2 is defined on Σ, and is called the *Hopf differential* of f. By definition, the umbilic points of f coincide with the zeros of Q. Defining a meromorphic function on $\widetilde{\Sigma}$ by the ratio of canonical forms

$$(4.10) \qquad \rho = \frac{\theta}{\omega},$$

then $|\rho|\colon \Sigma \to [0, +\infty]$ is well-defined on Σ, and $p \in \Sigma$ is a singular point if and only if $|\rho(p)| = 1$.

Note that the $(1,1)$-part of the first fundamental form

$$(4.11) \qquad ds_{1,1}^2 = |\omega|^2 + |\theta|^2$$

is positive definite on Σ because it is the pull-back of the canonical Hermitian metric of $SL(2, \mathbf{C})$. Moreover, $2ds_{1,1}^2$ coincides with the pull-back of the Sasakian metric on $T_1^*\mathbf{H}^3$ by the Legendrian lift L_f of f (which is the sum of the first and third fundamental forms in this case, see [27, Section 2] for details). The complex structure on Σ is compatible with the conformal metric $ds_{1,1}^2$. Note that any flat front is orientable ([24, Theorem B]). In this section, for each flat front $f\colon \Sigma \to \mathbf{H}^3$, we always regard Σ as a Riemann surface with this complex structure.

The two *hyperbolic Gauss maps* are defined by

$$(4.12) \qquad G = \frac{E_{11}}{E_{21}}, \quad G_* = \frac{E_{12}}{E_{22}}, \quad \text{where} \quad \mathcal{E}_f = (E_{ij}).$$

By identifying the ideal boundary $\mathbf{S}_\infty^2$ of $\mathbf{H}^3$ with the Riemann sphere $\mathbf{C} \cup \{\infty\}$, the geometric meaning of G and G_* is given as follows ([11], [25, Appendix A], [38]): The hyperbolic Gauss maps G and G_* represent the intersection points in $\mathbf{S}_\infty^2$ for the two oppositely-oriented normal geodesics emanating from f. In particular, G and G_* are meromorphic functions on Σ and parallel fronts have the same hyperbolic Gauss maps. We have already obtained an estimate for the totally ramified value numbers of the hyperbolic Gauss maps of complete flat fronts in $\mathbf{H}^3$ in [20]. This estimate is similar to the case of the Gauss map of pseudo-algebraic minimal surfaces in Euclidean four-space (see [18]). Let z be a local complex coordinate on Σ. Then we have the following identities (see [27]):

$$(4.13) \qquad s(\omega) - S(G) = 2Q, \quad s(\theta) - S(G_*) = 2Q,$$

where $S(G)$ is the Schwarzian derivative of G with respect to z as in

$$(4.14) \qquad S(G) = \left\{ \left(\frac{G''}{G'} \right)' - \frac{1}{2} \left(\frac{G''}{G'} \right)^2 \right\} dz^2 \qquad \left(' = \frac{d}{dz} \right),$$

and $s(\omega)$ and $s(\theta)$ is the Schwarzian derivative of the integral of ω and θ, respectively.

Here, we note on the interchangeability of the canonical forms and the hyperbolic Gauss maps. The canonical forms (ω, θ) have the $U(1)$-ambiguity $(\omega, \theta) \mapsto (e^{is}\omega, e^{-is}\theta)$ $(s \in \mathbf{R})$, which corresponds to

$$(4.15) \qquad \mathcal{E}_f \longmapsto \mathcal{E}_f \begin{pmatrix} e^{is/2} & 0 \\ 0 & e^{-is/2} \end{pmatrix}.$$

For a second ambiguity, defining the *dual* $\mathcal{E}_f^{\natural}$ of $\mathcal{E}_f$ by

$$\mathcal{E}_f^{\natural} = \mathcal{E}_f \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

then $\mathcal{E}_f^{\natural}$ is also Legendrian with $f = \mathcal{E}_f^{\natural}\mathcal{E}_f^{\natural *}$. The hyperbolic Gauss maps $G^{\natural}$, $G_*^{\natural}$ and canonical forms $\omega^{\natural}$, $\theta^{\natural}$ of $\mathcal{E}_f^{\natural}$ satisfy

$$G^{\natural} = G_*, \quad G_*^{\natural} = G, \quad \omega^{\natural} = \theta, \quad \theta^{\natural} = \omega.$$

Namely, the operation $\natural$ interchanges the roles of ω and θ and also G and G_*.

A flat front $f \colon \Sigma \to \mathbf{H}^3$ is said to be *weakly complete* (resp. *of finite type*) if the metric $ds_{1,1}^2$ as in (4.11) is complete (resp. of finite total curvature). We note that the universal cover of a weakly complete flat front is also weakly complete, but completeness is not preserved when lifting to the universal cover.

Fact 4.1 ([24, Proposition 3.2]). *If a flat front $f \colon \Sigma \to \mathbf{H}^3$ is weakly complete and of finite type, then Σ is biholomorphic to $\overline{\Sigma}_\gamma \backslash \{p_1, \ldots, p_k\}$, where $\overline{\Sigma}_\gamma$ is a closed Riemann surface of genus γ and $p_j \in \overline{\Sigma}_\gamma$ $(j = 1, \ldots, k)$.*

Each puncture point p_j $(j = 1, \ldots, k)$ is called an *WCF-end* of f. We can assume that a neighborhood of p_j is biholomorphic to the punctured disk $\mathbf{D}^* = \{z \in \mathbf{C} \,;\, 0 < |z| < 1\}$.

Fact 4.2 ([11], [27], [24, Proposition 3.2]). *Let $f \colon \mathbf{D}^* \to \mathbf{H}^3$ be a WCF-end of a flat front. Then the canonical forms ω and θ are expressed*

$$\omega = z^{\mu}\hat{\omega}(z)dz, \quad \theta = z^{\mu_*}\hat{\theta}(z)dz, \quad (\mu, \mu_* \in \mathbf{R}, \, \mu + \mu_* \in \mathbf{Z}),$$

where $\hat{\omega}$ and $\hat{\theta}$ are holomorphic functions in z which do not vanish at the origin. In particular, the function $|\rho| \colon \mathbf{D}^* \to [0, \infty]$ as in (4.10) can be extended across the end.

Here, $|\omega|^2$ and $|\theta|^2$ are considered as conformal flat metrics on $\mathbf{D}_\varepsilon^*$ for sufficiently small $\varepsilon > 0$. The real numbers μ and μ_* are the order of the metrics $|\omega|^2$ and $|\theta|^2$ at the origin respectively, that is,

$$(4.16) \qquad \mu = \mathrm{ord}_0|\omega|^2, \quad \mu_* = \mathrm{ord}_0|\theta|^2.$$

Since $ds_{1,1}^2 = |\omega|^2 + |\theta|^2$ is complete at the origin, it holds that

$$(4.17) \qquad \min\{\mu, \mu_*\} = \min\left\{\mathrm{ord}_0|\omega|^2, \mathrm{ord}_0|\theta|^2\right\} \leq 1.$$

for a WCF-end. By (4.9), the order of the Hopf differential is

$$(4.18) \qquad \operatorname{ord}_0 Q = \mu + \mu_* = \operatorname{ord}_0 |\omega|^2 + \operatorname{ord}_0 |\theta|^2.$$

We call the WCF-end *regular* if both G and G_* have at most poles. Then the following fact holds.

Fact 4.3 ([11], [24, Proposition 4.2])**.** *A WCF-end $f \colon \mathbf{D}^* \to \mathbf{H}^3$ of a flat front is regular if and only if the Hopf differential has a pole of order at most two at the origin, that is, $\operatorname{ord}_0 Q \geq -2$ holds.*

Now we investigate the value distribution of the ratio of canonical forms for weakly complete flat fronts in $\mathbf{H}^3$. We consider the case where the ratio is constant.

Proposition 4.4. *Let $f \colon \Sigma \to \mathbf{H}^3$ be a weakly complete flat front. If the meromorphic function ρ defined by (4.10) is constant, then f is congruent to a horosphere or a hyperbolic cylinder. Here, a surface equidistance from a geodesic is called a hyperbolic cylinder [27].*

Proof. In general, the function ρ is defined on the universal covering surface $\widetilde{\Sigma}$ of Σ. However, in this case, we can consider that ρ is constant on Σ. Then the metric $ds_{1,1}^2$ defined by (4.11) is represented as

$$(4.19) \qquad ds_{1,1}^2 = |\omega|^2 + |\theta|^2 = \left(1 + \left|\frac{\theta}{\omega}\right|^2\right)|\omega|^2 = (1 + |\rho|^2)|\omega|^2.$$

Thus the Gaussian curvature $K_{ds_{1,1}^2}$ of $ds_{1,1}^2$ vanishes identically on Σ. By Fact 4.1, Σ is biholomorphic to a closed Riemann surface of genus γ with k points removed, that is, $\Sigma = \overline{\Sigma}_\gamma \backslash \{p_1, \ldots, p_k\}$. Moreover we obtain the formula ([24, (3.2)])

$$\frac{1}{2\pi}\int_\Sigma (-K_{ds_{1,1}^2}) dA = -\chi(\overline{\Sigma}_\gamma) - \sum_{j=1}^{k} \operatorname{ord}_{p_j}(ds_{1,1}^2),$$

where dA denotes the area element of $ds_{1,1}^2$ and $\chi(\overline{\Sigma}_\gamma)$ the Euler number of $\overline{\Sigma}_\gamma$. Since the metric $ds_{1,1}^2$ is complete, for each WCF-end p_j, $\operatorname{ord}_{p_j} ds_{1,1}^2 \leq -1$ holds. Thus, in this case, we get $\gamma = 0$ and

$$(4.20) \qquad \sum_{j=1}^{k} \operatorname{ord}_{p_j}(ds_{1,1}^2) = -2.$$

Since $ds_{1,1}^2$ is well-defined on Σ, we need to consider the following two cases:

    (a) The flat front f has two WCF-ends p and q, and $\operatorname{ord}_p ds_{1,1}^2 = \operatorname{ord}_q ds_{1,1}^2 = -1$,

    (b) The flat front f has one WCF-end p, and $\operatorname{ord}_p ds_{1,1}^2 = -2$.

In the case (a), f is congruent to a hyperbolic cylinder. In fact, the WCF-ends are asymptotic to a finite cover of a hyperbolic cylinder ([11], [25]). In the case (b), then $\rho \equiv 0$. Because, if not, then it holds that $\operatorname{ord}_p Q = -4$ by (4.18). On the other hand, the identities (4.13) imply that the WCF-end p is regular. However, by Fact 4.3, it does not

occur. Hence the Hopf differential $Q = \omega\theta$ also vanishes identically on Σ, and then f is a horosphere. $\square$

Applying the same argument as in the proof of Theorem 3.2 to the ratio ρ of weakly complete flat fronts in $\mathbf{H}^3$, we give the following result for ρ.

Theorem 4.5. *Let $f \colon \Sigma \to \mathbf{H}^3$ be a weakly complete flat front and ρ the meromorphic function on $\widetilde{\Sigma}$ defined by (4.10). If ρ is nonconstant, then ρ can omit at most three values.*

As a corollary of Theorem 4.5, we can obtain the uniqueness theorem of weakly complete flat surfaces in $\mathbf{H}^3$. Note that Sasaki [41], Volkov and Vladimirova [46] have already obtained the same result for complete flat surfaces in $\mathbf{H}^3$ (See also [11, Theorem 3]).

Corollary 4.6. *Any weakly complete flat surface in $\mathbf{H}^3$ must be congruent to a horosphere or a hyperbolic cylinder.*

Proof. Because a weakly complete flat surface has no singularities, the complement of the image of ρ contains at least the circle $\{|\rho| = 1\} \subset \mathbf{C} \cup \{\infty\}$. From Proposition 4.4 and Theorem 4.5, it is a horosphere or a hyperbolic cylinder. $\square$

References

[1] S. Bernstein, Sur un théorème de géométrie et ses applications aux équations aux dérivées partielles du type elliptique, Comm. de la Soc. Math. de Kharkov (2éme sér.) **15**, 38–45 (1915–1917)

[2] E. Calabi, Improper affine hypersurfaces of convex type and a generalization of a theorem by K. Jörgens, Mich. Math. J. **5**, 108–126 (1958)

[3] E. Calabi, Examples of Bernstein problems for some non-linear equations, Proc. Sympos. Pure Math. **15**, 223–230 (1970)

[4] E. Calabi, Convex affine maximal surfaces, Results. Math. **13**, 209–223 (1988)

[5] B. Y. Chen and J. M. Morvan, Géométrie des surfaces lagrangiennes de $\mathbf{C}^2$, J. Math. Pures Appl. **66**, 321–335 (1987)

[6] Y. Fang, The Minding formula and its applications, Arch. Math. **72**, 473–480 (1999)

[7] L. Ferrer, A. Martínez and F. Milán, An extension of a theorem by K. Jörgens and a maximum principle at infinity for parabolic affine spheres, Math. Z. **230**, 471–486 (1999)

[8] H. Fujimoto, On the number of exceptional values of the Gauss map of minimal surfaces, J. Math. Soc. Japan **40**, 235–247 (1988)

[9] H. Fujimoto, Value Distribution Theory of the Gauss Map of Minimal Surfaces in $\mathbf{R}^m$, Aspects of Mathematics, E21. Friedr. Vieweg & Sohn, Braunschweig (1993).

[10] H. Fujimoto, Nevanlinna theory and minimal surfaces, Geometry V, 95–151, 267–272, Encyclopaedia Math. Sci., 90, Springer, Berlin (1997)

[11] J. A. Gálvez, A. Martínez and F. Milán, Flat surfaces in hyperbolic 3-space, Math. Ann. **316**, 419–435 (2000)

[12] R. Harvey and B. Lawson, Calibrated Geometries, Acta Math. **148**, 47–157 (1982)

[13] A. Huber, On subharmonic functions and differential geometry in the large, Comment. Math. Helv. **32**, 13–72 (1957)

[14] G. Ishikawa and Y. Machida, Singularities of improper affine spheres and surfaces of constant Gaussian curvature, Internat. J. Math. **17**, 269–293 (2007)

[15] K. Jörgens, Über die Lösungen der differentialgleichung $rt - s^2 = 1$, Math. Ann. **127**, 130–134 (1954)

[16] D. Joyce, Lectures on special Lagrangian geometry, In: Global theory of minimal surfaces, Clay Math. Proc. **2**, Amer. Math. Soc., Providence, RI, 667–695 (2005)

[17] L. Jin and M. Ru, Algebraic curves and the Gauss map of algebraic minimal surfaces, Differ. Geom. Appl. **25**, 701–712 (2007)

[18] Y. Kawakami, The Gauss map of pseudo-algebraic minimal surfaces in $\mathbf{R}^4$, Math. Nachr. **282**, 211–218 (2009)

[19] Y. Kawakami, Ramification estimates for the hyperbolic Gauss map, Osaka J. Math. **46**, 1059–1076 (2009)

[20] Y. Kawakami, Value distribution of the hyperbolic Gauss maps for flat fronts in hyperbolic three-space, Houston J. Math., **38**, 115–130 (2012)

[21] Y. Kawakami, R. Kobayashi and R. Miyaoka, The Gauss map of pseudo-algebraic minimal surfaces, Forum Math. **20**, 1055–1069 (2008)

[22] R. Kobayashi, Toward Nevanlinna theory as a geometric model of Diophantine approximation, Sugaku Expositions **16**, 39–79 (2003)

[23] M. Kokubu, W. Rossman, K. Saji, M. Umehara and K. Yamada, Singularities of flat fronts in hyperbolic space, Pacific J. Math. **221**, 303–351 (2005)

[24] M. Kokubu, W. Rossman, M. Umehara and K. Yamada, Flat fronts in hyperbolic 3-space and their caustics, J. Math. Soc. Japan **59**, 265–299 (2007)

[25] M. Kokubu, W. Rossman, M. Umehara and K. Yamada, Asymptotic behavior of flat surfaces in hyperbolic 3-space, J. Math. Soc. Japan, **61**, 799–852 (2009)

[26] M. Kokubu, M. Umehara and K. Yamada, An elementary proof of Small's formula for null curves in PSL(2, $\mathbf{C}$) and an analogue for Legendrian curves in PSL(2, $\mathbf{C}$), Osaka J. Math., **40**, 697–715 (2003)

[27] M. Kokubu, M. Umehara and K. Yamada, Flat fronts in hyperbolic 3-space, Pacific J. Math. **216**, 149–175 (2004)

[28] A.-M. Li, U. Simon and Z. Zhao, Global affine differential geometry of hypersurfaces, Walter de Gruyter, Berlin-New York (1993)

[29] A. Martínez, Improper affine maps, Math. Z. **249**, 755–766 (2005)

[30] A. Martínez, Relatives of Flat Surfaces in $\mathbf{H}^3$, In: Proceedings of International Workshop on Integrable systems, Geometry and Visualization (November 19–24 at Kyushu university, Fukuoka, Japan), 115–132 (2005)

[31] D. Nakajo, A representation formula for indefinite improper affine spheres, Results. Math. **55**, 139–159 (2009)

[32] R. Nevanlinna, Analytic Function, Translated from the second German edition by Phillip Emig. Die Grundlehren der mathematischen Wissenschaften, Springer, New York (1970)

[33] J. Noguchi and T. Ochiai, Geometric Function Theory in Several Complex Variables, Transl. Math. Monog. **80**, Amer. Math. Soc. (1990)

[34] K. Nomizu and T. Sasaki, Affine differential geometry, Cambridge University Press (1994)

[35] R. Osserman, Proof of a conjecture of Nirenberg, Comm. on Pure and Applied Math. **12**, 229–232 (1959)

[36] R. Osserman, Global properties of minimal surfaces in E^3 and E^n, Ann. of Math. **80**, 340–364 (1964)

[37] R. Osserman, A survey of minimal surfaces, second edition, Dover Publications Inc. (1986)

[38] P. Roitman, Flat surfaces in hyperbolic 3-space as normal surfaces to a congruence of geodesics, Tohoku Math. J. **59**, 21–37 (2007)

[39] A. Ros, The Gauss map of minimal surfaces, In: Diffrential Geometry, Valencia 2002, Proceedings of the Conference in Honour of Antonio M. Naveria, World Scientific, 235–252 (2002)

[40] M. Ru, Nevanlinna theory and its relation to Diophantine approximation, World Sci. Publ. Co. Inc., River Edge, NJ (2001)

[41] S. Sasaki, On complete flat surfaces in hyperbolic 3-space, Kōdai Math Sem. Rep. **25**, 449–457 (1973)

[42] K. Shiohama, Total curvatures and minimal areas of complete open surfaces, Proc. Amer. Math. Soc. **94**, 310–316 (1985)

[43] K. Saji, M. Umehara and K. Yamada, The geometry of fronts, Ann. of Math. **169**, 491–529 (2009)

[44] N. S. Trudinger and X. J. Wang, Affine complete locally convex hypersurfaces, Invent. math. **150**, 45–60 (2002)

[45] M. Umehara and K. Yamada, Applications of a complete lemma in minimal surface theory to various classes of surfaces, to appear in Bulletin of the London Mathematical Society, arXiv:0909.1128

[46] Y. A. Volkov and S. M. Vladimirova, Isometric immersions of the Euclidean plane in Lobačevskiĭ space(Russian), Mat. Zametki **10**, 327–332 (1971)

Graduate School of Science and Engineering,
Yamaguchi university,
Yamaguchi, 753-8512, Japan
E-mail address: `y-kwkami@yamaguchi-u.ac.jp`

Faculty of Mathematics,
Kyushu university,
744, Motooka, Nishiku, Fukuoka-city, 819-0395, Japan
E-mail address: `nakajo@math.kyushu-u.ac.jp`