ON THE cd-INDEX AND γ-VECTOR OF S*-SHELLABLE CW-SPHERES

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Abstract. We show that the γ-vector of the order complex of any polytope is the f-vector of a balanced simplicial complex. This is done by proving this statement for a subclass of Stanley's S-shellable CW-spheres which includes all polytopes. The proof shows that certain parts of the **cd**-index, when specializing $\mathbf{c} = 1$ and considering the resulted polynomial in $\mathbf{d}$, are the f-polynomials of simplicial complexes that can be colored with "few" colors. We conjecture that the **cd**-index of a regular CW-sphere is itself the *flag* f-vector of a colored simplicial complex in a certain sense.

1. Introduction

Let P be an $(n-1)$-dimensional regular CW-sphere (that is, a regular CW-complex which is homeomorphic to an $(n-1)$-dimensional sphere). In face enumeration, one of the most important combinatorial invariants of P is the **cd**-index. The **cd**-index $\Phi_P(\mathbf{c},\mathbf{d})$ of P is a non-commutative polynomial in the variables $\mathbf{c}$ and $\mathbf{d}$ that encodes the flag f-vector of P. By the result of Stanley [St1] and Karu [Ka], it is known that the **cd**-index $\Phi_P(\mathbf{c},\mathbf{d})$ has non-negative integer coefficients. On the other hand, a characterization of the possible **cd**-indices for regular CW-spheres, or other related families, e.g. Gorenstien* posets, is still beyond reach. In this paper we take a step in this direction and establish some non-trivial upper bounds, as we detail now.

If we substitute 1 for $\mathbf{c}$ in $\Phi_P(\mathbf{c},\mathbf{d})$, we obtain a polynomial of the form

$$\Phi_P(1,\mathbf{d}) = \delta_0 + \delta_1\mathbf{d} + \cdots + \delta_{\lfloor \frac{n}{2}\rfloor}\mathbf{d}^{\lfloor \frac{n}{2}\rfloor},$$

where $\lfloor \frac{n}{2}\rfloor$ is the integer part of $\frac{n}{2}$, such that each δ_i is a non-negative integer. In other words, δ_i is the sum of coefficients of monomials in $\Phi_P(\mathbf{c},\mathbf{d})$ for which $\mathbf{d}$ appears i times.

Let Δ be a (finite abstract) simplicial complex on the vertex set V. We say that Δ is *k-colored* if there is a map $c : V \to [k] = \{1,2,\dots,k\}$, called a *$k$-coloring map of* Δ, such that if $\{x,y\}$ is an edge of Δ then $c(x) \neq c(y)$. Let $f_i(\Delta)$ denote the number of elements $F \in \Delta$ having cardinality $i+1$, where $f_{-1}(\Delta) = 1$. The main result of this paper is the following.

Theorem 1.1. *Let P be an $(n-1)$-dimensional S*-shellable regular CW-sphere, and let $\Phi_P(1,\mathbf{d}) = \delta_0 + \delta_1\mathbf{d} + \cdots + \delta_{\lfloor \frac{n}{2}\rfloor}\mathbf{d}^{\lfloor \frac{n}{2}\rfloor}$. Then there exists an $\lfloor \frac{n}{2}\rfloor$-colored simplicial complex Δ such that*

$$\delta_i = f_{i-1}(\Delta) \quad \textit{for } i = 0,1,\dots,\lfloor \frac{n}{2}\rfloor.$$

Key words and phrases. polytope, γ-vector, cd-index, balanced simplicial complex.

Research of the first author was partially supported by KAKENHI 22740018. Research of the second author was partially supported by an NSF Award DMS-0757828.

The precise definition of S*-shellability is given in Section 2. The most important class of S*-shellable CW-spheres is the class of the boundary complexes of polytopes. By the Kruskal-Katona Theorem (see e.g. [St2, II, Theorem 2.1]), the above theorem gives a certain upper bound on δ_i in terms of δ_{i-1}. Better upper bounds are given by Frankl-Füredi-Kalai theorem which characterizes the f-vectors of k-colored complexes [FFK].

The numbers $\delta_0, \delta_1, \delta_2, \ldots$ relate to the γ-vector (see Section 4 for the definition) of the barycentric subdivision (order complex) of P, namely the simplicial complex whose elements are the chains of nonempty cells in P ordered by inclusion. Indeed, as an application of Theorem 1.1 we prove the following.

Theorem 1.2. *Let P be an $(n-1)$-dimensional S*-shellable regular CW-sphere and let $\mathrm{sd}(P)$ be the barycentric subdivision of P. Then there exists an $\lfloor \frac{n}{2} \rfloor$-colored simplicial complex Γ such that*

$$\gamma_i(\mathrm{sd}(P)) = f_{i-1}(\Gamma) \quad \textit{for } i = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor.$$

Recall that an $(n-1)$-dimensional simplicial complex is said to be *balanced* if it is n-colored. If P is the boundary complex of an arbitrary convex n-dimensional polytope, then $\delta_{\lfloor \frac{n}{2} \rfloor} > 0$ and we conclude the following.

Corollary 1.3. *Let P be the boundary complex of an n-dimensional polytope. Then the γ-vector of $\mathrm{sd}(P)$ is the f-vector of a balanced simplicial complex.*

The above corollary supports the conjecture of Nevo and Petersen [NP, Conjecture 6.3] which states that the γ-vector of a flag homology sphere is the f-vector of a balanced simplicial complex. This conjecture was verified for the barycentric subdivision of simplicial homology spheres (in this case all the cells are simplices) in [NPT].

It would be natural to ask if the above theorems hold for all regular CW-spheres (or more generally, Gorenstein* posets). We conjecture a stronger statement on the **cd**-index, see Conjecture 4.3.

This paper is organized as follows: in Section 2 we recall some known results on the **cd**-index and define S*-shellability, in Section 3 we prove our main theorem, Theorem 1.1, in Section 4 we derive consequences for γ-vectors and present a conjecture on the **cd**-index, Conjecture 4.3.

2. **cd**-index of S*-shellable CW-spheres

In this section we recall some known results on the **cd**-index.

Let P be a graded poset of rank $n+1$ with the minimal element $\hat{0}$ and the maximal element $\hat{1}$. Let ρ denote the rank function of P. For $S \subset [n] = \{1, 2, \ldots, n\}$, a chain $\hat{0} = \sigma_0 < \sigma_1 < \sigma_2 < \cdots < \sigma_{k+1} = \hat{1}$ of P is called an *S-flag* if $\{\rho(\sigma_1), \ldots, \rho(\sigma_k)\} = S$. Let $f_S(P)$ be the number of S-flags of P. Define $h_S(P)$ by

$$h_S(P) = \sum_{T \subset S} (-1)^{|S|-|T|} f_T(P),$$

where $|X|$ denotes the cardinality of a finite set X. The vectors $(f_S(P) : S \subset [n])$ and $(h_S(P) : S \subset [n])$ are called the *flag f-vector* and *flag h-vector* of P respectively.

Now we recall the definition of the **cd**-index. For $S \subset [n]$, we define a non-commutative monomial $u_S = u_1 u_2 \cdots u_n$ in variables $\mathbf{a}$ and $\mathbf{b}$ by $u_i = \mathbf{a}$ if $i \notin S$ and $u_i = \mathbf{b}$ if $i \in S$. Let

$$\Psi_P(\mathbf{a}, \mathbf{b}) = \sum_{S \subset [n]} h_S(P) u_S.$$

For a graded poset P, let $\mathrm{sd}(P)$ be the order complex of $P - \{\hat{0}, \hat{1}\}$. Thus

$$\mathrm{sd}(P) = \{\{\sigma_1, \sigma_2, \ldots, \sigma_k\} \subset P - \{\hat{0}, \hat{1}\} : \sigma_1 < \sigma_2 < \cdots < \sigma_k\}.$$

We say that P is *Gorenstein** if the simplicial complex $\mathrm{sd}(P)$ is a homology sphere. It is known that if P is Gorenstein* then $\Psi_P(\mathbf{a}, \mathbf{b})$ can be written as a polynomial $\Phi_P(\mathbf{c}, \mathbf{d})$ in $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$ [BK], and this non-commutative polynomial $\Phi_P(\mathbf{c}, \mathbf{d})$ is called the **cd**-*index* of P. Moreover, by the celebrated results due to Stanley [St1] (for convex polytopes) and Karu [Ka] (for Gorenstein* posets), the coefficients of $\Phi_P(\mathbf{c}, \mathbf{d})$ are non-negative integers.

Next, we define S*-shellability of regular CW-spheres by slightly modifying the definition of S-shellability introduced by Stanley [St1, Definition 2.1].

Let P be a regular CW-sphere (a regular CW-complex which is homeomorphic to a sphere) and $\mathcal{F}(P)$ its face poset. Then the order complex of $\mathcal{F}(P)$ is a triangulation of a sphere, so the poset $\mathcal{F}(P) \cup \{\hat{0}, \hat{1}\}$ is Gorenstein*. We define the **cd**-index of P by $\Phi_P(\mathbf{c}, \mathbf{d}) = \Phi_{\mathcal{F}(P) \cup \{\hat{0}, \hat{1}\}}(\mathbf{c}, \mathbf{d})$. For any cell σ of P, we write $\bar{\sigma}$ for the closure of σ. For an $(n-1)$-dimensional regular CW-sphere P, let ΣP be the *suspension of* P, namely, ΣP is the n-dimensional regular CW-sphere obtained from P by attaching two n-dimensional cells τ_1 and τ_2 such that $\partial \bar{\tau}_1 = \partial \bar{\tau}_2 = P$. Also, for an $(n-1)$-dimensional regular CW-ball P (a regular CW-complex which is homeomorphic to an $(n-1)$-dimensional ball), let P' be the $(n-1)$-dimensional regular CW-sphere which is obtained from P by adding an $(n-1)$-dimensional cell τ so that $\partial \bar{\tau} = \partial P$.

Definition 2.1. Let P be an $(n-1)$-dimensional regular CW-sphere. We say that P is *S*-shellable* if either $P = \{\emptyset\}$ or there is an order $\sigma_1, \sigma_2, \ldots, \sigma_r$ of the facets of P such that the following conditions hold.

(a) $\partial \bar{\sigma}_1$ is S*-shellable.

(b) For $1 \leq i \leq r-1$, let

$$\Omega_i = \bar{\sigma}_1 \cup \bar{\sigma}_2 \cup \cdots \cup \bar{\sigma}_i$$

and for $2 \leq i \leq r-1$ let

$$\Gamma_i = \overline{\left[\partial \bar{\sigma}_i \setminus \left(\partial \bar{\sigma}_i \cap \Omega_{i-1}\right)\right]}.$$

Then both Ω_i and Γ_i are regular CW-balls of dimension $(n-1)$ and $(n-2)$ respectively, and Γ_i' is S*-shellable with the first facet of the shelling being the facet which is not in Γ_i.

Remark 2.2. The difference between the above definition and Stanley's S-shellability is that S-shellability only assume that P and Γ_i' are Eulerian and assume no conditions on Ω_i. However, S*-shellable regular CW-spheres are S-shellable, and the boundary complex of convex polytopes are S*-shellable by the line shelling [BM]. We leave the verification of this fact to the readers.

The next recursive formula is due to Stanley [St1].

Lemma 2.3 (Stanley)**.** *With the same notation as in Definition 2.1, for $i = 1, 2, \dots, r-2$, one has*

$$\Phi_{\Omega'_{i+1}}(\mathbf{c},\mathbf{d}) = \Phi_{\Omega'_i}(\mathbf{c},\mathbf{d}) + \Big\{\Phi_{\Gamma'_{i+1}}(\mathbf{c},\mathbf{d}) - \Phi_{\Sigma(\partial\Gamma_{i+1})}(\mathbf{c},\mathbf{d})\Big\}\mathbf{c} + \Phi_{\partial\Gamma_{i+1}}(\mathbf{c},\mathbf{d})\mathbf{d}.$$

Since $\Omega'_{r-1} = P$ the above formula gives a way to compute the **cd**-index of P recursively.

Next, we recall a result of Ehrenborg and Karu proving that the **cd**-index increases by taking subdivisions. Let P and Q be regular CW-complexes, and let $\phi : \mathcal{F}(P) \to \mathcal{F}(Q)$ be a poset map. For a subcomplex $Q' = \sigma_1 \cup \cdots \cup \sigma_s \subset Q$, where each σ_i is a cell of Q, we write $\phi^{-1}(Q') = \phi^{-1}(\sigma_1) \cup \cdots \cup \phi^{-1}(\sigma_s)$.

Following [EK, Definition 2.6], for $(n-1)$-dimensional regular CW-spheres P and $\hat{P}$, we say that $\hat{P}$ is a *subdivision* of P if there is an order preserving surjective poset map $\phi : \mathcal{F}(\hat{P}) \to \mathcal{F}(P)$, satisfying that for any cell σ of P, $\phi^{-1}(\bar{\sigma})$ is a homology ball having the same dimension as σ and $\phi^{-1}(\partial\bar{\sigma}) = \partial(\phi^{-1}(\bar{\sigma}))$.

The following result was proved in [EK, Theorem 1.5].

Lemma 2.4 (Ehrenborg-Karu)**.** *Let P and $\hat{P}$ be $(n-1)$-dimensional regular CW-spheres. If $\hat{P}$ is a subdivision of P then one has a coefficientwise inequality $\Phi_{\hat{P}}(\mathbf{c},\mathbf{d}) \geq \Phi_P(\mathbf{c},\mathbf{d})$*

Back to S*-shellable regular CW-spheres, with the same notation as in Definition 2.1, Ω'_i is a subdivision of $\Sigma(\partial\Omega_i)$ and $\partial\Omega_i$ is a subdivision of $\Sigma(\partial\Gamma_{i+1})$. Indeed, for the first statement, if τ_1 and τ_2 are the facets of $\Sigma(\partial\Omega_i)$ then define $\phi : \mathcal{F}(\Omega'_i) \to \mathcal{F}(\Sigma(\partial\Omega_i))$ by

$$\phi(\sigma) = \begin{cases} \sigma, & \text{if } \sigma \in \partial\Omega_i, \\ \tau_1, & \text{if } \sigma \text{ is an interior face of } \Omega_i, \\ \tau_2, & \text{if } \sigma \notin \Omega_i. \end{cases}$$

Similarly, for the second statement, if τ_1 and τ_2 are the facets of $\Sigma(\partial\Gamma_{i+1})$ then define $\phi : \mathcal{F}(\partial\Omega_i) \to \mathcal{F}(\Sigma(\partial\Gamma_{i+1}))$ by

$$\phi(\sigma) = \begin{cases} \sigma, & \text{if } \sigma \in \partial\Gamma_{i+1}, \\ \tau_1, & \text{if } \sigma \in \bar{\sigma}_{i+1} \backslash \partial\Gamma_{i+1}, \\ \tau_2, & \text{otherwise.} \end{cases}$$

Since $\Phi_{\Sigma P}(\mathbf{c},\mathbf{d}) = \Phi_P(\mathbf{c},\mathbf{d})\mathbf{c}$ for any regular CW-sphere P (see [St1, Lemma 1.1]), Lemma 2.4 shows

Lemma 2.5. *With the same notation as in Definition 2.1, for $i = 2, 3, \dots, r-2$, one has $\Phi_{\Omega'_i}(\mathbf{c},\mathbf{d}) \geq \Phi_{\partial\Gamma_{i+1}}(\mathbf{c},\mathbf{d})\mathbf{c}^2$.*

3. Proof of the main theorem

In this section, we prove Theorem 1.1.

For a homogeneous **cd**-polynomial Φ (i.e., a homogeneous polynomial of $\mathbb{Z}\langle\mathbf{c},\mathbf{d}\rangle$ with $\deg\mathbf{c} = 1$ and $\deg\mathbf{d} = 2$) of degree n, we define $\Phi_0, \Phi_2, \dots, \Phi_n$ by

$$\Phi = \Phi_0 + \Phi_2\mathbf{d}\mathbf{c}^{n-2} + \Phi_3\mathbf{d}\mathbf{c}^{n-3} + \cdots + \Phi_{n-1}\mathbf{d}\mathbf{c} + \Phi_n\mathbf{d}$$

where $\Phi_0 = \alpha\mathbf{c}^n$ for some $\alpha \in \mathbb{Z}$ and each Φ_k is a **cd**-polynomial of degree $k-2$ for $k \geq 2$. Also, we write $\Phi_{\leq k} = \Phi_0 + \Phi_2\mathbf{d}\mathbf{c}^{n-2} + \cdots + \Phi_k\mathbf{d}\mathbf{c}^{n-k}$.

Definition 3.1.

- A vector $(\delta_0, \delta_1, \dots, \delta_s) \in \mathbb{Z}^{s+1}$ is said to be *k-FFK* if there is a k-colored simplicial complex Δ such that $\delta_i = f_{i-1}(\Delta)$ for $i = 0, 1, \dots, s$. ($\{\emptyset\}$ is a 0-colored simplicial complex.) A homogeneous $\mathbf{cd}$-polynomial $\Phi = \Phi(\mathbf{c}, \mathbf{d})$ is said to be k-FFK if, when we write $\Phi(1, \mathbf{d}) = \delta_0 + \delta_1 \mathbf{d} + \cdots + \delta_s \mathbf{d}^s$, the vector $(\delta_0, \delta_1, \dots, \delta_s)$ is k-FFK.
- A homogeneous $\mathbf{cd}$-polynomial Φ of degree n is said to be *primitive* if the coefficient of $\mathbf{c}^n$ in Φ is 1.
- Let Φ be a homogeneous $\mathbf{cd}$-polynomial. A primitive homogeneous $\mathbf{cd}$-polynomial Ψ is said to be *k-good for Φ* if Ψ is k-FFK and $\Phi(1, \mathbf{d}) \geq \Psi(1, \mathbf{d})$. Also, we say that a homogeneous $\mathbf{cd}$-polynomial Ψ is k-good for Φ if it is the sum of primitive homogeneous $\mathbf{cd}$-polynomials that are k-good for Φ.

Next, we recall Frankl-Füredi-Kalai theorem [FFK], which characterizes all possible f-vectors of colored complexes. Let $\mathbb{N}_i^{(k)} = \{i + jk : j \in \mathbb{Z}_{\geq 0}\}$ for $i = 1, 2, \dots, k$ and

$$\mathcal{C}^{(k)} = \{F \subset \mathbb{N} : |F \cap \mathbb{N}_i^{(k)}| \leq 1 \text{ for } i = 1, 2, \dots, k\},$$

where $\mathbb{N}$ is the set of positive integers. Let $>_{\mathrm{rev}}$ be the reverse lexicographic order induced by $1 >_{\mathrm{rev}} 2 >_{\mathrm{rev}} \cdots$. Thus, for finite subsets $F \subset \mathbb{N}$ and $G \subset \mathbb{N}$ with $|F| = |G|$, one has $F >_{\mathrm{rev}} G$ if the largest integer in the symmetric difference $(F \setminus G) \cup (G \setminus F)$ is contained in G. A *k-colored compressed complex* is a simplicial complex Δ such that $\Delta \subset \mathcal{C}^{(k)}$ and that, for every $F \in \Delta$ and $G \in \mathcal{C}^{(k)}$ with $|G| = |F|$ and $G >_{\mathrm{rev}} F$, one has $G \in \Delta$. Since $>_{\mathrm{rev}}$ is a total order on the set of finite subsets of $\mathbb{N}$ having the same cardinality, k-colored compressed complex is uniquely determined by its f-vector.

Theorem 3.2 (Frankl-Füredi-Kalai)**.** *A vector $(\delta_0, \delta_1, \dots, \delta_s) \in \mathbb{Z}^{s+1}$ is k-FFK if and only if there is a k-colored compressed complex Δ such that $f_{i-1}(\Delta) = \delta_i$ for $i = 0, 1, \dots, s$.*

We will use the following observation, which follows from [NPT, Lemma 3.1]:

Lemma 3.3. *If Φ is a k-FFK homogeneous $\mathbf{cd}$-polynomial of degree n, and if Ψ' and Ψ'' are homogeneous $\mathbf{cd}$-polynomials of degree n' and n'' respectively, where $n', n'' \leq n - 2$, which are k-good for Φ then*

$$\Phi + \Psi' \mathbf{d} \mathbf{c}^{n-n'-2} \text{ and } \Phi + \Psi' \mathbf{d} \mathbf{c}^{n-n'-2} + \Psi'' \mathbf{d} \mathbf{c}^{n-n''-2}$$

are $(k+1)$-FFK.

Proof. For a simplicial complex Γ, we write $f(\Gamma, \mathbf{d}) = 1 + f_0(\Gamma)\mathbf{d} + f_1(\Gamma)\mathbf{d}^2 + \cdots$. There are k-colored complexes $\Delta, \Delta^{(1)}, \cdots, \Delta^{(m)}, \cdots, \Delta^{(s)}$ such that $f(\Delta, \mathbf{d}) = \Phi(1, \mathbf{d})$, $\sum_{1 \leq i \leq m} f(\Delta^{(i)}, \mathbf{d}) = \Psi'(1, \mathbf{d})$, $\sum_{m+1 \leq i \leq s} f(\Delta^{(i)}, \mathbf{d}) = \Psi''(1, \mathbf{d})$ and $\Phi(1, d) \geq f(\Delta^{(i)}, d)$ for all $1 \leq i \leq s$. By Frankl-Füredi-Kalai theorem, we may assume that all these complexes are k-colored compressed. Then, since $\Phi(1, \mathbf{d}) \geq \Psi'(1, \mathbf{d})$ and $\Phi(1, \mathbf{d}) \geq \Psi''(1, \mathbf{d})$, each $\Delta^{(i)}$ is a subcomplex of Δ. For $i = 1, 2, \dots, s$, let

$$\Gamma^{(i)} = \Delta \bigcup \left\{ \bigcup_{j=1}^{i} \{F \cup \{v_j\} : F \in \Delta^{(j)}\} \right\},$$

where $v_1,\dots,v_s$ are new vertices. Since each $\Delta^{(j)}$ is a subcomplex of Δ, $\Gamma^{(i)}$ is a simplicial complex. Also, $f(\Gamma^{(m)},\mathbf{d})=(\Phi+\Psi'\mathbf{dc}^{n-n'-2})(1,\mathbf{d})$ and $f(\Gamma^{(s)},\mathbf{d})=(\Phi+\Psi'\mathbf{dc}^{n-n'-2}+\Psi''\mathbf{dc}^{n-n''-2})(1,\mathbf{d})$. We claim that each $\Gamma^{(i)}$ is $(k+1)$-colored. Let V be the vertex set of Δ and $c:V\to[k]$ a k-coloring map of Δ. Then the map $\hat{c}:V\cup\{v_1,\dots,v_i\}\to[k+1]$ defined by $\hat{c}(x)=c(x)$ if $x\in V$ and $\hat{c}(x)=k+1$ if $x\notin V$ is a $(k+1)$-coloring map of $\Gamma^{(i)}$. □

Let P be an $(n-1)$-dimensional S^*-shellable regular CW-sphere with the shelling $\sigma_1,\dots,\sigma_r$. Keeping the notation in Definition 2.1, to simplify notations, we use the following symbols.

$$\begin{aligned}
\Phi^{(i)} &= \Phi^{(i)}(\mathbf{c},\mathbf{d})=\Phi_{\Omega_i'}(\mathbf{c},\mathbf{d})\\
\Phi &= \Phi_P(\mathbf{c},\mathbf{d})=\Phi^{(r-1)}\\
\Psi^{(i)} &= \Phi_{\Gamma_{i+1}'}(\mathbf{c},\mathbf{d})-\Phi_{\Sigma(\partial\Gamma_{i+1})}(\mathbf{c},\mathbf{d})\\
\Psi &= \sum_{i=1}^{r-2}\Psi^{(i)}\\
\Pi &= \Phi-\Phi^{(1)}.
\end{aligned}$$

Thus Stanley's recursive formula, Lemma 2.3, says

$$\Phi^{(i+1)}=\Phi^{(i)}+\Psi^{(i)}\mathbf{c}+\Phi_{\partial\Gamma_{i+1}}(\mathbf{c},\mathbf{d})\mathbf{d}$$

and

$$\Pi=\Psi\mathbf{c}+\sum_{i=1}^{r-2}\Phi_{\partial\Gamma_{i+1}}(\mathbf{c},\mathbf{d})\mathbf{d}.$$

The last part of the following proposition is a restatement of Theorem 1.1.

Proposition 3.4. *With notation as above, the following holds.*

(1) *For $2\le k\le n$, $\Psi_k^{(i)}$ is $\lfloor\frac{k}{2}-1\rfloor$-good for $\Phi^{(i)}_{\le k-2}+\Psi^{(i)}_{\le k-2}\mathbf{c}$.*
(2) *For $2\le k\le n$, Π_k is $\lfloor\frac{k}{2}-1\rfloor$-good for $\Phi^{(1)}_{\le k-2}+\Pi_{\le k-2}$.*
(3) *For $2\le k\le n$, Φ_k is $\lfloor\frac{k}{2}-1\rfloor$-good for $\Phi_{\le k-2}$.*
(4) *For $0\le k\le n$, $\Phi_{\le k}$ is $\lfloor\frac{k}{2}\rfloor$-FFK. In particular, the* **cd***-index of P is $\lfloor\frac{n}{2}\rfloor$-FFK.*

Proof. The proof is by induction on dimension, where all statements clearly hold for $n=0,1$. Suppose that all statements are true up to dimension $n-2$. To simplify notations, for a regular CW-sphere Q, we write $\Phi_Q=\Phi_Q(\mathbf{c},\mathbf{d})$.

Proof of (1). By applying the induction hypothesis to Γ'_{i+1} (use statement(2)), each $\Psi_k^{(i)}$ is $\lfloor\frac{k}{2}-1\rfloor$-good for $(\Phi_{\Sigma(\partial\Gamma_{i+1})})^{(i)}_{\le k-2}+\Psi^{(i)}_{\le k-2}$. Thus, $\Psi_k^{(i)}$ is $\lfloor\frac{k}{2}-1\rfloor$-good for $(\Phi_{\Sigma(\partial\Gamma_{i+1})})^{(i)}_{\le k-2}\mathbf{c}+\Psi^{(i)}_{\le k-2}\mathbf{c}$. By Lemma 2.5,

$$\Phi_{\Sigma(\partial\Gamma_{i+1})}\mathbf{c}=\Phi_{\partial\Gamma_{i+1}}\mathbf{c}^2\le\Phi_{\Omega_i'}=\Phi^{(i)}.$$

Since $(\Upsilon\mathbf{c})_j=\Upsilon_j$ for any homogeneous **cd**-polynomial Υ, $\Psi_k^{(i)}$ is $\lfloor\frac{k}{2}-1\rfloor$-good for $\Phi^{(i)}_{\le k-2}+\Psi^{(i)}_{\le k-2}\mathbf{c}$.

Proof of (2). By the definition of Π,

$$\Pi_k = \sum_{i=1}^{r-2} \Psi_k^{(i)} \text{ for } k < n$$

and

$$\Pi_n = \sum_{i=1}^{r-2} \Phi_{\partial\Gamma_{i+1}}.$$

By (1), each $\Psi_k^{(i)}$ is $\lfloor \frac{k}{2} - 1 \rfloor$-good for $\Phi_{\le k-2}^{(i)} + \Psi_{\le k-2}^{(i)}\mathbf{c}$. Then since

$$\Phi_{\le k-2}^{(i)} + \Psi_{\le k-2}^{(i)}\mathbf{c} \le \Phi_{\le k-2} = \Phi_{\le k-2}^{(1)} + \Pi_{\le k-2},$$

Π_k is $\lfloor \frac{k}{2} - 1 \rfloor$-good for $\Phi_{\le k-2}^{(1)} + \Pi_{\le k-2}$ for $k < n$. Also, each $\Phi_{\partial\Gamma_{i+1}}$ is $\lfloor \frac{n}{2} - 1 \rfloor$-FFK by the induction hypothesis (use (4)), and $\Phi_{\partial\Gamma_{i+1}}\mathbf{c}^2 \le \Phi^{(i)}$ by Lemma 2.5. The latter condition clearly says

$$\Phi_{\partial\Gamma_{i+1}}\mathbf{c}^2 \le \Phi_{\le n-2}^{(i)} \le \Phi_{\le n-2} = \Phi_{\le n-2}^{(1)} + \Pi_{\le n-2}.$$

Hence Π_n is $\lfloor \frac{n}{2} - 1 \rfloor$-good for $\Phi_{\le n-2}^{(1)} + \Pi_{\le n-2}$.

Proof of (3). Observe that since $\Phi^{(1)} = \Phi_{\partial\bar{\sigma}_1}\mathbf{c}$,

$$\Phi_k = \Phi_k^{(1)} + \Psi_k \text{ for } k < n$$

and

$$\Phi_n = \Pi_n.$$

We already proved that $\Phi_n = \Pi_n$ is $\lfloor \frac{n}{2} - 1 \rfloor$-good for $\Phi_{\le n-2}$ in the proof of (2). Suppose $k < n$. Since $\Phi^{(1)} = \Phi_{\partial\bar{\sigma}_1}\mathbf{c}$, by the induction hypothesis (use (3)), $\Phi_k^{(1)}$ is $\lfloor \frac{k}{2} - 1 \rfloor$-good for $\Phi_{\le k-2}^{(1)}$. Since $\Phi_{\le k-2}^{(1)} \le \Phi_{\le k-2}$ and since we already proved that $\Psi_k = \Pi_k$ is $\lfloor \frac{k}{2} - 1 \rfloor$-good for $\Phi_{\le k-2}$ in the proof of (2), Φ_k is $\lfloor \frac{k}{2} - 1 \rfloor$-good for $\Phi_{\le k-2}$.

Proof of (4). This statement easily follows from (3). For $k = 0, 1$, the statement is obvious (as $\Phi_{\le 0} = \Phi_{\le 1} = \mathbf{c}^n$). Suppose that $\Phi_{\le 2m+1}$ is m-FFK, where $m \in \mathbb{Z}_{\ge 0}$. Then both Φ_{2m+2} and Φ_{2m+3} are m-good for $\Phi_{\le 2m+1}$ by (3), and therefore $\Phi_{\le 2m+2}$ and $\Phi_{\le 2m+3}$ are $(m+1)$-FFK by Lemma 3.3. □

4. γ-vectors of polytopes and a conjecture on the cd-index

γ-vectors and the cd-index. Let Δ be an $(n-1)$-dimensional simplicial complex. Then the h-vector $h(\Delta) = (h_0, h_1, \dots, h_n)$ of Δ is defined by the relation

$$\sum_{i=0}^{n} h_i x^{n-i} = \sum_{i=0}^{n} f_{i-1}(\Delta)(x-1)^{n-i}.$$

If Δ is a simplicial sphere (that is, a triangulation of a sphere), or more generally a homology sphere, then $h_i = h_{n-i}$ for all i by the Dehn-Sommerville equations, and in this case the γ-vector $(\gamma_0, \gamma_1, \dots, \gamma_{\lfloor \frac{n}{2} \rfloor})$ of Δ is defined by the relation

$$\sum_{i=0}^{n} h_i x^i = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_i x^i (1+x)^{n-2i}.$$

It was conjectured by Gal [Ga] that if Δ is a flag homology sphere then its γ-vector is non-negative. Recently Nevo and Peterson [NP] further conjectured that the γ-vector of a flag homology sphere is the f-vector of a balanced simplicial complex. These conjectures are open in general, the latter conjecture was verified for barycentric subdivisions of simplicial homology spheres [NPT], and Gal's conjecture is known to be true for barycentric subdivisions of regular CW-spheres by the following fact, combined with Karu's result on the nonnegativity of the **cd**-index for Gorenstien* posets:

Let P be an $(n-1)$-dimensional regular CW-sphere. The *barycentric subdivision* $\mathrm{sd}(P)$ of P is the order complex of $\mathcal{F}(P)$. Let $(h_0,h_1,\dots,h_n)$ and $(\gamma_0,\gamma_1,\dots,\gamma_{\lfloor\frac{n}{2}\rfloor})$ be the h-vector and γ-vector of $\mathrm{sd}(P)$, respectively. Then it is easy to see that $h_i=\sum_{S\subset[n],\ |S|=i} h_S(P)$. Thus if $\Phi_P(1,\mathbf{d})=\delta_0+\delta_1\mathbf{d}+\delta_2\mathbf{d}^2+\cdots+\delta_{\lfloor\frac{n}{2}\rfloor}\mathbf{d}^{\lfloor\frac{n}{2}\rfloor}$, then for all $i\geq 0$,

$$\gamma_i=2^i\delta_i.$$

Since δ_i is non-negative, we conclude that γ_i is also non-negative.

The next simple statement, combined with Theorem 1.1, proves Theorem 1.2.

Lemma 4.1. *With the same notation as above, if $(\delta_0,\delta_1,\dots,\delta_{\lfloor\frac{n}{2}\rfloor})$ is k-FFK then $(\gamma_0,\gamma_1,\dots,\gamma_{\lfloor\frac{n}{2}\rfloor})$ is also k-FFK.*

Proof. Let Δ be a k-colored simplicial complex on the vertex set V with $f_{i-1}(\Delta)=\delta_i$ for all $i\geq 0$ and let $c:V\to[k]$ be a k-coloring map of Δ. Consider a collection of subsets of $W=\{x_v : v\in V\}\cup\{y_v : v\in V\}$

$$\hat{\Delta}=\{x_G\cup y_{F\setminus G} : F\in\Delta,\ G\subset F\},$$

where $x_H=\{x_v : v\in H\}$ and $y_H=\{y_v : v\in H\}$ for any $H\subset V$. Then $\hat{\Delta}$ is a simplicial complex with $f_{i-1}(\hat{\Delta})=2^i f_{i-1}(\Delta)=\gamma_i$ for all i. The map $\hat{c}:W\to[k]$, $\hat{c}(x_v)=\hat{c}(y_v)=c(v)$, shows that $\hat{\Delta}$ is k-colored. □

Proof of Corollary 1.3. By Theorem 1.2, in order to prove Corollary 1.3 it is enough to show that $\delta_{\lfloor\frac{n}{2}\rfloor}(P)>0$ where P is the boundary complex of an n-polytope. Billera and Ehrenborg showed that the **cd**-index of n-polytopes is minimized (coefficientwise) by the n-simplex, denoted σ^n [BE]. Thus, it is enough to verify that $\delta_{\lfloor\frac{n}{2}\rfloor}(\sigma^n)>0$. It is known that *all* the **cd**-coefficients of σ^n are positive (e.g., by using the Ehrenborg-Readdy formula for the **cd**-index of a pyramid over a polytope [ER, Theorem 5.2]). □

A conjecture on the cd-index. It would be natural to ask if Theorems 1.1 and 1.2 hold for all regular CW-spheres (or all Gorenstein* posets). We phrase a conjecture on the **cd**-index, that, if true, immediately implies Theorem 1.1, as well as the entire Proposition 3.4(4).

For an arbitrary **cd**-monomial $w=\mathbf{c}^{s_0}\mathbf{d}\mathbf{c}^{s_1}\mathbf{d}\cdots\mathbf{d}\mathbf{c}^{s_k}$ of degree n (where $0\leq s_i$ for all i and $s_0+\cdots+s_k+2k=n$), let F_w be the following subset of $[n-1]$:

$$F_w=\{s_0+1,s_0+s_1+3,s_0+s_1+s_2+5,\dots,s_0+\cdots+s_{k-1}+2k-1\}.$$

Note that F_w contains no two consecutive numbers. For example, $F_{\mathbf{c}^n}=\emptyset$, $F_{\mathbf{d}^k}=\{1,3,\dots,2k-1\}$ and $F_{\mathbf{c}\mathbf{d}^k}=\{2,4,\dots,2k\}$. Let $\mathcal{A}$ be the set of subsets of $[n-1]$ that have no two consecutive numbers, and let $\mathcal{B}$ be the set of **cd**-monomials of degree n.

Then $w \mapsto F_w$ is a bijection from $\mathcal{B}$ to $\mathcal{A}$ (as $k = |F_w|$ and $s_k = n-2k-s_{k-1}-\cdots-s_0$ we see that the inverse map exists).

Let Δ be a k-colored simplicial complex with the vertex set V and a k-coloring map $c : V \to [k]$. For any subset $S \subset [k]$, let $f_S(\Delta) = |\{F \in \Delta : \ c(F) = S\}|$. The vector $(f_S(\Delta) : S \subset [k])$ is called the *flag f-vector of* Δ. Note that the flag f-vector of a poset P is equal to the flag f-vector of $\mathrm{sd}(P)$ by the coloring map defined by the rank function.

Definition 4.2. Let $\Phi = \sum_w a_w w$ be a homogeneous **cd**-polynomial of degree n with w the **cd**-monomials and $a_w \in \mathbb{Z}$. For $S \subset [n-1]$, we define

$$\alpha_S(\Phi) = \begin{cases} a_w, & \text{if } S = F_w \text{ for some } w \in \mathcal{B} \\ 0, & \text{if } S \notin \mathcal{A}. \end{cases}$$

Conjecture 4.3. Let P be an $(n-1)$-dimensional regular CW-sphere (or more generally, Gorenstein* poset of rank $n+1$). Then there exists an $(n-1)$-colored simplicial complex Δ such that $f_S(\Delta) = \alpha_S(\Phi_P)$ for all $S \subset [n-1]$.

Thus the above conjecture states that the **cd**-index is itself the flag f-vector of a colored complex. If the above conjecture is true then $\Phi_P(1, \mathbf{d}) = 1 + f_0(\Delta)\mathbf{d} + \cdots + f_{\lfloor \frac{n}{2} \rfloor -1}(\Delta)\mathbf{d}^{\lfloor \frac{n}{2} \rfloor}$. Although Δ is $(n-1)$-colored, this fact implies Theorem 1.1. Indeed, since $f_S(\Delta) = \alpha_S(\Phi_P) = 0$ if S has consecutive numbers, if $c : V \to [n-1]$ is an $(n-1)$-coloring map of Δ then the map $\hat{c} : V \to [\lfloor \frac{n}{2} \rfloor]$ defined by $\hat{c}(v) = \lfloor \frac{c(v)+1}{2} \rfloor$ is an $\lfloor \frac{n}{2} \rfloor$-coloring map of Δ.

The next result supports the conjecture in low dimension.

Proposition 4.4. *Let P be a Gorenstein* poset of rank $n+1$. For all $i, j \in [n-1]$,*

$$\alpha_{\{i\}}(\Phi_P)\alpha_{\{j\}}(\Phi_P) \geq \alpha_{\{i,j\}}(\Phi_P).$$

Proof. Let $(h_S(P) : S \subset [n])$ be the flag h-vector of P. Let $\{i, i+j\} \subset [n-1]$ with $j \geq 2$. What we must prove is $\alpha_{\{i\}}(\Phi_P)\alpha_{\{i+j\}}(\Phi_P) \geq \alpha_{\{i,i+j\}}(\Phi_P)$.

Observe that

$$\begin{aligned} h_{[i]\cup\{i+j+1,\dots,n\}}(P) &= \alpha_{\{i,i+j\}}(\Phi_P) + \alpha_{\{i\}}(\Phi_P) + \alpha_{\{i+j\}}(\Phi_P) + \alpha_{\emptyset}(\Phi_P), \\ h_{[i]}(P) &= \alpha_{\{i\}}(\Phi_P) + \alpha_{\emptyset}(\Phi_P), \\ h_{\{i+j+1,\dots,n\}}(P) &= \alpha_{\{i+j\}}(\Phi_P) + \alpha_{\emptyset}(\Phi_P) \end{aligned}$$

(as $h_{[i]\cup\{i+j+1,\dots,n\}}(P)$ is the coefficient of $\mathbf{b}^i\mathbf{a}^j\mathbf{b}^{n-i-j}$ in $\Psi_P(\mathbf{a}, \mathbf{b})$, etc.). Since $\alpha_{\emptyset} = 1$, it is enough to prove that

$$h_{[i]}(P)h_{\{i+j+1,\dots,n\}}(P) \geq h_{[i]\cup\{i+j+1,\dots,n\}}(P).$$

It follows from [St2, III, Theorem 4.6] that there is an n-colored simplicial complex Δ with a coloring map $c : V \to [n]$ such that $f_S(\Delta) = h_S(P)$ for all $S \subset [n]$, where V is the vertex set of Δ. Let

$$\Delta_S = \{F \in \Delta : c(F) = S\}$$

for $S \subset [n]$. Then it is clear that

$$\Delta_{[i]\cup\{i+j+1,\dots,n\}} \subset \{F \cup G : F \in \Delta_{[i]},\ G \in \Delta_{\{i+j+1,\dots,n\}}\},$$

which implies the desired inequality. ∎

It is straightforward that the above proposition proves the next statement.

Corollary 4.5. *Conjecture 4.3 holds for $n \le 5$.*

Non-existence of d-polynomials. For a Gorenstein* poset P, we call $\Phi_P(1,\mathbf{d})$ *the* $\mathbf{d}$*-polynomial of* P. It is a challenging problem to classify all possible $\mathbf{d}$-polynomials of Gorenstein* posets, which give a complete characterization of all possible face vectors of Gorenstein* order complexes since knowing $\mathbf{d}$-polynomials is equivalent to knowing γ-vectors. The problem is open even for the 3-dimensional case. To study this problem, by virtue of Theorem 1.1, it is natural to ask which FFK vector is realizable as the $\mathbf{d}$-polynomial of a Gorenstein* poset. We show that not all $\lfloor \frac{n}{2} \rfloor$-FFK vectors are realizable as the $\mathbf{d}$-polynomial of a Gorenstein* poset of rank $n+1$.

First recall that the ordinal sum $Q_1 + Q_2$ of two disjoint posets Q_1 and Q_2 is the poset whose elements are the union of elements in Q_1 and Q_2 and whose relations are those in Q_1 union those in Q_2 union all $q_1 < q_2$ where $q_1 \in Q_1$ and $q_2 \in Q_2$. For Gorenstein* posets Q_1 and Q_2, the poset $Q_1 * Q_2 = (Q_1 - \{\hat{1}\}) + (Q_2 - \{\hat{0}\})$ is called the *join* of Q_1 and Q_2, and $\Sigma Q_1 = Q_1 * B_2$, where B_2 is a Boolean algebra of rank 2, is called the *suspension* of Q_1. By [St1, Lemma 1.1], $\Phi_{Q_1 * Q_2}(\mathbf{c},\mathbf{d}) = \Phi_{Q_1}(\mathbf{c},\mathbf{d}) \cdot \Phi_{Q_2}(\mathbf{c},\mathbf{d})$.

Proposition 4.6. *Let P be a Gorenstein* poset of rank 5, and let*

$$\Phi_P(\mathbf{c},\mathbf{d}) = \mathbf{c}^4 + \alpha_{\{1\}}\mathbf{dc}^2 + \alpha_{\{2\}}\mathbf{cdc} + \alpha_{\{3\}}\mathbf{c}^2\mathbf{d} + \alpha_{\{1,3\}}\mathbf{d}^2$$

be its $\mathbf{cd}$-index. Suppose $\alpha_{\{2\}} = 0$. Then there are Gorenstein posets P_1 and P_2 of rank 3 such that $P = P_1 * P_2$. In particular, $\alpha_{\{1,3\}} = \alpha_{\{1\}}\alpha_{\{3\}}$.*

Proof. Let r denote the rank function $r : P \to \{0,1,\dots,5\}$ ($r(\hat{0}) = 0, r(\hat{1}) = 5$). Let $P_1 := \{F \in P : \ r(F) \le 2\}$ and $P_2 := \{F \in P : \ r(F) \ge 3\}$.

As P is Gorenstien*, to show that $P = P_1 + P_2$ it is enough to show that $P_2 \cup \{\hat{0}\}$ is Gorenstien* (as a Gorenstien* poset contains no proper subposet which is Gorenstien* of the same rank, and each interval $[F,\hat{1}]$ with $r(F) = 2$ in P is Gorenstien*). For this, it is enough to show that any rank 4 element in P covers exactly two rank 3 elements in P. Indeed, this guarantees that the dual poset to P_2, denoted P_2^*, is the face poset of a union of CW 1-spheres, and as P is Gorenstien* so is its dual P^*, hence P_2^* is Cohen-Macaulay since P_2^* is a rank selected poset [St2, III, Theorem 4.5], which implies that P_2^* is the face poset of one CW 1-sphere, i.e. $P_2 \cup \{\hat{0}\}$ is Gorenstien*.

Let F be a rank 4 element of P. Then P is a subdivision of $\Sigma([\hat{0},F])$ (Recalling [EK, Definition 2.6], this is shown by the map $\phi : P \to \Sigma([\hat{0},F])$, $\phi(\sigma) = \sigma$ if $\sigma < F$, $\phi(\sigma) = \sigma_1$ if σ and F are incomparable, and $\phi(F) = \sigma_2$, where σ_1, σ_2 are the rank 4 elements in $\Sigma([\hat{0},F])$). Thus, by Lemma 2.4, the coefficient of $\mathbf{cdc}$ in the $\mathbf{cd}$-index of $\Sigma([\hat{0},F])$ is zero, hence the coefficient of the monomial $\mathbf{cd}$ in the $\mathbf{cd}$-index of $[\hat{0},F]$ is zero.

This fact implies, when expanding the $\mathbf{cd}$-index of $[\hat{0},F]$ in terms of $\mathbf{a},\mathbf{b}$, that $h_{\{3\}}([\hat{0},F])$ equals the coefficient of $\mathbf{c}^3$, namely $h_{\{3\}}([\hat{0},F]) = 1$. Switching to the flag f-vector of $[\hat{0},F]$ we get $f_{\{3\}}([\hat{0},F]) = h_\emptyset([\hat{0},F]) + h_{\{3\}}([\hat{0},F]) = 1 + 1 = 2$. Thus, F covers exactly two rank 3 elements in P. □

Example 4.7. Consider the 2-FFK vector $(1,6,7)$. We claim that $\Phi_P(1,\mathbf{d}) \neq 1+6\mathbf{d}+7\mathbf{d}^2$ for all Gorenstein* poset P of rank 5. Indeed, if $\Phi_P(1,\mathbf{d}) = 1+6\mathbf{d}+7\mathbf{d}^2$, then $\alpha_{\{1,3\}} = 7$. Then $\alpha_{\{1\}} + \alpha_{\{3\}} = 6$ and $\alpha_{\{2\}} = 0$ by Proposition 4.4, which contradicts Proposition 4.6.

A similar argument shows that $(1, 2a, a^2 - 2)$, where $a \geq 3$, is 2-FFK, but not realizable as the **d**-polynomial of a Gorenstein* poset of rank 5.

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