A variational problem for pullback metrics

(引き戻した計量に関する変分問題)

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Abstract

Variational problems on Riemannian manifolds have been playing a significant role in geometry and analysis. In this thesis, we give a research on a variational problem from a viewpoint of *pullback metrics.*

Let M and N be Riemannian manifolds without boundary, and let g and h be Riemannian metrics of M and N respectively. Let f be a smooth map from M into N and df be its differential map.

We consider a metric pulled back by the map f. We call it a pullback metric and denote it by f^*h. The pullback metric f^*h is defined by

$$(f^*h)(X, Y) = h\big(df(X),\, df(Y)\big),$$

where X and Y are vector fields on M. The pullback metric f^*h is a $(0,2)$-type tensor and it is natural to take its *trace* and *norm.*

The *trace* of the pullback metric f^*h is given by

$$\begin{aligned}\mathrm{Tr}_g(f^*h) &:= \sum_i h(df(e_i), df(e_i)) \\ &= \|df\|^2,\end{aligned}$$

where $\{e_i\}$ is a local orthonormal frame on M. A critical point of the energy functional

$$E(f) = \int_M \|df\|^2 dv_g$$

is called a harmonic map, where dv_g is the volume form on M. The research on harmonic maps originated with Eells and Sampson. The theory of harmonic maps has been making tremendous progress in the last fifty years. Many researches on harmonic maps have brought interesting results, not only of the properties of harmonic maps themselves but also of the applications of them. Well-known important examples of harmonic maps are geodesics, minimal surfaces, harmonic functions, and so on.

We pay attention to the *norm* of the pullback metric

$$\|f^*h\|^2 = \sum_{i,j} h\big(df(e_i),\, df(e_j)\big)^2,$$

and consider the functional

$$\Phi(f) = \int_M \|f^*h\|^2\, dv_g\,.$$

Generally speaking, the norm contains more information than the trace does. This doctoral thesis deals with variational problems for the functional Φ.

We give a summary of the contents of this thesis.

In Chapter 0, we give a brief review of background materials for this thesis with some notations. To describe our results, we need some basic notions in differential geometry, analysis and global analysis. We also give a lemma which is used in our argument. We define a $(0,1)$-type tensor σ_f by

$$\sigma_f(X) = \sum_i h(df(X), df(e_i))df(e_i)\,.$$

The tensor σ_f plays an important role in our arguments.

In Chapter 1, we give a first variation formula for the functional Φ and introduce the notion of stationary maps. By the first variation formula, we get the Euler-Lagrange equation

$$\mathrm{div}_g \sigma_f \;=\; 0\,,$$

where $\mathrm{div}_g \sigma_f$ denotes the divergence of σ_f. For any smooth map f, we call f a *stationary map* if its first variation vanishes, that is, it satisfies the Euler-Lagrange equation. The notion of stationary maps is a central theme in this thesis.

In Chapter 2, we give some examples of stationary maps. Geodesics, 4-harmonic functions and isometric harmonic maps are stationary maps. They illustrate the class of stationary maps, and show that this class contains many important examples.

In Chapter 3, we give a second variation formula. The second variation formula contains the term of curvature tensor of N. This formula is used in the argument of the stability of stationary maps.

In Chapter 4, we prove the existence of minimizers of the functional Φ in each 3-homotopy class of the Sobolev space $\mathrm{L}^{1,4}(M, N)$. We say that two maps in $\mathrm{L}^{1,4}(M, N)$ are 3-homotopic if they are homotopic on the three dimensional skeletons of a triangulation of M. By the results of White, we see that the 3-homotopy is well-defined on $\mathrm{L}^{1,4}(M, N)$. We give a proof of this existence theorem by using this fact.

In Chapter 5, we give a monotonicity formula for stationary map. We prove this formula under a weaker condition. Monotonicity formulas are utilized in the regularity theory of solutions of variational problems.

In Chapter 6, we give a Bochner type formula. The Bochner type formula contains the Ricci curvature of M and the curvature tensor of N. Bochner type formulas play an important role in the Bochner technique.

In Chapter 7, in the case that the target manifold is a Lie group with bi-invariant metric, we describe the Euler-Lagrange equation through the Maurer-Cartan form.

要旨

リーマン多様体上の変分問題は, 幾何学や解析学において, 重要な役割を担ってきた. 本論文では, pullback metric の観点から, 変分問題についての研究を行う.

M と N を境界なしのリーマン多様体とし, g と h をそれぞれ M と N のリーマン計量とする. f を M から N への滑らかな写像とし, df をその微分写像とする.

f によって引き戻した計量を考える. それを pullback metric と呼び, f^*h と表す. Pullback metric f^*h は,

$$(f^*h)(X, Y) = h\big(df(X), df(Y)\big)$$

により定義される. ただし, X と Y は, M 上のベクトル場とする. Pullback metric f^*h は, $(0,2)$-型テンソルであるので, その**トレース**と**ノルム**を考えるのが自然である.

Pullback metric f^*h の **トレース** は,

$$\begin{aligned}\mathrm{Tr}_g(f^*h) &:= \sum_i h(df(e_i), df(e_i)) \\ &= \|df\|^2\end{aligned}$$

で与えられる. ただし, $\{e_i\}$ は M の local orthonormal frame とする. エネルギー汎関数

$$E(f) = \int_M \|df\|^2 dv_g$$

の臨界点は, 調和写像と呼ばれる. ただし, dv_g は M の体積要素とする. 調和写像の研究は, Eells と Sampson により始められた. 調和写像理論は, この 50 年間で非常に大きな進歩を遂げている. 調和写像に関する多くの研究が, 調和写像そのものの性質に関してだけではなく, 応用に関しても興味深い結果をもたらした. よく知られている調和写像の重要な例としては, 測地線, 極小曲面, 調和関数などが挙げられる.

本論文では, pullback metric の **ノルム**

$$\|f^*h\|^2 = \sum_{i,j} h\big(df(e_i), df(e_j)\big)^2$$

に着目し, 汎関数

$$\Phi(f) = \int_M \|f^*h\|^2 \, dv_g$$

を考察する.

一般的に言って, ノルムはトレースより多くの情報を含む. 本論文では, 汎関数 Φ に関する変分問題を扱う.

各章の内容の要約を述べる.

第 0 章では, 本論文の背景となる基本的事項を記号とともに簡単に説明する. 得られた結果を述べるにあたり, 微分幾何学, 解析学, 大域解析学における基本的概念が必要となる. また, 本論文の議論でよく用いる補題を与える. さらに, $(0,1)$-型テンソル σ_f を

$$\sigma_f(X) = \sum_i h(df(X), df(e_i))df(e_i)$$

により定義する. このテンソル σ_f は, 本論文の議論において重要な役割を果たす.

第 1 章では, 汎関数 Φ に対する第一変分公式を与え, stationary map の概念を導入する. 第一変分公式より, Euler-Lagrange 方程式

$$\mathrm{div}_g \sigma_f \ = \ 0$$

が得られる. ただし, $\mathrm{div}_g \sigma_f$ は, σ_f の divergence を表している. 本論文では, 写像 f の第一変分がゼロになるとき, f が *stationary map* であると言う. 言い換えると, f が上記の Euler-Lagrange 方程式を満たすことである. Stationary map の概念は, 本論文の中心的主題である.

第 2 章では, stationary map の例をいくつか与える. 測地線, 4-調和関数, 等長調和写像は, stationary map である. これらは, stationary map のクラスがどのようなものかを表し, このクラスに多くの重要な例が含まれていることを示す.

第 3 章では, 第二変分公式を与える. 第二変分公式には, N の曲率項があらわれる. 第二変分公式は, stationary map の安定性を議論するときに用いられる.

第 4 章では, Sobolev 空間 $\mathrm{L}^{1,4}(M, N)$ の各 3-ホモトピークラスにおける汎関数 Φ の最小解の存在を証明する. $\mathrm{L}^{1,4}(M, N)$ の 2 つの写像が 3-ホモトピックであるとは, それらが M の三角形分割の 3 次元骨格上でホモトピックであるときに言う. White の結果より, 3-ホモトピーが $\mathrm{L}^{1,4}(M, N)$ 上, well-defined であることが言える. この事実を用いて, 最小解の存在定理の証明を与える.

第 5 章では, stationary map に対して, monotonicity formula を与える. この formula をより弱い仮定の下で証明する. Monotonicity formula は, 変分問題の解の regularity theory に使われる.

第6章では, Bochner type formula を与える. Bochner type formula には, M の Ricci 曲率や, N の曲率テンソルが含まれている. Bochner type formula は, Bochner technique において重要な役割を果たす.

第7章では, target manifold が両側不変計量を持つ Lie 群の場合に, Euler-Lagrange 方程式を Maurer-Cartan form を通して記述する.

Contents

Basic notations

Throughout this thesis, we use the following notations:

M, N : Riemannian manifolds

g, h : Riemannian metrics of M and N respectively

f : a smooth map from M into N

$\{e_i\}$: a local orthonormal frame on M

X, Y, Z : vector fields on M

U, V, W : vector fields on N

Introduction

In this thesis, we consider the functional

$$\Phi(f) = \int_M \|f^*h\|^2 \, dv_g \,,$$

where dv_g is the volume form on a Riemannian manifold M.

First of all, we give the definition of this functional Φ more precisely. Let M and N be Riemannian manifolds without boundary, and let g and h be Riemannian metrics of M and N respectively. Let f be a smooth map from M into N and df be its differential map. The pullback metric f^*h is a $(0,2)$-type tensor defined by

$$(f^*h)(X, Y) = h\big(df(X),\, df(Y)\big)$$

for any smooth vector fields X and Y on M. We take the *norm* of the pullback metric

$$\|f^*h\|^2 = \sum_{i,j} h\big(df(e_i),\, df(e_j)\big)^2 \,,$$

where $\{e_i\}$ is a local orthonormal frame on M. The functional Φ defined by integrating $\|f^*h\|^2$ on M is the very functional which we consider.

We introduce the notion of harmonic maps, including the historical aspect. The notion of harmonic maps originated with Eells and Sampson. In their paper published in 1964, they proved that there exists a harmonic map between compact Riemannian manifolds if the sectional curvature of the target manifold is non-positive, by the heat equation method.

Many researchers have been researching on harmonic maps and reporting various results such as the existence, the uniqueness and the stability. Well-known important examples of harmonic maps are geodesics, minimal surfaces, harmonic functions, and so on. The research on minimal surfaces has made progress independently of that on harmonic maps. As the research on harmonic maps developed, the relation between harmonic maps and minimal surfaces became clear. If M is two-dimensional, under the assumption that f is conformal, f is harmonic if and only if it is minimal.

The application of harmonic maps is researched in various regions of research. Rigidity is known as an example of applications. An object is rigid, if the object is not deformed with keeping its properties. The rigidity theory has made remarkable progress by strong rigidity theorems by Siu. For further results on harmonic maps, see Eells-Lemaire [2] and [3].

A harmonic map is a critical point of

$$E(f) = \int_M \|df\|^2 dv_g,$$

where

$$\|df\|^2 = \sum_{i=1}^{m} h(df(e_i), df(e_i)),$$

and dv_g denotes the volume form on M. The functional E is called an energy functional in the theory of harmonic maps. Let f_t denote a compactly supported deformation of f. We get the first variation formula

$$\left.\frac{dE(f_t)}{dt}\right|_{t=0} = -2\int_M h(\tau_f, V)\, dv_g,$$

where V denotes the variation vector field, and τ_f is the tension field of f.

We see the energy functional E from a different viewpoint. The energy functional E is the integral of

$$\|df\|^2 = \sum_{i=1}^{m} h(df(e_i), df(e_i)),$$

which is the norm of the differential map df. From the viewpoint of the pullback, we regard $\|df\|^2$ as the *trace* of the pullback metric f^*h.

Generally speaking, the norm contains more information than the trace does. This doctoral thesis deals with variational problems for the functional Φ, which is an integral of the *norm*.

We introduce the notion of stationary maps. We define a $(0,1)$-type tensor σ_f by

$$\sigma_f(X) = \sum_i h(df(X), df(e_i))df(e_i).$$

The tensor σ_f plays an important role in our arguments. For the functional Φ, we get a first variation formula

$$\left.\frac{d\Phi(f_t)}{dt}\right|_{t=0} = -4\int_M h(\mathrm{div}_g\sigma_f, V)\, dv_g$$

for any compactly supported deformation f_t, where $\mathrm{div}_g\sigma_f$ denotes the divergence of σ_f. We call f a *stationary map* if its first variation vanishes. By the first variation formula, a smooth map f is a stationary map if and only if

$$\mathrm{div}_g\sigma_f = 0,$$

which is called the Euler-Lagrange equation for the functional Φ. The notion of stationary maps is a central theme in this thesis. Geodesics, 4-harmonic functions and isometric harmonic maps are stationary maps. The class of stationary maps contains important examples.

We give a second variation formula for the functional Φ. Let $f_{s,t}$ be a compactly supported deformation of f. Then we have

$$\begin{aligned}
\frac{1}{4}\frac{\partial^2\Phi(f_{s,t})}{\partial s\partial t}\bigg|_{s,t=0} &= -\int_M h\big(\mathrm{Hess}_f\big(\tfrac{\partial}{\partial s},\, \tfrac{\partial}{\partial t}\big),\, \mathrm{div}_g\sigma_f\big)dv_g \\
&+ \int_M \sum_{i,j} h\big(\nabla_{e_i}V,\, \nabla_{e_j}W\big)\, h(df(e_i),\, df(e_j))dv_g \\
&+ \int_M \sum_{i,j} h\big(\nabla_{e_i}V,\, df(e_j)\big)\, h\big(\nabla_{e_i}W,\, df(e_j)\big)dv_g \\
&+ \int_M \sum_{i,j} h\left(\nabla_{e_i}V,\, df(e_j)\right)\, h\left(df(e_i),\, \nabla_{e_j}W\right) dv_g \\
&- \int_M \sum_{i,j} h\big({}^N\!R\,(df(e_i),\, V)\,W,\, df(e_j)\big)h\big(df(e_i),\, df(e_j)\big)dv_g\,,
\end{aligned}$$

where V and W denote the variation vector fields, and Hess_f is the Hessian of f.

We prove that there exist minimizers in any 3-homotopy class. By Nash's isometric embedding, we may assume that N is a submanifold of a Euclidean space $\mathbb{R}^q$. Let

$$\mathrm{L}^{1,4}(M,\, N) \;=\; \{\; f \in \mathrm{L}^{1,4}(M,\, \mathbb{R}^q) \;|\; f(x) \in N \text{ a.e. } \}\,,$$

where $\mathrm{L}^{1,4}(M,\, \mathbb{R}^q)$ denotes the Sobolev space of $\mathbb{R}^q$-valued L^4-functions on M whose weak derivatives are in L^4. By Theorem 3.4 in White [11], we see the following properties.

(1) The 3-homotopy is well-defined for any map $f \in \mathrm{L}^{1,4}(M,\, N)$.
(2) If f_i converges weakly to f_∞ in $\mathrm{L}^{1,4}(M,\, N)$, then f_i and f_∞ are 3-homotopic for sufficiently large i.

The functional Φ is defined on $\mathrm{L}^{1,4}(M,\, N)$, in which the 3-homotopy is well-defined. Then we want to minimize the functional Φ in each 3-homotopy class, i.e., in the following class:

$$\mathcal{F}_{f_0} \;=\; \{\; f \in \mathrm{L}^{1,4}(M,\, N) \mid f \text{ is 3-homotopic to } f_0 \;\}$$

for any given continuous map f_0 from M into N. Under the assumption that M and N are compact, we conclude that there exists a minimizer of the functional Φ in $\mathcal{F}_{f_0}$.

We give a monotonicity formula for stationary maps. We prove this formula holds under the weaker condition that f is a stationary map with respect to diffeomorphisms on M, i.e., f satisfies

$$\frac{d}{dt}\Phi(f \circ \varphi_t)\bigg|_{t=0} \;=\; 0$$

for any compactly supported 1-parameter family φ_t of diffeomorphisms on M. Under this weaker condition, we give the following monotonicity formula.

Fix $x_0 \in M$ and let $\rho > 0$. Let f be a stationary map with respect to diffeomorphisms on M. Then it satisfies

$$\frac{d}{d\rho}\left\{ e^{C\rho}\rho^{4-m} \int_{B_\rho(x_0)} \|f^*h\|^2 dv_g \right\} \;\geq\; 0\,,$$

where C is a constant and $B_\rho(x_0)$ is the ball of radius ρ centered at x_0.

Let f be a smooth map from M into N. Then the Bochner type formula is as follows:

$$\begin{aligned}
\frac{1}{4}\triangle\,\|f^*h\|^2 \;&=\; \mathrm{div}_g\alpha_f \;-\; h\big(\tau_f,\, \mathrm{div}_g\sigma_f\big) \;+\; \frac{1}{2}\|\nabla(f^*h)\|^2 \\
&+ \sum_{i,j,k} h\big(\nabla df(e_k,\, e_i),\, \nabla df(e_k,\, e_j)\big)h\big(df(e_i),\, df(e_j)\big) \\
&+ \sum_{i,j} h\big(df\big({}^M\!R(e_i,\, e_j)\, e_j\big),\, \sigma_f(e_i)\big) \\
&- \sum_{i,j} h\big({}^N\!R\big(df(e_i),\, df(e_j)\big)df(e_j),\, \sigma_f(e_i)\big)\,,
\end{aligned}$$

where

$$\alpha_f(X) \;=\; h\big(\sigma_f(X),\, \tau_f\big)\,.$$

Bochner type formulas play an important role in the Bochner technique. The Bochner technique is useful to research on properties of manifolds.

In the case that the target manifold is a Lie group with bi-invariant metric, we describe the Euler-Lagrange equation through the Maurer-Cartan form.

序論

本論文では, 汎関数

$$\Phi(f) = \int_M \|f^*h\|^2 \, dv_g$$

を考察する. ただし, dv_g をリーマン多様体 M の体積要素とする.

初めに, 汎関数 Φ のより正確な定義を与える. M と N を境界なしのリーマン多様体とし, g と h をそれぞれ M と N のリーマン計量とする. f を M から N への滑らかな写像とし, df をその微分写像とする. Pullback metric f^*h は, M 上の滑らかなベクトル場 X と Y に対し,

$$(f^*h)(X, Y) = h\big(df(X), df(Y)\big)$$

と定義される $(0, 2)$-型テンソルである. Pullback metric の **ノルム** をとると,

$$\|f^*h\|^2 = \sum_{i,j} h\big(df(e_i), df(e_j)\big)^2$$

となる. ただし, $\{e_i\}$ は, M の local orthonormal frame とする. $\|f^*h\|^2$ を M 上で積分することにより定義される汎関数 Φ は, まさに本論文において考察する汎関数である.

調和写像理論を, 歴史的側面も含めて紹介する. 調和写像の研究は, Eells と Sampson により始められた. 彼らは, 1964 年に発表された論文において, target manifold の断面曲率が非正のとき, リーマン多様体間の調和写像が存在することを, 熱方程式の方法を用いて証明した.

多くの研究者が調和写像の研究を行っており, 調和写像の存在や一意性, また, 安定性などの様々な結果が報告されている. よく知られている調和写像の重要な例としては, 測地線, 極小曲面, 調和関数などが挙げられる. 一方, 極小曲面の研究は, 調和写像とは独立に発展してきた. 調和写像の研究が進むにつれて, 極小曲面との関連性が明らかになってきた. M が 2 次元のとき, f が共形という仮定の下では, f が調和であることと極小であることは, 同値である.

調和写像の応用は, 様々な研究分野において, 研究されている. 剛性は, 応用例の一つとして知られている. 対象が剛性であるとは, その対象が性質を保ったまま変形できないときに言う. Siu による強剛性定理により, 剛性理論が大きく発展した. 調和写像に関するさらなる結果は, Eells-Lemaire [2], [3] を参照せよ.

調和写像とは,

$$E(f) = \int_M \|df\|^2 dv_g$$

の臨界点のことである．ここで，

$$\|df\|^2 = \sum_{i=1}^{m} h(df(e_i), df(e_i))$$

で，dv_g は，M の体積要素を表している．調和写像理論において，汎関数 E は，エネルギー汎関数と呼ばれている．コンパクトなサポートを持つ変分 f_t に対して，第一変分公式

$$\left.\frac{dE(f_t)}{dt}\right|_{t=0} = -2\int_M h\left(\tau_f,\, V\right)\, dv_g$$

が得られる．ここで，V は変分ベクトル場であり，また，τ_f は f のテンション場である．

エネルギー汎関数 E を異なる観点から見る．エネルギー汎関数 E は，微分写像 df のノルム

$$\|df\|^2 = \sum_{i=1}^{m} h(df(e_i), df(e_i))$$

の積分である．引き戻しの観点から見ると，$\|df\|^2$ を pullback metric の トレース と見ることができる．

一般的に言って，ノルムはトレースより多くの情報を含む．本論文では，pullback metric の**ノルム** の積分である汎関数 Φ に関する変分問題を扱う．

Stationary map の概念を導入する．$(0,1)$-型テンソル σ_f を

$$\sigma_f(X) = \sum_i h\big(df(X),\, df(e_i)\big)df(e_i)$$

と定義する．本論文の議論において，テンソル σ_f は重要な役割を果たす．汎関数 Φ に対する第一変分公式は，以下のようになる．コンパクトなサポートを持つ変分 f_t に対して，

$$\left.\frac{d\Phi(f_t)}{dt}\right|_{t=0} = -4\int_M h\left(\mathrm{div}_g\sigma_f,\, V\right)\, dv_g$$

が成り立つ．ここで，$\mathrm{div}_g\sigma_f$ は，σ_f の divergence を表している．第一変分公式がゼロになるとき，f を *stationary map* と呼ぶ．第一変分公式より，滑らかな写像 f が stationary map であることと，f が方程式

$$\mathrm{div}_g\sigma_f = 0$$

を満たすことは同値であることが言える．この方程式は, 汎関数 Φ の Euler-Lagrange 方程式である．Stationary map の概念は, 本論文の中心的主題である．測地線, 4-調和関数, 等長調和写像は, stationary map である．このように, stationary map のクラスは重要な例を含んでいる．

汎関数 Φ の第二変分公式を与える．コンパクトなサポートを持つ変分 $f_{s,t}$ に対して,

$$
\begin{aligned}
\frac{1}{4}\left.\frac{\partial^2\Phi(f_{s,t})}{\partial s\partial t}\right|_{s,t=0} &= -\int_M h\big(\mathrm{Hess}_f\big(\tfrac{\partial}{\partial s},\tfrac{\partial}{\partial t}\big),\ \mathrm{div}_g\sigma_f\big)dv_g \\
&+ \int_M \sum_{i,j} h\big(\nabla_{e_i}V,\ \nabla_{e_j}W\big)\, h(df(e_i),\ df(e_j))dv_g \\
&+ \int_M \sum_{i,j} h\big(\nabla_{e_i}V,\ df(e_j)\big)\, h\big(\nabla_{e_i}W,\ df(e_j)\big)dv_g \\
&+ \int_M \sum_{i,j} h\left(\nabla_{e_i}V,\ df(e_j)\right)\, h\left(df(e_i),\ \nabla_{e_j}W\right)dv_g \\
&- \int_M \sum_{i,j} h\big({}^N\!R\left(df(e_i),\ V\right)W,\ df(e_j)\big)h\big(df(e_i),\ df(e_j)\big)dv_g
\end{aligned}
$$

が成り立つ．ここで, V と W は変分ベクトル場であり, また, Hess_f は f の Hessian を表している．

各 3-ホモトピークラスにおける最小解の存在を証明する．Nash の等長埋め込みにより, N はユークリッド空間 $\mathbb{R}^q$ の部分多様体と仮定することができる．このとき,

$$
\mathrm{L}^{1,4}(M,\ N) \ = \ \{\ f \in \mathrm{L}^{1,4}(M,\ \mathbb{R}^q) \mid f(x) \in N \text{ a.e. } \}
$$

とする．ただし, $\mathrm{L}^{1,4}(M, \mathbb{R}^q)$ は, M 上の $\mathbb{R}^q$-値 L^4-関数で, その関数の弱微分が L^4 に入っているような Sobolev 空間を表している．White [11] の Theorem 3.4 により,

(1) 3-ホモトピーは, 任意の写像 $f \in \mathrm{L}^{1,4}(M,\ N)$ に対して, well-defined である．
(2) $\mathrm{L}^{1,4}(M,\ N)$ で f_i が f_∞ に弱収束するならば, 十分大きな i に対して f_i と f_∞ は, 3-ホモトピックである．

ということが分かる．汎関数 Φ は $\mathrm{L}^{1,4}(M,\ N)$ 上で定義されるが, そこでは, 3-ホモトピーが well-defined である．各 3-ホモトピークラスで, 汎関数 Φ の最小解を見つけたい．3-ホモトピークラスとは,

$$
\mathcal{F}_{f_0} \ = \ \left\{\ f \in \mathrm{L}^{1,4}(M,\ N) \mid f \text{ is 3-homotopic to } f_0\ \right\}
$$

というクラスである．ただし, f_0 は, M から N への連続写像とする．M と N はコンパクトとすると, $\mathcal{F}_{f_0}$ 内に汎関数 Φ の最小解が存在する．

Stationary map に対して, monotonicity formula を与える．f が M の diffeomorphism に関する stationary map というより弱い条件, すなわち, M 上の任意のコンパクトなサポートを持つ 1-parameter family φ_t に対して,

$$\frac{d}{dt}\Phi(f \circ \varphi_t)\bigg|_{t=0} = 0$$

を満たすという条件の下で, 以下のような monotonicity formula が成り立つことを証明する．

$x_0 \in M$ を固定し, $\rho > 0$ とする．f が M の diffeomorphism に関する stationary map とする．このとき, f は,

$$\frac{d}{d\rho}\left\{ e^{C\rho}\rho^{4-m}\int_{B_\rho(x_0)} \|f^*h\|^2 dv_g \right\} \geq 0$$

を満たす．ただし, C は定数で, $B_\rho(x_0)$ は, x_0 を中心とした半径 ρ の球である．

f を M から N への滑らかな写像とする．このとき, Bochner type formula は,

$$\begin{aligned}
\frac{1}{4}\triangle\|f^*h\|^2 &= \mathrm{div}_g\alpha_f - h\big(\tau_f, \mathrm{div}_g\sigma_f\big) + \frac{1}{2}\|\nabla(f^*h)\|^2 \\
&\quad + \sum_{i,j,k} h\big(\nabla df(e_k, e_i), \nabla df(e_k, e_j)\big) h\big(df(e_i), df(e_j)\big) \\
&\quad + \sum_{i,j} h\big(df\big({}^M R(e_i, e_j)\, e_j\big), \sigma_f(e_i)\big) \\
&\quad - \sum_{i,j} h\big({}^N R\big(df(e_i), df(e_j)\big) df(e_j), \sigma_f(e_i)\big)
\end{aligned}$$

となる．ただし,

$$\alpha_f(X) = h\big(\sigma_f(X), \tau_f\big)$$

とする．Bochner type formula は, Bochner technique において重要な役割を果たす．Bochner technique は, 多様体の性質を調べるのに役立つ．

Target manifold が両側不変計量を持つ Lie 群の場合に, Euler-Lagrange 方程式を Maurer-Cartan form を通して記述する．

0 Preliminaries

0.1 Riemannian geometry

A Riemannian manifold is a smooth manifold M with a Riemannian metric g. A Riemannian metric g is an inner product on the tangent space T_xM at any point $x \in M$. That means that g is bilinear, symmetry and positive definite.

Throughout this thesis, M and N are Riemannian manifolds without boundary. Let g and h denote the Riemannian metrics on M and N respectively.

Let $\mathfrak{X}(M)$ be the set of vector fields on M. We define a connection ∇ on M by a map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ with the following properties
(1) $\nabla_{X+Y}Z = \nabla_X Z + \nabla_Y Z$,
(2) $\nabla_{fX}Z = f\nabla_X Z$,
(3) $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$,
(4) $\nabla_X(fY) = (Xf)Y + f\nabla_X Y$
for X, Y, $Z \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$. We define a torsion T by $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$, where $[X,Y] = XY - YX$. A connection ∇ is called torsion free if $T = 0$. A Levi-Civita connection is a torsion free connection defined by $\nabla g = 0$, i.e., $\nabla(g(X,Y)) = g(\nabla X, Y) + g(X, \nabla Y)$. We know the fact that there exists a Levi-Civita connection uniquely on any Riemannian manifold.

Let M be an m-dimensional Riemannian manifold and ∇ be a Levi-Civita connection. A curvature tensor R is defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z$$

for X, Y, $Z \in \mathfrak{X}(M)$. By the curvature tensor R, we define a sectional curvature K by

$$K(X, Y) = \frac{g(R(X, Y)Y, X)}{|X \wedge Y|^2},$$

where $|X \wedge Y|$ denotes the area of the plane spanned by X and Y, i.e., $|X \wedge Y|^2 = g(X, X)\, g(Y, Y) - g(X, Y)^2$. We see that the sectional curvature determines the curvature tensor from the definition. The curvature of M is called positive if K is positive definite, and negative if K is negative definite. The trace of the curvature tensor is the Ricci curvature

$$\mathrm{Ric}(X,Y) = \sum_{i=1}^{m} g(R(X, e_i)e_i, Y),$$

where $\{e_i\}$ denotes a local orthonormal frame. In this thesis, we use this notation for a local orthonormal frame. Furthermore the trace of the Ricci curvature is the scalar curvature

$$\mathrm{Scal} = \sum_{i=1}^{m} \mathrm{Ric}(e_i, e_i).$$

Let X be a vector field associated with a 1-parameter family φ_t of diffeomorphisms on M. For any differential form ω on M, the Lie derivative of ω is defined by

$$\mathcal{L}_X \omega = \frac{d}{dt}(\varphi_t^* \omega)\Big|_{t=0} ,$$

where $\varphi_t^* \omega$ denotes the pullback of ω by φ_t.

Let f be a smooth map from M into N and df be its differential map. The pullback metric f^*h is a $(0,2)$-type tensor defined by

$$(f^*h)(X,\, Y) = h\big(df(X),\, df(Y)\big)$$

for $X,\, Y \in \mathfrak{X}(M)$.

A map f is an isometric embedding if it is an embedding with the condition of $f^*h = g$.

0.2 Lie groups

We call G a Lie group if it is a smooth manifold with a smooth group structure, i.e., $G \times G \ni (g,\, h) \mapsto g \cdot h^{-1} \in G$ is smooth.

Let V be a vector space. We call $[\,,\,]$ a Lie bracket if a bilinear map $[\,,\,] : V \times V \to V$ satisfies

(1) (linearity) $[X,\, aY + bZ] = a[X,\, Y] + b[X,\, Z]$,
(2) (skew-symmetry) $[X,\, Y] = -[Y,\, X]$,
(3) (Jacobi identity) $[[X,\, Y],\, Z] + [[Y,\, Z],\, X] + [[Z,\, X],\, Y] = 0$
for $X,\, Y,\, Z \in V$ and $a,\, b \in \mathbb{R}$.

Let $\mathfrak{g}$ denote the Lie algebra associated with the Lie group G. A 1-form θ of G is called a Maurer-Cartan form if $\theta_x(X_x) = X$, where $x \in G,\, X_x \in T_xG,\, X \in \mathfrak{g}$.

0.3 Sobolev spaces

The set of maps of class C^∞ from M into $\mathbb{R}^q$ is denoted by $\mathrm{C}^\infty(M,\, \mathbb{R}^q)$. The norm $\|\;\|_{1,p}$ is defined by

$$\|f\|_{1,p} = \Big(\int_M |f|^p \, dv_g + \int_M |df|^p \, dv_g\Big)^{1/p}$$

for $1 < p < \infty$. The Banach space $\mathrm{L}^{1,p}(M,\, \mathbb{R}^q)$ is defined by taking the completion of $\mathrm{C}^\infty(M,\, \mathbb{R}^q)$ with respect to the norm $\|\;\|_{1,p}$.

Let X be a Banach space and X^* be the dual space of X. Let $\langle\, ,\, \rangle$ be the pairing of X and X^*. A sequence $\{x_i\}_{i=1}^{\infty}$ in X converges weakly to $x \in X$ if

$$\langle x_i\, , y\rangle \to \langle x\, , y\rangle \qquad (i \to \infty)$$

for $y \in X^*$.

0.4 Homotopy

Let X be a topological space and A be a subspace of X. We call $(X,\, A)$ a topological pair. We define a map $f : (X,\, A) \to (Y\, , B)$ by $f : X \to Y$ such that $f(A) \subset B$. Let $f_1, f_2 : (X,\, A) \to (Y,\, B)$ and $I = [0,\, 1]$. We call f_1 and f_2 homotopic if a continuous map $H : (X,\, A) \times I \to (Y,\, B)$ satisfies
(1) $H(x,\, 0) = f_1(x)$,
(2) $H(x,\, 1) = f_2(x)$,
(3) $H(A,\, t) \subset B$ for any $t \in I$.
Then we write $f_1 \simeq f_2$. If $A = B = \emptyset$, we say that f_1 and f_2 are (freely) homotopic. If $X = I^m$, $A = \partial I^m$ and $B = \{x_0\}$, f_1 and f_2 are homotopic based at x_0. We define a homotopy group by $\pi_m(Y,\, x_0) = \{f : (I^m,\, \partial I^m) \to (Y,\, \{x_0\})\}/\simeq$.

0.5 Harmonic maps

Let M and N be Riemannian manifolds without boundary, and let f be a smooth map from M into N. A harmonic map is a critical point of the energy functional

$$E(f) = \int_M \|df\|^2 dv_g,$$

where

$$\|df\|^2 = \sum_{i=1}^{m} h(df(e_i), df(e_i))\, ,$$

and dv_g denotes the volume form on M. The tensor

$$\tau_f = \sum_i (\nabla_{e_i} df)(e_i) \tag{1}$$

is called a *tension field* of f. Then f is a harmonic map if and only if $\tau_f = 0$.

0.6 A preliminary lemma

We give a preliminary lemma, which we use in our argument. Let M and N be Riemannian manifolds without boundary with Riemannian metrics g and h respectively.

Let f be a smooth map from M into N. We take the *norm* of the pullback metric

$$\|f^*h\|^2 = \sum_{i,j} h\big(df(e_i),\, df(e_j)\big)^2,$$

and consider the functional

$$\Phi(f) = \int_M \|f^*h\|^2\, dv_g\,.$$

We define the tensor σ_f, which plays an important role in our arguments, as follows:

$$\sigma_f(X) = \sum_i h(df(X), df(e_i))df(e_i) \tag{2}$$

for any vector field X on M. We give the following lemma:

Lemma 0.1 Let X be a vector field on M and U be a vector field on N. Then we have

$$\sum_i h\big(U,\, dF(e_i)\big)\, h\big(dF(X),\, dF(e_i)\big) \;=\; h\big(U,\, \sigma_F(X)\big)\,. \tag{3}$$

In particular,

$$\sum_i h\big(U,\, df(e_i)\big)\, h\big(df(X),\, df(e_i)\big) \;=\; h\big(U,\, \sigma_f(X)\big) \tag{4}$$

$$\|f^*h\|^2 \;=\; \sum_i h\big(df(e_i),\, \sigma_f(e_i)\big)\,. \tag{5}$$

Proof. The equality (3) easily follows from the definition of σ_F. Indeed, since $h(A,\, B)h(C,\, D) = h(A,\, h(C,\, D)B)$, we have

$$\begin{aligned}\sum_i h\big(U,\, dF(e_i)\big)\, h\big(dF(X),\, dF(e_i)\big) &= h\big(U,\, \sum_i h\big(dF(X),\, dF(e_i)\big)dF(e_i)\big)\\ &= h\big(U,\, \sigma_F(X)\big)\,.\end{aligned}$$

Furthermore let $U = df(e_j)$ and let $X = e_j$ in (4), and sum with respect to j, and then we have (5). □

1 First variation formula

In this chapter we give a first variation formula for the functional Φ. Let M and N be Riemannian manifolds without boundary and f be a smooth map from M into N.

1.1 First variation formula

Take any smooth deformation F of f, i.e., any smooth map

$$F \,:\, (-\varepsilon,\, \varepsilon) \times M \longrightarrow N \text{ s.t. } F(0,\, x) = f(x),$$

where ε is a positive constant. Suppose that the deformation F is compactly supported, i.e., $F(t\,, x) = f(x)$ for any t outside a fixed compact set. Let $f_t(x) = F(t,\, x)$, and then $f_0(x) = f(x)$. We often say a deformation $f_t(x)$ instead of a deformation $F(t,\, x)$. Let $V = dF(\frac{\partial}{\partial t})\big|_{t=0}$ denote the variation vector field of the deformation F (see Urakawa [8] for notations). Then we have

Theorem 1.1 (first variation formula)

$$\left.\frac{d\Phi(f_t)}{dt}\right|_{t=0} = -4\int_M h\,(\mathrm{div}_g \sigma_f,\, V)\; dv_g\,,$$

where dv_g denotes the volume form on M and $\mathrm{div}_g\sigma_f$ is the divergence of σ_f, i.e., $\mathrm{div}_g\sigma_f = \sum_i (\nabla_{e_i}\sigma_f)(e_i)$.

Proof. We calculate $\frac{\partial}{\partial t}\,\|f_t^* h\|^2$ at any fixed point $x_0 \in M$. The connection ∇ is trivially extended to a connection on $(-\varepsilon,\, \varepsilon) \times M$. We use the same notation ∇ for this connection. The frame e_i is also trivially extended to a frame on $(-\varepsilon,\, \varepsilon) \times$ (the domain of the frame), and we use the same notation e_i. Then we see $\nabla_{e_i}\frac{\partial}{\partial t} = \nabla_{\frac{\partial}{\partial t}} e_i = 0$ on $(-\varepsilon,\, \varepsilon) \times M$. Use a normal coordinate at x_0, and we can assume $\nabla_{e_i} e_j = 0$ at x_0 for any $i,\, j$. Since $(dF)_{(t,x)}\big((e_i)_{(t,x)}\big) = (df_t)_x\big((e_i)_{(t,x)}\big)$, we denote it by $dF(e_i)$ simply. Note that

$$\nabla_{\frac{\partial}{\partial t}}(dF(e_i)) = \Big(\nabla_{\frac{\partial}{\partial t}} dF\Big)(e_i) = (\nabla_{e_i} dF)\left(\tfrac{\partial}{\partial t}\right) = \nabla_{e_i}\left(dF\left(\tfrac{\partial}{\partial t}\right)\right),$$

since $\left[\frac{\partial}{\partial t}, e_i\right] = 0$. Then we have

$$
\begin{aligned}
(6)\quad \frac{1}{4}\frac{\partial}{\partial t}\|f_t^*h\|^2 &= \frac{1}{4}\frac{\partial}{\partial t}\sum_{i,j} h\big(df_t(e_i),\, df_t(e_j)\big)\, h\big(df_t(e_i),\, df_t(e_j)\big) \\
&= \frac{1}{4}\frac{\partial}{\partial t}\sum_{i,j} h\big(dF(e_i),\, dF(e_j)\big)\, h\big(dF(e_i),\, dF(e_j)\big) \\
&= \sum_{i,j} h\big(\nabla_{\frac{\partial}{\partial t}}\big(dF(e_i)\big),\ dF(e_j)\big)\, h\big(dF(e_i),\, dF(e_j)\big) \\
&= \sum_{i,j} h\big(\nabla_{e_i}\big(dF\big(\tfrac{\partial}{\partial t}\big)\big)\,,\ dF(e_j)\big)\, h\big(dF(e_i),\, dF(e_j)\big) \\
&\overset{\text{Lemma 0.1 (3)}}{=} \sum_{i,j} h\big(\nabla_{e_i}\big(dF\big(\tfrac{\partial}{\partial t}\big)\big)\,,\ \sigma_F(e_i)\big)\,.
\end{aligned}
$$

The last equality follows from Lemma 0.1 (3) for $U = \nabla_{e_i}\big(dF\big(\frac{\partial}{\partial t}\big)\big)$ and for $X = e_i$. Integrate the both sides of (6) over M and use integration by parts

$$
\begin{aligned}
\int_M \frac{\partial}{\partial t}\|f_t^*h\|^2\, dv_g &= 4\int_M \sum_i h\big(\nabla_{e_i}\big(dF\big(\tfrac{\partial}{\partial t}\big)\big)\,,\ \sigma_F(e_i)\big)\, dv_g \\
&= -4\int_M \sum_i h\big(dF\big(\tfrac{\partial}{\partial t}\big),\ \nabla_{e_i}(\sigma_F(e_i))\big)\, dv_g\,.
\end{aligned}
$$

Then let $t = 0$, and we obtain the first variation formula. □

1.2 Stationary maps

We give here the notion of stationary maps for the functional Φ.

Definition 1.1 We call a smooth map f a stationary map if the first variation of F (at f) identically vanishes, i.e.,

$$
\left.\frac{d\Phi(f_t)}{dt}\right|_{t=0} = 0
$$

for any smooth deformation f_t of f.

By the first variation formula (Theorem 1.1), we have

Proposition 1.1 A smooth map f is a stationary map if and only if it satisfies the equation

$$\mathrm{div}_g \sigma_f = 0, \tag{7}$$

which is called the Euler-Lagrange equation for the functional Φ, where σ_f is the covariant tensor defined by (2).

1.3 Euler-Lagrange equation in the case of surfaces

We consider the case of immersed surfaces in $\mathbb{R}^3$. Let D be a domain of $\mathbb{R}^2$ with the standard metric. Let f be a smooth map from D into $\mathbb{R}^3$. We write $(\ ,\ )$ for the inner product with respect to the standard metric. Put $e_1 = \frac{\partial}{\partial x}$ and $e_2 = \frac{\partial}{\partial y}$. In this case, (7) is represented by the following form.

Theorem 1.2 In the case of surfaces, the Euler-Lagrange equation is represented as

$$\begin{aligned} &E\frac{\partial^2 f}{\partial x^2} + 2F\frac{\partial^2 f}{\partial x \partial y} + G\frac{\partial^2 f}{\partial y^2} \\ &+ \frac{\partial E}{\partial x}\frac{\partial f}{\partial x} + \frac{\partial F}{\partial x}\frac{\partial f}{\partial y} + \frac{\partial F}{\partial y}\frac{\partial f}{\partial x} + \frac{\partial G}{\partial y}\frac{\partial f}{\partial y} = 0, \end{aligned}$$

where E, F, G are the first fundamental quantities, i.e.,

$$E = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial x}\right), F = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right), G = \left(\frac{\partial f}{\partial y}, \frac{\partial f}{\partial y}\right).$$

Proof. We calculate

$$
\begin{aligned}
\mathrm{div}_g \sigma_f &= (\nabla_{e_1}\sigma_f)(e_1) + (\nabla_{e_2}\sigma_f)(e_2) \\
&= \nabla_{e_1}(\sigma_f(e_1)) + \nabla_{e_2}(\sigma_f(e_2)) \\
&= \nabla_{e_1}(h(df(e_1), df(e_1))\, df(e_1) + h(df(e_1), df(e_2))\, df(e_2)) \\
&\quad + \nabla_{e_2}(h(df(e_2), df(e_1))\, df(e_1) + h(df(e_2), df(e_2))\, df(e_2)) \\
&= \nabla_{\frac{\partial}{\partial x}}\Big(h\Big(df\Big(\tfrac{\partial}{\partial x}\Big), df\Big(\tfrac{\partial}{\partial x}\Big)\Big)\, df\Big(\tfrac{\partial}{\partial x}\Big) + h\Big(df\Big(\tfrac{\partial}{\partial x}\Big), df\Big(\tfrac{\partial}{\partial y}\Big)\Big)\, df\Big(\tfrac{\partial}{\partial y}\Big)\Big) \\
&\quad + \nabla_{\frac{\partial}{\partial y}}\Big(h\Big(df\Big(\tfrac{\partial}{\partial y}\Big), df\Big(\tfrac{\partial}{\partial x}\Big)\Big)\, df\Big(\tfrac{\partial}{\partial x}\Big) + h\Big(df\Big(\tfrac{\partial}{\partial y}\Big), df\Big(\tfrac{\partial}{\partial y}\Big)\Big)\, df\Big(\tfrac{\partial}{\partial y}\Big)\Big) \\
&= \tfrac{\partial}{\partial x}\Big(\Big(\tfrac{\partial f}{\partial x}, \tfrac{\partial f}{\partial x}\Big)\tfrac{\partial f}{\partial x} + \Big(\tfrac{\partial f}{\partial x}, \tfrac{\partial f}{\partial y}\Big)\tfrac{\partial f}{\partial y}\Big) + \tfrac{\partial}{\partial y}\Big(\Big(\tfrac{\partial f}{\partial y}, \tfrac{\partial f}{\partial x}\Big)\tfrac{\partial f}{\partial x} + \Big(\tfrac{\partial f}{\partial y}, \tfrac{\partial f}{\partial y}\Big)\tfrac{\partial f}{\partial y}\Big) \\
&= \tfrac{\partial}{\partial x}\Big(E\,\tfrac{\partial f}{\partial x} + F\,\tfrac{\partial f}{\partial y}\Big) + \tfrac{\partial}{\partial y}\Big(F\,\tfrac{\partial f}{\partial x} + G\,\tfrac{\partial f}{\partial x}\Big) \\
&= \tfrac{\partial E}{\partial x}\tfrac{\partial f}{\partial x} + E\,\tfrac{\partial^2 f}{\partial x^2} + \tfrac{\partial F}{\partial x}\tfrac{\partial f}{\partial y} + F\,\tfrac{\partial^2 f}{\partial x \partial y} + \tfrac{\partial F}{\partial y}\tfrac{\partial f}{\partial x} + F\,\tfrac{\partial^2 f}{\partial x \partial y} + \tfrac{\partial G}{\partial y}\tfrac{\partial f}{\partial y} + G\,\tfrac{\partial^2 f}{\partial y^2}.
\end{aligned}
$$

This proves the theorem. □

2 Examples

In this chapter we give some examples of stationary maps.

2.1 4-harmonic functions

We first give the definition of a p-harmonic map. Let f be a map from M into N. We call f p-harmonic if f is a weak solution of

$$\mathrm{Tr}_g\,(\nabla(\|df\|^{p-2}df)) = 0.$$

Here we consider the case that the target manifold is the one-dimensional Euclidean space.

Lemma 2.1 Let f be a function from M into $\mathbb{R}$ with the standard metric. Then f is a stationary map if and only if it is 4-harmonic.

Proof. This result is given by Kawai-Nakauchi [4] without proof. We give a full proof for reader's convenience. Since

$$\begin{aligned}
\mathrm{div}_g \sigma_f &= \sum_{i,j} \nabla_{e_i}\big(h(df(e_i),\, df(e_j))df(e_j)\big) \\
&= \sum_{i,j} \nabla_{e_i}\big((df(e_i)\, df(e_j)^2\big) \\
&= \sum_{i} \nabla_{e_i}\Big(\sum_{j} df(e_j)^2\, df(e_i)\Big) \\
&= \sum_{i} \nabla_{e_i}(\|df\|^2\, df(e_i)) \\
&= \mathrm{Tr}_g\left(\nabla(\|df\|^2 df)\right),
\end{aligned}$$

we see that f is a stationary map if and only if it is 4-harmonic. □

2.2 Geodesics

We discuss the case of curves. In this case, we see that all stationary curves are geodesics by an arclength parameter.

Lemma 2.2 When f is parameterized by the arclength parameter, it is a stationary map if and only if it is geodesic.

Proof. Let s be an arclength parameter, i.e., $\|f'(s)\|^2 = 1$, where f' is the differential by s. Exchange the parameter t for the arc length parameter s. Since

$$\begin{aligned}
\sigma_f\left(\tfrac{\partial}{\partial s}\right) &= h\left(df\left(\tfrac{\partial}{\partial s}\right), df\left(\tfrac{\partial}{\partial s}\right)\right) df\left(\tfrac{\partial}{\partial s}\right) \\
&= df\left(\tfrac{\partial}{\partial s}\right),
\end{aligned}$$

we have

$$\begin{aligned}
\mathrm{div}_g \sigma_f &= \left(\nabla_{\frac{\partial}{\partial s}} df\right)\left(\tfrac{\partial}{\partial s}\right) \\
&= \nabla_{df\left(\frac{\partial}{\partial s}\right)}\left(df\left(\tfrac{\partial}{\partial s}\right)\right).
\end{aligned}$$

Then, when f is parameterized by the arclength parameter, it is a stationary map if and only if it is geodesic. □

2.3 Radially symmetric maps with singularities

Lemma 2.3 (See Kawai-Nakauchi [4].)
Let

$$\begin{array}{cccc} f_0 : & \mathbb{B}^m & \to & \mathbb{S}^{m-1} \\ & \Psi & & \Psi \\ & x & \mapsto & \dfrac{x}{|x|} = f_0(x)\,, \end{array}$$

where $\mathbb{B}^m$ is an m-dimensional ball and $\mathbb{S}^{m-1}$ is an $(m-1)$-dimensional sphere. Then the map f_0 is a stationary map.

The above map f_0 is well-known in the theory of harmonic maps. This map gives a structure of singularities of harmonic maps.

2.4 Other examples

Lemma 2.4 Let M and N be Riemannian manifolds and $f : M \to N$. The following two maps are stationary maps:

(1) f : isometric harmonic.

(2) f : totally geodesic.

Proof. We give a proof of (1). If f is isometric, i.e., $f^*h = g$, then we see

$$\begin{aligned} \mathrm{div}_g \sigma_f &= \sum_{i,j} \nabla_{e_i}(h(df(e_i),\, df(e_j))df(e_j)) \\ &= \sum_{i,j} \nabla_{e_i}(g(e_i, e_j)df(e_j)) \\ &= \mathrm{Tr}_g(\nabla df)\,. \end{aligned}$$

Hence, when f is isometric, it is a stationary map if and only if it is a harmonic one.

We give a proof of (2). We have

$$\begin{aligned}\mathrm{div}_g \sigma_f &= \sum_{i,j} \nabla_{e_i}(h(df(e_i),\, df(e_j))df(e_j)) \\ &= \sum_{i,j} h((\nabla_{e_i} df)(e_i), df(e_j))df(e_j) \\ &\quad + \sum_{i,j} h(df(e_i), (\nabla_{e_i} df)(e_j))df(e_j) \\ &\qquad + \sum_{i,j} h(df(e_i), df(e_j))(\nabla_{e_i} df)(e_j) \\ &= 0\,,\end{aligned}$$

since f is totally geodesic, i.e., $\nabla df = 0$. $\square$

3 Second variation formula

In this chapter we give a second variation formula for the functional Φ. Let M and N be Riemannian manifolds without boundary and f be a smooth map from M into N.

3.1 Second variation formula

Take any smooth deformation F of f with two parameters, i.e., any smooth map

$$F \,:\, (-\varepsilon,\, \varepsilon) \times (-\delta,\, \delta) \times M \longrightarrow N \;\text{ s.t. }\; F(0,\, 0,\, x) = f(x)\,.$$

Suppose that the deformation F is compactly supported, i.e., $F(s,\, t,\, x) = f(x)$ for any s and t outside a fixed compact set. Let $f_{s,t}(x) = F(s,\, t,\, x)$, and then $f_{0,0}(x) = f(x)$. We often say a deformation $f_{s,t}(x)$ instead of a deformation $F(s,\, t,\, x)$. Let

$$V = dF(\tfrac{\partial}{\partial s})\big|_{s,t=0},\quad W = dF(\tfrac{\partial}{\partial t})\big|_{s,t=0}$$

denote the variation vector fields of the deformation F. Then we have

Theorem 3.1 (second variation formula)

$$\frac{1}{4}\left.\frac{\partial^2 \Phi(f_{s,t})}{\partial s \partial t}\right|_{s,t=0} = -\int_M h\big(\mathrm{Hess}_f(\tfrac{\partial}{\partial s}, \tfrac{\partial}{\partial t}), \operatorname{div}_g \sigma_f\big) dv_g$$
$$+ \int_M \sum_{i,j} h\big(\nabla_{e_i} V, \nabla_{e_j} W\big)\, h(df(e_i), df(e_j)) dv_g$$
$$+ \int_M \sum_{i,j} h\big(\nabla_{e_i} V, df(e_j)\big)\, h\big(\nabla_{e_i} W, df(e_j)\big) dv_g$$
$$+ \int_M \sum_{i,j} h\left(\nabla_{e_i} V, df(e_j)\right)\, h\left(df(e_i), \nabla_{e_j} W\right) dv_g$$
$$- \int_M \sum_{i,j} h\big({}^N\!R\,(df(e_i), V)\, W, df(e_j)\big) h\big(df(e_i), df(e_j)\big) dv_g\,,$$

where Hess_f denotes the Hessian of f, i.e., $\mathrm{Hess}_f(X, Y) = (\nabla_X df)(Y) = (\nabla_Y df)(X)$.

Proof. The connection ∇ is trivially extended to a connection on $(-\varepsilon, \varepsilon) \times (-\delta, \delta) \times M$. We use the same notation ∇ for this connection. The frame e_i is also trivially extended to a frame on $(-\varepsilon, \varepsilon) \times (-\delta, \delta) \times$ (the domain of the frame), denoted by the same notation e_i. Then we see

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial s}} e_i &= \nabla_{e_i} \tfrac{\partial}{\partial s} = 0 \\
\nabla_{\frac{\partial}{\partial t}} e_i &= \nabla_{e_i} \tfrac{\partial}{\partial t} = 0 \\
\nabla_{\frac{\partial}{\partial s}} \tfrac{\partial}{\partial t} &= \nabla_{\frac{\partial}{\partial t}} \tfrac{\partial}{\partial s} = 0
\end{aligned}$$

on $(-\varepsilon, \varepsilon) \times M$. Take and fix any point $x_0 \in M$, and we calculate $\frac{\partial^2}{\partial s \partial t} \|f_{s,t}^* h\|^2$ at $(s, t, x) = (0, 0, x_0)$. Using a normal coordinate we can assume $\nabla_{e_i} e_j = 0$ at x_0 for any i, j. Then we see

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial s}} (dF(e_i)) &= \left(\nabla_{\frac{\partial}{\partial s}} dF\right)(e_i) = (\nabla_{e_i} dF)\left(\tfrac{\partial}{\partial s}\right) = \nabla_{e_i} \left(dF\left(\tfrac{\partial}{\partial s}\right)\right) \\
\nabla_{\frac{\partial}{\partial t}} (dF(e_i)) &= \left(\nabla_{\frac{\partial}{\partial t}} dF\right)(e_i) = (\nabla_{e_i} dF)\left(\tfrac{\partial}{\partial t}\right) = \nabla_{e_i} \left(dF\left(\tfrac{\partial}{\partial t}\right)\right) .
\end{aligned}$$

Then we have

$$(8)\quad \frac{1}{4}\frac{\partial^2}{\partial s\partial t}\|f_{s,t}^{*}h\|^2 = \frac{1}{4}\frac{\partial^2}{\partial s\partial t}\sum_{i,j} h\left(df_{s,t}(e_i),\, df_{s,t}(e_j)\right)^2$$

$$= \frac{1}{4}\frac{\partial^2}{\partial s\partial t}\sum_{i,j} h\left(dF(e_i),\, dF(e_j)\right)^2$$

$$= \sum_{i,j} h\big(\nabla_{\frac{\partial}{\partial s}}\nabla_{\frac{\partial}{\partial t}}\left(dF(e_i)\right),\, dF(e_j)\big)\, h\big(dF(e_i),\, dF(e_j)\big)$$

$$+ \sum_{i,j} h\big(\nabla_{\frac{\partial}{\partial s}}\left(dF(e_i)\right),\, \nabla_{\frac{\partial}{\partial t}}\left(dF(e_j)\right)\big)\, h\big(dF(e_i),\, dF(e_j)\big)$$

$$+ \sum_{i,j} h\big(\nabla_{\frac{\partial}{\partial s}}\left(dF(e_i)\right),\, dF(e_j)\big)\, h\big(\nabla_{\frac{\partial}{\partial t}}\left(dF(e_i)\right),\, dF(e_j)\big)$$

$$+ \sum_{i,j} h\big(\nabla_{\frac{\partial}{\partial s}}\left(dF(e_i)\right),\, dF(e_j)\big)\, h\big(dF(e_i),\, \nabla_{\frac{\partial}{\partial t}}\left(dF(e_j)\right)\big).$$

We calculate the first term in the right hand side. Since

$$\begin{aligned}\nabla_{\frac{\partial}{\partial s}}\nabla_{\frac{\partial}{\partial t}}\left(dF(e_i)\right) &= \left(\nabla_{\frac{\partial}{\partial s}}\nabla_{\frac{\partial}{\partial t}}dF\right)(e_i) = \left(\nabla_{\frac{\partial}{\partial s}}\nabla_{e_i}dF\right)\left(\tfrac{\partial}{\partial t}\right)\\ &= \left(\nabla_{e_i}\nabla_{\frac{\partial}{\partial s}}dF\right)\left(\tfrac{\partial}{\partial t}\right) - {}^{N}R\left(dF(e_i),\, dF\left(\tfrac{\partial}{\partial s}\right)\right) dF\left(\tfrac{\partial}{\partial t}\right)\\ &= \nabla_{e_i}\mathrm{Hess}_F\left(\tfrac{\partial}{\partial s},\, \tfrac{\partial}{\partial t}\right) - {}^{N}R\left(dF(e_i),\, dF\left(\tfrac{\partial}{\partial s}\right)\right) dF\left(\tfrac{\partial}{\partial t}\right),\end{aligned}$$

we have

$$(9)\quad \sum_{i,j} h\big(\nabla_{\frac{\partial}{\partial s}}\nabla_{\frac{\partial}{\partial t}}\left(dF(e_i)\right),\, dF(e_j)\big)\, h\big(dF(e_i),\, dF(e_j)\big)$$

$$= \sum_{i,j} h\big(\nabla_{e_i}\mathrm{Hess}_F\left(\tfrac{\partial}{\partial s},\, \tfrac{\partial}{\partial t}\right),\, dF(e_j)\big)\, h\big(dF(e_i),\, dF(e_j)\big)$$

$$- \sum_{i,j} h\big({}^{N}R\left(dF(e_i),\, dF\left(\tfrac{\partial}{\partial s}\right)\right) dF\left(\tfrac{\partial}{\partial t}\right),\, dF(e_j)\big)\, h\big(dF(e_i),\, dF(e_j)\big).$$

On the other hand by Lemma 0.1 (3) for $U = \nabla_{e_i}\mathrm{Hess}_F\left(\frac{\partial}{\partial s},\, \frac{\partial}{\partial t}\right)$ and for $X = e_i$, we have

$$(10)\quad \sum_{i,j} h\big(\nabla_{e_i}\mathrm{Hess}_F\left(\tfrac{\partial}{\partial s},\, \tfrac{\partial}{\partial t}\right),\, dF(e_j)\big)\, h\big(dF(e_i),\, dF(e_j)\big) = \sum_{i} h\big(\nabla_{e_i}\mathrm{Hess}_F\left(\tfrac{\partial}{\partial s},\, \tfrac{\partial}{\partial t}\right),\, \sigma_F(e_i)\big).$$

Integrate (8) over M and use (9) and (10). Let $s = t = 0$, and then we have the second variation formula. □

3.2 Some remarks

We give two remarks for the second variation formula.

Remark 3.1 Note that the first term in the right hand side in Theorem 3.1 vanishes if f is a stationary map.

Remark 3.2 The last term of the right hand side is equal to

$$-\int_M \sum_i h({}^N\!R\,(df(e_i),\, V)\,W,\, \sigma_f(e_i))dv_g\,.$$

Indeed we have this equality by Lemma 0.1 (4) for $U = {}^N\!R\,(df(e_i),\, V)\,W$ and $X = e_i$.

4 Minimizers in 3-homotopy classes

In this chapter we utilize the notion of 3-homotopy in a Sobolev space, which is given by White, and construct a minimizer of the functional Φ in each 3-homotopy class of the Sobolev space.

4.1 Weak homotopy in Sobolev spaces

We assume, by Nash's isometric embedding, that N is a submanifold of a Euclidean space $\mathbb{R}^q$. Let

$$\mathrm{L}^{1,p}(M,\, N) \;=\; \{\; f \in \mathrm{L}^{1,p}(M,\, \mathbb{R}^q) \;|\; f(x) \in N \text{ a.e. } \}\,,$$

where $\mathrm{L}^{1,p}(M,\, \mathbb{R}^q)$ denotes the Sobolev space of $\mathbb{R}^q$-valued L^p-functions on M whose weak derivatives are in L^p. Then White proved that the notion of the $[p-1]$-homotopy is compatible with the Sobolev space $\mathrm{L}^{1,p}(M,\, N)$, where $[\ \ ]$ denotes the Gauss symbol, i.e., $[r]$ is the integer greater than or equal to r. Maps f_1 and f_2 are $[p-1]$-homotopic if they are homotopic on the $[p-1]$-dimensional skeletons of a triangulation of M.

Theorem W (Theorem 3.4 in White [11]. See also [10] and [1].)
(1) The $[p-1]$-homotopy is well-defined for any map $f \in \mathrm{L}^{1,p}(M,\, N)$.
(2) If f_i converges weakly to f_∞ in $\mathrm{L}^{1,p}(M,\, N)$, then f_i and f_∞ are $[p-1]$-homotopic for sufficiently large i.

4.2 Minimizers in 3-homotopy classes

In this section, we assume that the Riemannian manifolds M and N are compact. The functional Φ is defined on $\mathrm{L}^{1,4}(M, N)$, in which the 3-homotopy is well-defined. Then we want to minimize the functional Φ in each 3-homotopy class, i.e., in the following class:

$$\mathcal{F}_{f_0} = \{ f \in \mathrm{L}^{1,4}(M, N) \mid f \text{ is 3-homotopic to } f_0 \}$$

for any given continuous map f_0 from M into N.

Theorem 4.1 There exists a minimizer of the functional Φ in $\mathcal{F}_{f_0}$.

Remark 4.1 When M is 4-dimensional and $\pi_4(N) = 0$, any *continuous* minimizer in $\mathcal{F}_0$ minimizes Φ in its ordinary (free) homotopy class of f_0.

Proof of Theorem 4.1. Take a minimizing sequence f_i for Φ, i.e., $\Phi(f_i)$ converges to the infimum of Φ. Passing to a subsequence if necessary, we may assume that f_i converges weakly to a map f_∞ in $\mathrm{L}^{1,4}(M, N)$. The map f_∞ is 3-homotopic to f_0, since the weak convergence in $\mathrm{L}^{1,4}(M, N)$ preserves 3-homotopy. Then by the semi-continuity of Φ,

$$\Phi(f_\infty) \le \lim_{i\to\infty} \Phi(f_i) = \text{infimum.}$$

Hence we get $\Phi(f_\infty) = $ infimum, i.e., f_∞ is a minimizer. $\square$

5 Monotonicity formula

In this chapter we give a monotonicity formula for stationary maps. We prove this formula under a weaker condition.

5.1 Stationary maps with respect to diffeomorphisms

We assume the following weak notion of stationary maps.

Definition 5.1 We call f a *stationary map with respect to diffeomorphisms* on M if

$$\frac{d}{dt}\Phi(f \circ \varphi_t)\bigg|_{t=0} = 0$$

for any 1-parameter family φ_t of diffeomorphisms on M.

Note that the notion of stationary maps in Definition 5.1 is weaker than that of stationary ones in Definition 1.1, since $f_t(x) = f \circ \varphi_t(x)$ is a deformation in Theorem 1.1.

5.2 Another first variation formula

To prove the monotonicity formula, we give a first variation formula in the following form for variations by 1-parameter families of diffeomorphisms. Let $\overline{V}$ be a compactly supported smooth vector field on M, and let φ_t $(-\varepsilon < t < \varepsilon)$ be a 1-parameter family of diffeomorphisms on M for this vector field $\overline{V}$. Then we have

Theorem 5.1 (first variation formula)

$$\frac{d\Phi(f \circ \varphi_t)}{dt}\bigg|_{t=0} = \int_M \Big\{ - \|f^*h\|^2 \mathrm{div}_g \overline{V} + 4 \sum_{i=1}^{m} h\big(df(\nabla_{e_i}\overline{V}),\, \sigma_f(e_i)\big) \Big\}\, dv_g\,.$$

Proof. This formula follows from the general form of the first variation formula (Theorem 1.1). Let V be the vector field for the deformation f_t and set $f_t = f \circ \varphi_t$. Then we have

$$\begin{aligned} \nabla_{e_i} V &= (\nabla_{e_i} df)(\overline{V}) + df(\nabla_{e_i}\overline{V}) \\ &= (\nabla_{\overline{V}} df)(e_i) + df(\nabla_{e_i}\overline{V})\,, \end{aligned}$$

and hence

$$
\begin{aligned}
4\sum_i h\big(\sigma_f(e_i),\, \nabla_{e_i} V\big) & \\
&= \quad 4\sum_i h\big(\sigma_f(e_i),\, (\nabla_{\overline{V}} df)(e_i)\big) \;+\; 4\sum_i h\big(\sigma_f(e_i),\, df(\nabla_{e_i}\overline{V})\big) \\
&\overset{\text{Lemma 0.1 (4)}}{=} \quad 4\sum_{i,j} h\big((\nabla_{\overline{V}} df)(e_i),\, df(e_j)\big) h\big(df(e_i),\, df(e_j)\big) \;+\; 4\sum_i h\big(\sigma_f(e_i),\, df(\nabla_{e_i}\overline{V})\big) \\
&= \quad \mathcal{L}_{\overline{V}}\Big(\sum_{i,j} h\big(df(e_i),\, df(e_j)\big)^2\Big) \;+\; 4\sum_i h\big(\sigma_f(e_i),\, df(\nabla_{e_i}\overline{V})\big) \\
&= \quad \mathcal{L}_{\overline{V}} \|f^*h\|^2 \;+\; 4\sum_i h\big(\sigma_f(e_i),\, df(\nabla_{e_i}\overline{V})\big)\,,
\end{aligned}
$$

where $\mathcal{L}$ denotes the Lie derivative. By Theorem 1.1,

$$
\begin{aligned}
\frac{d}{dt}\Phi(f\circ\varphi_t)\Big|_{t=0} &= \frac{d\Phi(f_t)}{dt}\Big|_{t=0} \\
&= \int_M \{\mathcal{L}_{\overline{V}}\|f^*h\|^2 \;+\; 4\sum_i h\big(\sigma_f(e_i),\, df(\nabla_{e_i}\overline{V})\big)\}\, dv_g \\
&= -\int_M \|f^*h\|^2\, \mathcal{L}_{\overline{V}}\, dv_g \;+\; 4\int_M \sum_i h\big(\sigma_f(e_i),\, df(\nabla_{e_i}\overline{V})\big)\, dv_g \\
&= -\int_M \|f^*h\|^2\, \mathrm{div}_g \overline{V}\, dv_g \;+\; 4\int_M \sum_i h\big(\sigma_f(e_i),\, df(\nabla_{e_i}\overline{V})\big)\, dv_g\,,
\end{aligned}
$$

and then we have the formula. □

5.3 Monotonicity formula

Under the weaker condition in Definition 5.1, we give the following monotonicity formula.

Theorem 5.2 (monotonicity formula) Fix $x_0 \in M$ and let $\rho > 0$. Let $r = r(x)$ denote the distance function between x_0 and x. Let f be a stationary map with respect to diffeomorphisms on M. Then it satisfies

$$\frac{d}{d\rho}\left\{e^{C\rho}\rho^{4-m}\int_{B_\rho(x_0)} \|f^*h\|^2 dv_g\right\}$$
$$\geq\ 4e^{C\rho}\rho^{4-m}\frac{d}{d\rho}\int_{B_\rho(x_0)} \|h(df(grad\, r),\, df)\|^2\, dv_g \ \geq\ 0\,,$$

where m is the dimension of M and C is a constant, and

$$\|h(df(grad\, r),\, df)\|^2 \ =\ \sum_i h(df(grad\, r),\, df(e_i))^2\,.$$

Proof. We use the argument by Price [7] (see also Xin [12], p.43). Let $\overline{V}$ be a smooth vector field on M, which is supported compactly in $B_\rho(x_0)$. Take a 1-parameter family of diffeomorphisms φ_t $(-\varepsilon < t < \varepsilon)$ of M for this vector field $\overline{V}$. Then by the first variation formula (Theorem 5.1), we have

$$\text{(11)} \qquad \int_M \Big\{ -\|f^*h\|^2 \mathrm{div}_g \overline{V} + 4\sum_{i=1}^{m} h\big(df(\nabla_{e_i}\overline{V}),\, \sigma_f(e_i)\big)\Big\}\, dv_g \ =\ 0\,.$$

Let $r = r(x)$ denote the distance function between x_0 and x, and let $\frac{\partial}{\partial r}$ be the gradient vector field of the distance function r. We can take a local orthonormal frame $\{e_i\}$ such that $e_m = \frac{\partial}{\partial r}$. We adopt here a smooth vector field

$$\overline{V}(x) \ =\ \xi(r) r\frac{\partial}{\partial r} \ =\ \xi\big(r(x)\big) r(x)\frac{\partial}{\partial r}$$

in a coordinate neighborhood of x_0, and vanishes outside the neighborhood. The function $\xi(r)$ is defined later. We see, for $1 \leq i \leq m-1$,

$$\text{(12)} \qquad \nabla_{e_i}\frac{\partial}{\partial r} \ =\ \sum_{j=1}^{m-1} \mathrm{Hess}(r)(e_i,\, e_j)e_j\,,$$

where $\mathrm{Hess}(r)(X,\, Y) = (\nabla dr)(X,\, Y)$ denotes the Hessian of the function r. Indeed, (12) holds: Since $dr(e_j) = g(\frac{\partial}{\partial r},\, e_j) = 0$ $(j = 1, \cdots, m-1)$ and $g(\frac{\partial}{\partial r},\, \frac{\partial}{\partial r}) = 1$, we have

$$\begin{aligned}
\nabla_{e_i}\tfrac{\partial}{\partial r} &= \sum_{j=1}^{m-1} g(\nabla_{e_i}\tfrac{\partial}{\partial r},\, e_j)\, e_j \;+\; g(\nabla_{e_i}\tfrac{\partial}{\partial r},\, \tfrac{\partial}{\partial r})\,\tfrac{\partial}{\partial r} \;=\; -\sum_{j=1}^{m-1} g(\tfrac{\partial}{\partial r},\, \nabla_{e_i}e_j)\, e_j \\
&= -\sum_{j=1}^{m-1} dr\big(\nabla_{e_i}e_j\big)\, e_j \;=\; \sum_{j=1}^{m-1}(\nabla dr)(e_i,\, e_j)\, e_j\,.
\end{aligned}$$

Here we used

$$0 \;=\; \nabla_{e_i}\{g\big(\tfrac{\partial}{\partial r},\, e_j\big)\} \;=\; g\big(\nabla_{e_i}\tfrac{\partial}{\partial r},\, e_j\big) \;+\; g\big(\tfrac{\partial}{\partial r},\, \nabla_{e_i}e_j\big)\,.$$

Thus (12) is proved. Then we have

$$\nabla_{\frac{\partial}{\partial r}}\overline{V} \;=\; \nabla_{\frac{\partial}{\partial r}}\Big(\xi(r)r\tfrac{\partial}{\partial r}\Big) \;=\; (\xi(r)r)'\tfrac{\partial}{\partial r}\,, \tag{13}$$

and for $1 \le i \le m-1$,

$$\nabla_{e_i}\overline{V} \;=\; \xi(r)r\nabla e_i\tfrac{\partial}{\partial r} \;=\; \xi(r)r\sum_{j=1}^{m-1}\mathrm{Hess}(r)(e_i,\, e_j)e_j\,. \tag{14}$$

By the comparison theorem of Hessian, we know

$$\frac{1}{r}g(e_i,\, e_j)(1-cr) \;\le\; \mathrm{Hess}(r)(e_i,\, e_j) \;\le\; \frac{1}{r}g(e_i,\, e_j)(1+cr)\,, \tag{15}$$

where c is a constant which depends on the upper and lower bounds of the sectional curvature of M. We calculate $\mathrm{div}_g\,\overline{V}$ and $\sum_{i=1}^{m} h\big(df(\nabla_{e_i}\overline{V}),\, \sigma_f(e_i)\big)$. We have

$$\begin{aligned}
\mathrm{div}_g\,\overline{V} \;&=\; \sum_{i=1}^{m-1} g(\nabla_{e_i}\overline{V},\, e_i) \;+\; g(\nabla_{\frac{\partial}{\partial r}}\overline{V},\, \tfrac{\partial}{\partial r}) \\
&\overset{(13)\,\&\,(14)}{=} \xi(r)r\sum_{i,j=1}^{m-1}\mathrm{Hess}(r)(e_i,\, e_j)g(e_j,\, e_i) \;+\; (\xi(r)r)' \\
&\overset{(15)}{\ge} (m-1)\xi(r)(1-cr) \;+\; (\xi(r)r)' \\
&= \xi'(r)r + m\xi(r) - (m-1)c\xi(r)r\,.
\end{aligned} \tag{16}$$

We get

$$
\begin{aligned}
(17) \qquad & 4\sum_{i=1}^{m} h\big(df(\nabla_{e_i}\overline{V}),\, \sigma_f(e_i)\big) \\
= \quad & 4\sum_{i=1}^{m-1} h\big(df(\nabla_{e_i}\overline{V}),\, \sigma_f(e_i)\big) \;+\; 4h\big(df(\nabla_{\frac{\partial}{\partial r}}\overline{V}),\, \sigma_f(\tfrac{\partial}{\partial r})\big) \\
\overset{(13)\,\&\,(14)}{=} \quad & 4\xi(r)r\sum_{i,j=1}^{m-1} \mathrm{Hess}(r)(e_i,\, e_j)h\big(df(e_j),\, \sigma_f(e_i)\big) \;+\; 4(\xi(r)r)'h\big(df(\tfrac{\partial}{\partial r}),\, \sigma_f(\tfrac{\partial}{\partial r})\big) \\
\overset{(15)}{\leq} \quad & 4\xi(r)(1+cr)\sum_{i=1}^{m-1} h\big(df(e_i),\, \sigma_f(e_i)\big) \;+\; 4\big(\xi'(r)r+\xi(r)\big)\, h\big(df(\tfrac{\partial}{\partial r}),\, \sigma_f(\tfrac{\partial}{\partial r})\big) \\
= \quad & 4\xi'(r)rh\big(df(\tfrac{\partial}{\partial r}),\, \sigma_f(\tfrac{\partial}{\partial r})\big) \\
& \quad + 4\xi(r)\left\{\sum_{i=1}^{m-1} h\big(df(e_i),\, \sigma_f(e_i)\big) \;+\; h\big(df(\tfrac{\partial}{\partial r}),\, \sigma_f(\tfrac{\partial}{\partial r})\big)\right\} \\
& \qquad + 4c\xi(r)r\sum_{i=1}^{m-1} h\big(df(e_i),\, \sigma_f(e_i)\big) \\
= \quad & 4\xi'(r)rh\big(df(\tfrac{\partial}{\partial r}),\, \sigma_f(\tfrac{\partial}{\partial r})\big) \;+\; 4\xi(r)\|f^*h\|^2 \;+\; 4c\xi(r)r\sum_{i=1}^{m-1} h\big(df(e_i),\, \sigma_f(e_i)\big)\,.
\end{aligned}
$$

By Lemma 0.1 (4) for $U = df\left(\frac{\partial}{\partial r}\right)$ and $X = \frac{\partial}{\partial r}$, we see

$$
(18) \quad h\big(df(\tfrac{\partial}{\partial r}),\, \sigma_f(\tfrac{\partial}{\partial r})\big) \;=\; \sum_{j=1}^{m} h\big(df(\tfrac{\partial}{\partial r}),\, df(e_j)\big)^2 \;=\; \big\|h\big(df(\tfrac{\partial}{\partial r}),\, df\big)\big\|^2\,,
$$

and

$$
(19) \qquad \sum_{i=1}^{m-1} h\big(df(e_i),\, \sigma_f(e_i)\big) \;=\; \sum_{i=1}^{m-1}\sum_{j=1}^{m} h\big(df(e_i),\, df(e_j)\big)^2 \;\leq\; \|f^*h\|^2\,.
$$

We have, by (17), (18) and (19),

$$
\begin{aligned}
(20) \qquad & 4\sum_{i=1}^{m} h\big(df(\nabla_{e_i}\overline{V}),\, \sigma_f(e_i)\big) \\
& \leq\; 4\xi'(r)r\,\big\|h\big(df(\tfrac{\partial}{\partial r}),\, df\big)\big\|^2 \;+\; 4\xi(r)\|f^*h\|^2 \;+\; 4c\xi(r)r\|f^*h\|^2\,.
\end{aligned}
$$

Therefore by (11), (16) and (20), we get

$$(21) \qquad -\int_M \xi'(r)r\,\|f^*h\|^2\,dv_g + (4-m)\int_M \xi(r)\,\|f^*h\|^2\,dv_g$$
$$+\,C\int_M \xi(r)r\,\|f^*h\|^2\,dv_g \;\geq\; -4\int_M \xi'(r)r\,\left\|h\big(df(\tfrac{\partial}{\partial r}),\,df\big)\right\|^2 dv_g\,,$$

where $C = (m+3)c$.

Take and fix a positive number ε, and let φ be a smooth function on $[0,\infty)$ such that

$$\varphi(r) = \varphi_\varepsilon(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq 1 \\ 0 & \text{if } 1+\varepsilon \leq r \end{cases} \qquad \text{and} \qquad \varphi'(r) \leq 0\,.$$

We define

$$\xi(r) \;=\; \xi_\rho(r) \;:=\; \varphi\left(\frac{r}{\rho}\right),$$

and we can verify

$$(22) \qquad \xi'(r)r \;=\; -\rho\frac{d}{d\rho}\xi(r)\,.$$

Then (21) and (22) imply, since $\|f^*h\|^2$ is independent of ρ,

$$\rho\frac{d}{d\rho}\int_M \xi(r)\,\|f^*h\|^2 dv_g \;+\; (4-m)\int_M \xi(r)\|f^*h\|^2\,dv_g$$
$$+\,C\int_M \xi(r)r\,\|f^*h\|^2\,dv_g \;\geq\; 4\rho\frac{d}{d\rho}\int_M \left\|h\big(df(\tfrac{\partial}{\partial r}),\,df\big)\right\|^2 \xi(r)dv_g\,.$$

Let ε tend to zero, and then, since $\xi(r)$ converges to the characteristic function for the ball $B_\rho(x_0)$, we have

$$\rho\frac{d}{d\rho}\int_{B_\rho(x_0)} \|f^*h\|^2 dv_g \;+\; (4-m)\int_{B_\rho(x_0)} \|f^*h\|^2 dv_g$$
$$+\,C\rho\int_{B_\rho(x_0)} \|f^*h\|^2\,dv_g \geq 4\rho\frac{d}{d\rho}\int_{B_\rho(x_0)} \left\|h\big(df(\tfrac{\partial}{\partial r}),\,df\big)\right\|^2 dv_g\,.$$

Multiply $e^{C\rho}\rho^{3-m}$ to the both sides of this inequality, and we have

$$\frac{d}{d\rho}\left\{e^{C\rho}\rho^{4-m}\int_{B_\rho(x_0)} \|f^*h\|^2 dv_g\right\}$$
$$\geq\; 4e^{C\rho}\rho^{4-m}\frac{d}{d\rho}\int_{B_\rho(x_0)} \left\|h\big(df(\tfrac{\partial}{\partial r}),\,df\big)\right\|^2 dv_g\,.$$

Thus we obtain the monotonicity formula. □

6 Bochner type formula

In this chapter we give a Bochner type formula.

6.1 Bochner type formula

In the following Bochner type formula, we assume that f is a smooth map.

Theorem 6.1 (Bochner type formula)

$$
\begin{aligned}
(23) \qquad \frac{1}{4}\triangle \|f^*h\|^2 \;=\; & \operatorname{div}_g \alpha_f \;-\; h\big(\tau_f,\, \operatorname{div}_g \sigma_f\big) \;+\; \frac{1}{2}\|\nabla(f^*h)\|^2 \\
& + \sum_{i,j,k} h\big(\nabla df(e_k,\, e_i),\, \nabla df(e_k,\, e_j)\big) h\big(df(e_i),\, df(e_j)\big) \\
& + \sum_{i,j} h\big(df\big({}^M R(e_i,\, e_j)\, e_j\big),\, \sigma_f(e_i)\big) \\
& - \sum_{i,j} h\big({}^N R\big(df(e_i),\, df(e_j)\big) df(e_j),\, \sigma_f(e_i)\big)\,,
\end{aligned}
$$

where τ_f is the tension field (1) of f and $\alpha_f(X) = h\big(\sigma_f(X),\, \tau_f\big)$.

Proof. We calculate

$$
\begin{aligned}
(24) \quad \frac{1}{4}\triangle \|f^*h\|^2 \;=\; & \frac{1}{4}\sum_k \nabla_{e_k}\nabla_{e_k}\Big(\sum_{i,j} h(df(e_i),\, df(e_j))\Big)^2 \\
=\; & \frac{1}{2}\sum_k \nabla_{e_k}\Big(\sum_{i,j} h(df(e_i),\, df(e_j))\nabla_{e_k} h(df(e_i),\, df(e_j))\Big) \\
=\; & \frac{1}{2}\Big\{\sum_k \sum_{i,j} (\nabla_{e_k} h(df(e_i),\, df(e_j)))^2 \\
& + \sum_k \sum_{i,j} \nabla_{e_k}\nabla_{e_k} h(df(e_i),\, df(e_j)) h(df(e_i),\, df(e_j))\Big\} \\
=\; & \frac{1}{2}\|\nabla f^*h\|^2 \\
& + \sum_{i,j} h\Big(\sum_k \nabla_{e_k}\nabla_{e_k} df(e_i),\, df(e_j)\Big)\, h\big(df(e_i),\, df(e_j)\big) \\
& \quad + \sum_{i,j,k} h\big(\nabla_{e_k} df(e_i),\, \nabla_{e_k} df(e_j)\big)\, h\big(df(e_i),\, df(e_j)\big)\,.
\end{aligned}
$$

Note

$$\sum_k \nabla_{e_k}\nabla_{e_k} df(e_i) \;=\; \sum_k \nabla_{e_k}\nabla_{e_i} df(e_k)$$
$$=\; \sum_k \nabla_{e_i}\nabla_{e_k} df(e_k) \;+\; df\Big(\sum_k {}^{M}R(e_i,\, e_k)e_k\Big) \;-\; \sum_k {}^{N}R(df(e_i),\, df(e_k))df(e_k)\,.$$

Then we obtain

$$\begin{aligned}(25)\quad & \sum_{i,j} h\Big(\sum_k \nabla_{e_k}\nabla_{e_k} df(e_i),\, df(e_j)\Big)\, h\big(df(e_i),\, df(e_j)\big)\\ &= \sum_{i,j} h\Big(\sum_k \nabla_{e_i}\nabla_{e_k} df(e_k),\, df(e_j)\Big)\, h\big(df(e_i),\, df(e_j)\big)\\ &\quad + \sum_{i,j} h\Big(df\Big(\sum_k {}^{M}R(e_i,\, e_k)e_k\Big),\, df(e_j)\Big) h\big(df(e_i),\, df(e_j)\big)\\ &\quad - \sum_{i,j,k} h\big(df(e_i),\, df(e_j)\big) h\big({}^{N}R(df(e_i),\, df(e_k)\big)df(e_k),\, df(e_j)\big)\,.\end{aligned}$$

Using the tension field τ_f , we see by Lemma 0.1 (4) for $U = \nabla_{e_i}\tau_f$ and for $X = e_i$

$$\begin{aligned}(26)\qquad & \sum_{i,j} h\Big(\sum_k \nabla_{e_i}\nabla_{e_k} df(e_k),\, df(e_j)\Big)\, h\big(df(e_i),\, df(e_j)\big)\\ &= \sum_{i,j} h\big(\nabla_{e_i}\tau_f,\, df(e_j)\big)\, h\big(df(e_i),\, df(e_j)\big)\\ &\overset{\text{Lemma 0.1 (4)}}{=} \sum_{i,j} h\big(\nabla_{e_i}\tau_f,\, \sigma_f(e_j)\big)\\ &= \sum_i (\nabla_{e_i}\alpha_f)(e_i) \;-\; h\Big(\tau_f,\, \sum_i (\nabla_{e_i}\sigma_f)(e_i)\Big)\\ &= \mathrm{div}_g\alpha_f \;-\; h\big(\tau_f,\, \mathrm{div}_g\sigma_f\big)\,.\end{aligned}$$

By (24), (25) and (26), we have the equality. □

6.2 Some remarks

We give two remarks for the Bochner type formula.

Remark 6.1 The second term in the right hand side of (23) vanishes if f is a stationary map, i.e., $\mathrm{div}_g\sigma_f = 0$.

Remark 6.2 The last two terms in the right hand side of (23) is

$$(27) \qquad + \sum_{i,j,k} h\big(df\,({}^M R(e_i,\, e_k)\, e_k),\, df(e_j)\big)\, h\big(df(e_i),\, df(e_j)\big)$$
$$- \sum_{i,j,k} h\big({}^N R\big(df(e_i),\, df(e_k)\big)df(e_k),\, df(e_j)\big)\, h\big(df(e_i),\, df(e_j)\big)\,.$$

The first term in (27) is non-negative if the Ricci curvature of M is non-negative. Furthermore the second term in (27) is non-negative if the curvature of N is non-positive.

7 The case of Lie groups

In this chapter we give the Euler-Lagrange equation in the case that the target manifold is a Lie group. Let f be a smooth map from a Riemannian manifold M into a compact Lie group G with bi-invariant metric. Let θ be the Maurer-Cartan form on G.

7.1 The case of harmonic maps

In this section, to compare with our case in the next section, we give the following known fact for harmonic maps into Lie groups.

Theorem 7.1 A smooth map f is harmonic if and only if $\mathrm{div}_g \alpha_f = 0$, where $\alpha_f = f^*\theta$.

For reader's convenience, we give a proof of Theorem 7.1. To prove the theorem, we use the following lemma which is applied also in the proof of Theorem 7.2 later.

Lemma 7.1 (See Urakawa [9], for example.)

$$\theta(\nabla_X V) \;=\; X(\theta(V)) + \frac{1}{2}[\alpha_f(X),\, \theta(V)]\,,$$

where $X \in \mathfrak{X}(M)$ and V is the variation vector field.

Proof of Theorem 7.1. We have

$$\begin{aligned}
\theta(\tau_f) &= \theta\Big(\sum_i(\nabla_{e_i}df)(e_i)\Big)\\
&= \sum_i \theta(\nabla_{e_i}(df(e_i)))\\
&= \sum_i \Big\{e_i(\theta(df(e_i))) + \frac{1}{2}[\alpha_f(e_i), \theta(df(e_i))]\Big\}\\
&= \sum_i \Big\{e_i(\alpha_f(e_i)) + \frac{1}{2}[\alpha_f(e_i), \alpha_f(e_i)]\Big\}\\
&= \sum_i e_i(\alpha_f(e_i))\\
&= \sum_i (\nabla_{e_i}\alpha_f)(e_i)\\
&= \mathrm{div}_g\alpha_f\,,
\end{aligned}$$

since $\alpha_f(X) = (f^*\theta)(X) = \theta(df(X))$. This proves the theorem. □

7.2 The case of stationary maps

As we saw in Chapter 1, f is a stationary map if and only if $\mathrm{div}_g\sigma_f = 0$, where

$$\sigma_f(X) = \sum_i h(df(X), df(e_i))df(e_i)\,.$$

We calculate the Euler-Lagrange equation in our case and we have

Theorem 7.2 A smooth map f is stationary if and only if $\mathrm{div}_g\beta_f = 0$, where $\beta_f = \theta \circ \sigma_f$.

Proof. The main idea of the proof is similar to Theorem 7.1. For simplicity, we use the notation α instead of α_f. By Lemma 7.1, we get

$$\begin{aligned}
\theta(\mathrm{div}_g\sigma_f) &= \theta\left(\sum_i(\nabla_{e_i}\sigma_f)(e_i)\right)\\
&= \theta\left(\sum_i \nabla_{e_i}(\sigma_f(e_i))\right)\\
&= \sum_i e_i(\theta(\sigma_f(e_i))) + \sum_i \frac{1}{2}[\alpha(e_i), (\theta(\sigma_f(e_i)))]\,.
\end{aligned}$$

We calculate the second term of the right hand side. Note that

$$\begin{aligned}(\theta(\sigma_f(e_i))) &= \theta\left(\sum_j h(df(e_i), df(e_j))df(e_j)\right) \\ &= \sum_j h(df(e_i), df(e_j))\theta(df(e_j)) \\ &= \sum_j h(df(e_i), df(e_j))\alpha(e_j).\end{aligned}$$

We have

$$\begin{aligned}\sum_i [\alpha(e_i), (\theta(\sigma_f(e_i)))] &= \sum_i [\alpha(e_i), \sum_j h(df(e_i), df(e_j))\alpha(e_j)] \\ &= \sum_i \sum_j h(df(e_i), df(e_j))[\alpha(e_i), \alpha(e_j)] \\ &= \left(\sum_{i<j} + \sum_{i=j} + \sum_{i>j}\right) h(df(e_i), df(e_j))[\alpha(e_i), \alpha(e_j)] \\ &= \sum_{i<j} h(df(e_i), df(e_j))[\alpha(e_i), \alpha(e_j)] \\ &\quad - \sum_{i<j} h(df(e_i), df(e_j))[\alpha(e_i), \alpha(e_j)] \\ &= 0.\end{aligned}$$

Therefore

$$\begin{aligned}\theta(\mathrm{div}_g \sigma_f) &= \sum_i e_i(\theta(\sigma_f(e_i))) \\ &= \sum_i (\nabla_{e_i}(\theta \circ \sigma_f))(e_i) \\ &= \mathrm{div}_g \beta_f .\end{aligned}$$

This completes the proof. $\square$

Further developments and open problems

In this thesis we discuss several results under general assumptions. For further developments, more precise arguments are necessary. We give some open problems:

(1) Find other examples of stationary maps.

(2) Classify stationary maps in some special cases, for example, the case that the source manifold M is a surface, or the case that the target manifold N is a Lie group.

(3) Find minimizers in each homotopy class of smooth maps from M into N.

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