

# ON A HOLONOMIC STRUCTURE OF SOME DOUBLE COSET DECOMPOSITION

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(Received September 30, 2011)

1

Introduction. Let  $GL_n(\mathbb{C})$  be the complex general linear group whose elements are  $n \times n$  invertible matrices. Let  $B$  be the subgroup of  $GL_n(\mathbb{C})$  consisting of all upper triangular matrices, and  $B'$  the subgroup of  $GL_n(\mathbb{C})$  consisting of all lower triangular matrices. We denote by  $V$  the set of all  $n \times n$  matrices over a field  $\mathbb{C}$  and by  $\rho$  the action of  $G = B \times B'$  which is given by  $\rho(b, b')v = bv'b'$  for  $(b, b') \in B \times B', v \in V$ .

The purpose of this paper is to investigate the micro-local structure of the triplet  $(G, \rho, V)$  by constructing a main part of the holonomy diagram.

2

The  $\rho(B, B')$ -orbit decomposition of  $GL_n(\mathbb{C}) \subset V$  is given by

$$GL_n(\mathbb{C}) = \bigcup_{w \in W} \rho(B, B')w,$$

where  $W = \{(\delta_{i, \sigma(j)}) \in G : \sigma \in S_n\}$ . The dimension of an orbit  $\rho(B, B')w$  is given by

$$\dim(\rho(B, B')w) = \frac{1}{2}n(n+1) + l(w),$$

where  $l$  is the standard length function.

We identify the dual space  $V^*$  with  $V$  by  $\langle w, w^* \rangle = \text{tr}(ww^*)$ . For  $w = (\delta_{i, \sigma(j)}) \in W$ , let  $S_w$  be the set  $\{(i, k), (\sigma(k), j) : 1 \leq k \leq n, i \leq \sigma(k), k \leq j\}$ . Then the conormal vector space  $V_w^*$  is given by

$$V_w^* = \{y = (y_{l,m}) \in V : y_{l,m} = 0 \text{ for } (l,m) \in S_w\}.$$

3

The element  $w = (\delta_{i, \sigma(j)})$  of  $W$  is identified with the element  $\sigma$  of the symmetric group  $S_n$ , and we denote by  $\sigma(1)\sigma(2) \dots \sigma(n)$  that element.

The case where  $l(w) = 0$ . In this case, we have  $w = 123 \dots n$ . Then the triplet  $(G_w, \rho_w, V_w^*)$  is a prehomogeneous representation, and there exist  $n - 1$  one-codimensional orbits.

These orbits correspond with elements  $w_i = (i, i + 1)123 \dots n$  for  $1 \leq i \leq n - 1$ .

The case where  $l(w) = 1$ . Then we have  $w = w_i = (i, i + 1)123 \dots n$  for  $1 \leq i \leq n - 1$ . When  $w$  is  $w_1 = (12)123 \dots n = 2134 \dots n$  or  $w_{n-1} = (n - 1, n)123 \dots n = 12 \dots n - 2, n, n - 1$ , then the triplet  $(G_w, \rho_w, V_w^*)$  is a prehomogeneous representation.

Let  $w$  be a point of  $W$ . We denote by  $\Lambda_w$  the conormal bundle  $T(\rho(B, B')w)^\perp$  of an orbit  $\rho(B, B')w$ . Assume that we have the following three conditions:

- (1) the triplet  $(G_w, \rho_w, V_w^*)$  and  $(G_{w'}, \rho_{w'}, V_{w'}^*)$  are prehomogeneous representations,
- (2)  $\dim \rho(B, B')w = \dim \rho(B, B')w' + 1$ , that is,  $l(w) = l(w') + 1$ ,
- (3)  $V_w^* \subset V_{w'}^*$ .

Then we have  $\dim(\Lambda_w \cap \Lambda_{w'}) = n - 1$ . And a one-codimensional orbit of  $(G_w, \rho_w, V_w^*)$  corresponds with a element  $w'$ .

For  $w \in W$ , we obtain the point  $w'$  which satisfies the conditions (1), (2) as follows. if  $y = (y_{lm}) \in V_w^*$  and  $y_{ij}$  is a element such that  $y_{ij} \neq 0, y_{ij} = 0 (1 \leq l \leq i - 1), y_{im} = 0 (j + 1 \leq m \leq n)$ , then we have  $w' = (w(j), i)w$ .

For  $1 \leq i \leq n - 1, 1 \leq k \leq n - i$ , we denote by  $\{i, i + k\}$  the following element:

$$\begin{aligned} & (i, i + k)(i, i + k - 1) \cdots (i, i + 1) \\ & (i - 1, n)(i - 1, n - 1) \cdots (i - 1, i + 1)(i - 1, i) \\ & \dots \\ & \dots \dots (13)(12)123 \dots n. \end{aligned}$$

The case where  $w = \{i, i + k\}$ . Then the triplet  $(G_w, \rho_w, V_w^*)$  is a prehomogeneous representation, and a one-codimensional orbit of  $(G_w, \rho_w, V_w^*)$  corresponds with  $w' = (w(k + 1), i + k + 1)w = (i, i + k + 1)\{i, i + k\} = \{i, i + (k + 1)\}$ .

#### 4

We associate  $\Lambda_w$  with  $w$  and connect  $w$  and  $w'$  if and only if  $\dim(\Lambda_w \cap \Lambda_{w'}) = n - 1$ . Thus we obtain a diagram which is called the holonomy diagram of  $(G, \rho, V)$ .

Following the ideas of the microlocal analysis, a main part of the holonomy diagram of the action  $\rho$  is given by Figure 2.

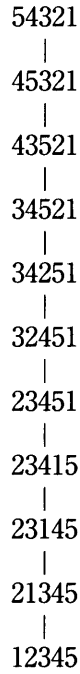


Figure 1. The case where  $n = 5$ .

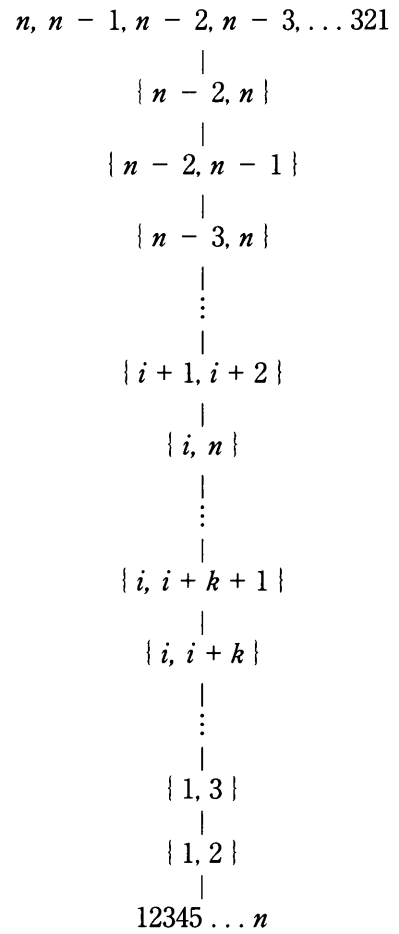


Figure 2. The general case.

### References

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