ON A HOLONOMIC STRUCTURE OF SOME DOUBLE COSET DECOMPOSITION

KASAI Shinichi

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1

Introduction. Let $GL_n(\mathbb{C})$ be the complex general linear group whose elements are $n \times n$ invertible matrices. Let B be the subgroup of $GL_n(\mathbb{C})$ consisting of all upper triangular matrices, and B' the subgroup of $GL_n(\mathbb{C})$ consisting of all lower triangular matrices. We denote by V the set of all $n \times n$ matrices over a field C and by ρ the action of $G = B \times B'$ which is given by $\rho(b, b')v = bv\,{}^tb'$ *for* $(b, b') \in B \times B', v \in V$.

The purpose of this paper is to investigate the micro-local structure of the triplet (G, ρ, V) by constructing a main part of the holonomy diagram.

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The $\rho(B, B')$-orbit decomposition of $GL_n(\mathbb{C}) \subset$ V is given by

$$GL_n(\mathbb{C}) = \bigcup_{w \in W} \rho(B, B')w,$$

where $W = \{(\delta_{i,\sigma(j)}) \in G : \sigma \in S_n\}$. The dimension of an orbit $\rho(B, B')w$ is given by

$$\dim(\rho(B, B')w) = \frac{1}{2}n(n+1) + l(w),$$

where l is the standard length function.

We identify the dual space V^* with V by $\langle w, w^* \rangle = \mathrm{tr}(ww^*)$. For $w = (\delta_{i,\sigma(j)}) \in$ W, let S_w be the set $\{(i, k), (\sigma(k), j) : 1 \leqq k \leqq n, i \leqq \sigma(k), k \leqq j\}$. Then the conornal vector space V_w^* is given by

$$V_w^* = \{ y = (y_{l,m}) \in V : y_{l,m} = 0 \text{ for } (l,m) \in S_w \}.$$

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The element $w = (\delta_{i,\sigma(j)})$ of W is identied with the element σ of the symmetric group S_n, and we denote by $\sigma(1)\sigma(2)\ldots\sigma(n)$ that element.

The case where $l(w) = 0$. In this case, we have $w = 123\ldots n$. Then the triplet (G_w, ρ_w, V_w^*) is a prehomogeneous representation, and there exist $n-1$ one-codimensional orbits.

These orbits correspond with elements $w_i = (i, i+1)123 \dots n$ for $1 \leqq i \leqq n-1$.

The case where $l(w) = 1$. Then we have $w = w_i = (i, i+1)123 \dots n$ for $1 \leqq i \leqq n-1$. When w is $w_1 = (12)123 \dots n = 2134 \dots n$ or $w_{n-1} = (n-1, n)123 \dots n = 12 \dots n-2, n, n-1$, then the triplet (G_w, ρ_w, V_w^*) is a prehomogeneous representation.

Let w be a point of W. We denote by Λ_w the conormal bundle $T(\rho(B,B')w)^{\perp}$ of an orbit $\rho(B,B')w$. Assume that we have the following three conditions:

(1) the triplet (G_w, ρ_w, V_w^*) and $(G_w', \rho_w', V_w^*{}')$ are prehomogeneous representations,

(2) $\dim \rho(B,B')w = \dim \rho(B,B')w' + 1$, that is, $l(w) = l(w') + 1$,

(3) $V_w^* \subset V_w^*{}'$.

Then we have $\dim(\Lambda_w \cap \Lambda_w{}') = n-1$. And a one-codimensional orbit of (G_w, ρ_w, V_w^*) corresponds with a element w'.

For $w \in W$, we obtain the point w' which satises the conditions (1), (2) as follows. if $y = (y_{lm}) \in V_w^*$ and y_{ij} is a element such that $y_{ij} \neq 0$, $y_{lj} = 0 (1 \leqq l \leqq i-1)$, $y_{im} = 0 (j+1 \leqq m \leqq n)$, then we have $w' = (w(j), i)w$.

For $1 \leqq i \leqq n-1$, $1 \leqq k \leqq n-i$, we denote by $\{i, i+k\}$ the following element:

$$
\begin{aligned}
&(i, i+k)(i, i+k-1) \cdots (i, i+1) \\
&(i-1, n)(i-1, n-1) \cdots (i-1, i+1)(i-1, i) \\
&\dots \\
&\cdots\cdots (13)(12)\ 123 \dots n.
\end{aligned}
$$

The case where $w = \{i, i+k\}$. Then the triplet (G_w, ρ_w, V_w^*) is a prehomogeneous representation, and a one-codimensional orbit of (G_w, ρ_w, V_w^*) corresponds with $w' = (w(k+1), i+k+1)w = (i, i+k+1)\{i, i+k\} = \{i, i+(k+1)\}$.

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We associate Λ_w with w and connect w and w' if and only if $\dim(\Lambda_w \cap \Lambda_w{}') = n-1$. Thus we obtain a diagram which is called the holonomy diagram of (G, ρ, V).

Following the ideas of the microlocal analysis, a main part of the holonomy diagram of the action ρ is given by Figure 2.

54321
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45321
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43521
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34521
|
34251
|
32451
|
23451
|
23415
|
23145
|
21345
|
12345

Figure 1. The case where n = 5.

$n, n-1, n-2, n-3, \ldots 321$
|
$\{n-2, n\}$
|
$\{n-2, n-1\}$
|
$\{n-3, n\}$
|
⋮
|
$\{i+1, i+2\}$
|
$\{i, n\}$
|
⋮
|
$\{i, i+k+1\}$
|
$\{i, i+k\}$
|
⋮
|
$\{1, 3\}$
|
$\{1, 2\}$
|
$12345 \ldots n$

Figure 2. The general case.

References

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