EQUIVARIANT SCHUBERT CALCULUS OF COXETER GROUPS

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ABSTRACT. We consider an equivariant extension for Hiller's Schubert calculus on the coinvariant ring of a finite Coxeter group.

1. INTRODUCTION

Throughout this note, all the cohomology ring is with the real coefficients unless otherwise stated. The primary goal of *Schubert calculus* is to describe the cohomology ring structure of the flag variety with respect to a distinguished basis consisting of the *Schubert classes*. Among many strategies for this subject is to reformulate the topological problem in an algebraic fashion.

Let G be a connected complex Lie group, B be its Borel sub-group. Then the homogeneous space G/B is called the *flag variety*. A family of cohomology classes indexed by the Weyl group W of G called the Schubert classes form a basis for the cohomology $H^*(G/B)$. On the other hand, $H^*(G/B)$ can be identified with the *coinvariant ring* of W, i.e. the polynomial ring divided by the ideal generated by the invariant polynomials of W. The relation between those two presentation of $H^*(G/B)$ was studied independently by [2] and [7]. Based on it, Hiller ([10]) rephrased and extended Schubert calculus purely in terms of the coinvariant ring of any finite Coxeter group including non-crystallographic ones, by defining a set of basis polynomials in the coinvariant ring corresponding to the Schubert classes.

We can impose another structure on G/B: it admits the canonical action of the maximal torus T and we can consider the equivariant cohomology with respect to this action. In this note, we investigate the equivariant cohomology $H_T^*(G/B)$ and develop an equivariant version of Hiller's Schubert calculus for the double coinvariant ring of a finite Coxeter group W. The main result is the construction of a Hiller-type double schubert polynomial given in a uniform manner for any finite Coxeter groups (see Definition 4.3).

The organization of this note is as follows: In §2 and §3, we recall basic notions of Schubert calculus. In §4, we define the double coinvariant ring for a finite Coxeter group and its equivariant Schubert classes. Using this definition, we prove Chevalley rule in §5, and a symmetry property in §6. We observe a relation between ordinary and equivariant setting in §7. §8 is devoted to examples.

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2. Coxeter groups

Here we collect some well-known facts on Coxeter groups, which are necessary in the later sections. Readers refer to [6] or [11] for detail.

A finite Coxeter group W is a generalization to a Weyl group of a Lie group. It is defined by generators and relations as:

$$W = \langle s_1, \dots, s_n \mid (s_i s_j)^{m_{ij}} = e \rangle,$$

where $m_{ii} = 1$ and $2 \le m_{ij} < \infty$. The number of generators n is called the *rank* of W. The complete classification for irreducible Coxeter groups is known and W is one of the

(1) crystallographic groups $A_n, B_n, C_n, D_n, G_2, F_4, E_6, E_7, E_8$, which correspond to the Weyl groups of Lie groups, or
(2) non-crystallographic groups $I_2(n), H_3, H_4$.

A finite Coxeter group of rank n coincides with a finite reflection group on $\mathbb{R}^n$: each generator s_i can be regarded as the reflection through the hyperplane defined by $\alpha_i = 0$, where $\alpha_i \in (\mathbb{R}^n)^*$ is called the *simple root*. $\beta \in (\mathbb{R}^n)^*$ is called the root if $\beta = w(\alpha_i)$ for some simple root α_i and $w \in W$. If a root β is a linear combination of simple roots with non-negative coefficients, it is called a *positive root*. The reflection through the hyperplane defined by $\beta = 0$ for a positive root β is denoted by s_β, i.e. $s_\beta = w s_i w^{-1} \in W$.

We fix this standard representation on $\mathbb{R}^n$ for each W and the action of W on the symmetric algebra over $(\mathbb{R}^n)^*$, which we denote by $\mathbb{R}[t_1, \dots, t_n]$, is defined by extending the representation. Namely, we define $w(f(t)) = f(w^{-1}(t))$ for $f(t) \in \mathbb{R}[t_1, \dots, t_n]$ and $w \in W$.

The following definitions are essential for our purpose:

Definition 2.1 (see [6, 11]). (1) *The* length $l(w) \in \mathbb{Z}_{\ge 0}$ *for $w \in W$ is the minimal length of the presentation of w by a product of $s_1, \dots, s_n$, which is called a* reduced word *for w.*
(2) *There is a unique element $w_0 \in W$ of the maximum length called the* longest element.
(3) *We denote $w <_\beta v$ iff $w = s_\beta v$ and $l(w) < l(v)$.*
(4) *The (strong) Bruhat order $w \le v$ is the closure relation of $w <_\beta v$.*

The following Lemma on the Bruhat order is used frequently in our discussion.

Lemma 2.2 (Exchange condition, see [6, 11]). *For any reduced word $v = s_{i_1} \cdots s_{i_{l(v)}}$, $w <_\beta v$ iff $w = s_{i_1} \cdots s_{i_{k-1}} s_{i_{k+1}} \cdots s_{i_{l(v)}}$ and $\beta = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})$ for some $1 \le k \le l(v)$.*

In particular, for a fixed $v \in W$, the number of positive roots β such that $w <_\beta v$ for some $w \in W$ is equal to $l(v)$.

3. Schubert calculus

In this section, we briefly recall the result by Berstein-Gelfand-Gelfand ([2]), which studys the ordinary cohomology of flag varieties. Let G be a connected complex Lie group of rank n, B be its Borel sub-group. Then the (right quotient) homogeneous space G/B is known to be a smooth projective variety with the T-action induced by the left multiplication and called the (generalized) *flag variety*. Denote by Π^+ the set of positive roots and by $\{\alpha_i \mid 1 \le i \le n\} \subset \Pi^+$ the set of simple roots. Then the Weyl group W of G is generated by the simple reflections $s_1, \dots, s_n$ corresponding to $\alpha_1, \dots, \alpha_n$.

Let B_- be the Borel sub-group opposite to B so that $B \cap B_-$ is the maximal algebraic torus. The Bruhat decomposition (see for example [4]) $G = \coprod_{w \in W} B_- w B$ induces a left T-stable cell decomposition $G/B = \coprod_{w \in W} B_- wB/B$. The class Z_w corresponding to the (dual of the) cell $B_- wB/B$ is called the *Schubert class* and having degree $2l(w)$. Since the cell decomposition involves even cells only, we have

$$H^*(G/B) = \bigoplus_{w \in W} \langle Z_w \rangle.$$

On the other hand, by the classical theorem by Borel [5], the cohomology ring has the form of so-called coinvariant ring $\mathbb{R}_W[x]$ of W:

$$H^*(G/B) = \mathbb{R}_W[x] := \frac{\mathbb{R}[x]}{(\mathbb{R}^+[x]^W)},$$

where $\mathbb{R}[x]$ is the polynomial ring $\mathbb{R}[x_1, \ldots, x_n]$ and $\left(\mathbb{R}^+[x]^W\right)$ is the ideal generated by the positive degree invariant polynomials. Here we regard $\mathbb{R}[x] \cong H^*(BT)$ as the symmetric algebra over the dual Lie algebra $\mathfrak{t}^*$, and the generators have degree 2.

In the fundamental work by Berstein-Gelfand-Gelfand ([2]), the relationship between the two presentations of $H^*(G/B)$ are revealed using the *divided difference operators*.

Definition 3.1 ([2]). *For a simple root α_i, define $\Delta_i : \mathbb{R}_W[x] \to \mathbb{R}_W[x]$ of degree -2 as*

$$\Delta_i f(x) = \frac{f(x) - f(s_i(x))}{-\alpha_i(x)}.$$

For a reduced word $w = s_{i_1} \cdots s_{i_k} \in W$, define

$$\Delta_w = \Delta_{i_1} \circ \cdots \circ \Delta_{i_k}.$$

Then it is independent of the choice of a reduced word for $w \in W$.

Theorem 3.2 ([2]). (1) *A polynomial $f \in R_W[x]$ represents the cohomology class*

$$\sum_{w \in W} \Delta_w(f)(0) Z_w.$$

(2) *A polynomial representative $\sigma_w(x)$ of Z_w is obtained by*

$$\sigma_w(x) = \begin{cases} \dfrac{(-1)^{|W|}}{|W|} \prod_{\beta \in \Pi^+} \beta(x) & (w = w_0) \\ \Delta_{w^{-1} w_0} \sigma_{w_0}(x) & (w \neq w_0) \end{cases}$$

The main problem in Schubert calculus is to give an algorithm for expressing the cup product of two Schubert classes by a linear combination of Schubert classes

$$Z_u \cup Z_v = \sum_{w \in W} c_{uv}^w Z_w, \quad c_{uv}^w \in \mathbb{Z},$$

where c_{uv}^w is called the *structure constant*. By the previous Theorem, this problem has an equivalent in the coinvariant ring setting. This point of view was pursued by Hiller ([10]) as follows:

Definition 3.3 ([10])**.** *Let W be a finite Coxeter group and $\mathbb{R}_W[x]$ be its coinvariant ring. Define Schubert classes in $\mathbb{R}_W[x]$ as*

$$\sigma_w(x) = \begin{cases} \dfrac{(-1)^{|W|}}{|W|} \prod_{\beta\in\Pi^+} \beta & (w = w_0) \\ \Delta_{w^{-1}w_0}\sigma_{w_0}(x) & (w \neq w_0) \end{cases}$$

Schubert classes form a vector space basis for $\mathbb{R}_W[x]$, so now the problem of structure constants is translated into an algebraic one, that is, to find an algorithm for c_{uv}^w in the following equation

$$\sigma_u \cdot \sigma_v = \sum_{w\in W} c_{uv}^w \sigma_w, \quad c_{uv}^w \in \mathbb{Z}.$$

Hiller showed, for example, the Chevalley rule in this setting.

4. Equivariant Schubert calculus

To generalize Hiller's Schubert calculus, what we concern is the (Borel) T-equivariant cohomology $H_T^*(G/B)$ with respect to the T-action induced by the left multiplication on G/B. (For a more detailed treatment in the topological aspect of our argument, readers refer to [12].) We consider $H_T^*(G/B)$ as an algebra over $H_T^*(pt) = \mathbb{R}[t] = \mathbb{R}[t_1, \dots, t_n]$ by the equivariant map $G/B \to pt$. Just as in the case of ordinary cohomology, $H_T^*(G/B)$ is a free $\mathbb{R}[t]$-module generated by Schubert classes, i.e.

$$H_T^*(G/B) \cong \bigoplus_{w\in W} \mathbb{R}[t_1, \dots, t_n]\langle Z_w \rangle.$$

On the other hand, the following description for the equivariant cohomology is well-known:

Proposition 4.1. *As $\mathbb{R}[t]$-algebras,*

$$H_T^*(G/B) \cong \frac{\mathbb{R}[t_1, \dots, t_n, x_1, \dots, x_n]}{I_W},$$

where I_W is the ideal generated by $f(t_1, \dots, t_r) - f(x_1, \dots, x_r)$ for all W-invariant polynomials f of positive degree.

Proof. The Borel construction associated to the T-action on G/B fits in the following pull-back diagram:

(4.1)
$$\begin{array}{ccccc} G/B & = & G/B & & \\ \downarrow & & \downarrow & & \\ ET \times_T G/B & \longrightarrow & EG \times_G G/B & = & BT \\ \downarrow & & \downarrow & & \\ BT & \longrightarrow & BG. & & \end{array}$$

The Eilenberg-Moore spectral sequence converges to $H_T^*(G/B) \cong H^*(ET \times_T G/B)$ with the E_2-term $\mathrm{Tor}_{H^*(BG)}(H^*(BT), H^*(BT))$. Recall from [5] that $H^*(BG) \cong H^*(BT)^W$. Since $H^*(BT)$ is free over $H^*(BG)$, there are only non-trivial entries in the 0-th column and so $E_2 \cong H_T^*(G/B)$ as $H^*(BT)$-algebras. Here $E_2 \cong \mathrm{Tor}_{H^*(BG)}(H^*(BT), H^*(BT))$ is just the tensor product $H^*(BT) \otimes_{H^*(BG)} H^*(BT)$. □

We call $\dfrac{\mathbb{R}[t_1, \ldots, t_n, x_1, \ldots, x_n]}{I_W}$ the *double coinvariant ring* of W and denote it by $\mathbb{R}_W[t; x]$.

The equivariant cohomology $H_T^*(G/B)$ has yet another description by *GKM-theory* [9]. The fixed points set of the T-action is $\{wB/B \mid w \in W\}$ so we have the localization map

$$H_T^*(G/B) \xrightarrow{\bigoplus_{w\in W} i_w^*} \bigoplus_{w\in W} H_T^*(wB/B) \cong \bigoplus_{w\in W} H^*(BT) \cong \bigoplus_{w\in W} \mathbb{R}[t].$$

It is known that this is an injection and the image is described by a certain combinatorial condition called GKM condition. The relation between these three descriptions are summarized as follows.

Proposition 4.2 ([15]). (1) *For a Schubert class $Z_w \in H_T^*(G/B)$,*

$$i_v^*(Z_w) = \begin{cases} 0 & (l(v) \le l(w) \text{ and } v \ne w) \\ \prod_{\beta\in\Pi^+, \exists v <_\beta w} \beta & (v = w). \end{cases}$$

(2) *For $f(t; x) \in \mathbb{R}_W[t; x]$, $i_w^*(f(t; x)) = f(t; w^{-1}(x))$.*

Now just as Hiller did, we bring equivariant Schubert calculus into the double coinvariant ring setting for any finite Coxeter group W. Using the divided difference operators extended by $\mathbb{R}[t]$-linearity, we can define the Schubert classes in $\mathbb{R}_W[t; x]$.

Definition 4.3. *Let W be a finite Coxeter group. For $w \in W$, we define the partition set of w as*

$$P_i(w) = \{(w_1, w_2, \ldots, w_i) \in W^i \mid w_1 \cdot w_2 \cdots w_i = w, l(w_k) > 0\ \forall k, l(w_1) + \cdots + l(w_i) = l(w)\}$$

Then the Schubert classes in $\mathbb{R}_W[t; x]$ are defined to be

$$\mathfrak{S}_{w_0}(t; x) = \sigma_{w_0}(x) + \sum_{v\in W} \sum_{i=1}^{l(w_0v^{-1})} \sum_{(w_1,w_2,\ldots,w_i)\in P_i(w_0v^{-1})} (-1)^i \sigma_{w_1}(t)\sigma_{w_2}(t)\cdots\sigma_{w_i}(t)\sigma_v(x) \in \mathbb{R}_W(t; x),$$

and

$$\mathfrak{S}_w(t; x) = \Delta_{w^{-1}w_0}\mathfrak{S}_{w_0}(x) = \sigma_w(x) + \sum_{v<w} \sum_{i=1}^{l(wv^{-1})} \sum_{(w_1,w_2,\ldots,w_i)\in P_i(wv^{-1})} (-1)^i \sigma_{w_1}(t)\sigma_{w_2}(t)\cdots\sigma_{w_i}(t)\sigma_v(x),$$

where σ_w is Hiller's Schubert class given in Definition 3.3.

Notice that $\mathfrak{S}_w(t; t) = \begin{cases} 0 & (w \ne e) \\ 1 & (w = e) \end{cases}$ since $\sigma_e(x) = 1$ and we can rewrite

$$\mathfrak{S}_w(t; x) = \sigma_w(x) - \sum_{i=1}^{l(w)} \sum_{(w_1,w_2,\ldots,w_i)\in P_i(w)} (-1)^i \sigma_{w_1}(t)\sigma_{w_2}(t)\cdots\sigma_{w_{i-1}}(t)(\sigma_{w_i}(t) - \sigma_{w_i}(x)). \tag{4.2}$$

And so by definition, $(\Delta_v\mathfrak{S}_w)(t; t) = \begin{cases} 0 & (v \ne w) \\ 1 & (v = w) \end{cases}$. This is the key property of the definition.

Remark 4.4. *A representative for an element in $\mathbb{R}_W[t;x]$ is determined up to the ideal I_W. We can choose another representative by replacing Hiller's Schubert class σ_w by other Schubert class. For example, when W is of type A_{n-1}, we can take Lascoux and Schützenberger's Schubert polynomial*

$$\sigma'_w(x) = \Delta_{w^{-1}w_0} x_1^{n-1} x_2^{n-2} \cdots x_{n-1}.$$

And then

$$\begin{aligned}\mathfrak{S}_{[121]}(t;x) =& \sigma_{[121]}(x) - \sigma_{[1]}(t)\sigma_{[21]}(x) - \sigma_{[2]}(t)\sigma_{[12]}(x)\\ &+(\sigma_{[1]}(t)\sigma_{[2]}(t) - \sigma_{[12]}(t))\sigma_{[1]}(x) + (\sigma_{[2]}(t)\sigma_{[1]}(t) - \sigma_{[21]}(t))\sigma_{[2]}(x)\\ &+(-\sigma_{[121]}(t) + \sigma_{[1]}(t)\sigma_{[21]}(t) + \sigma_{[2]}(t)\sigma_{[12]}(t)\\ &+\sigma_{[12]}(t)\sigma_{[1]}(t) + \sigma_{[21]}(t)\sigma_{[2]}(t)\\ &-\sigma_{[2]}(t)\sigma_{[1]}(t)\sigma_{[2]}(t) - \sigma_{[1]}(t)\sigma_{[2]}(t)\sigma_{[1]}(t))\\ =&(x_1 - t_1)(x_1 - t_2)(x_2 - t_1)\end{aligned}$$

These Schubert classes form a free $\mathbb{R}[t]$-basis for $\mathbb{R}_W[t;x]$.

Theorem 4.5.

$$\mathbb{R}_W[t;x] \cong \bigoplus_{w\in W} \mathbb{R}[t_1,\ldots,t_n]\langle \mathfrak{S}_w(t;x)\rangle$$

Proof. Note that $\mathbb{R}[t]$ is a local ring whose maximal ideal is $\mathbb{R}[t]^+$ with $\mathbb{R}[t]/\mathbb{R}[t]^+ = \mathbb{R}$ and $\mathbb{R}_W[t;x]/\mathbb{R}[t]^+\mathbb{R}_W[t;x] = \mathbb{R}_W[x]$. We apply the following form of Nakayama's Lemma [1, Prop. 2.8]:

$f_1(t;x),\ldots,f_N(t;x)$ generate $\mathbb{R}_W[t;x]$ over $\mathbb{R}[t] \Leftrightarrow f_1(0;x),\ldots,f_N(0;x)$ generate $\mathbb{R}_W[x]$ over $\mathbb{R}$.

Since $\mathfrak{S}_w(0;x) = \sigma_w(x)$, $\{\mathfrak{S}_w(t;x) \mid w \in W\}$ generate $\mathbb{R}_W[t;x]$ over $\mathbb{R}[t]$.

We now show that $\{\mathfrak{S}_w(t;x) \mid w \in W\}$ are free over $\mathbb{R}[t]$. Assume that $\sum_{v\in W} c_v(t)\mathfrak{S}_v(t;x) = 0$. For any $w \in W$, applying Δ_w and evaluating at $x = t$, we obtain

$$0 = \sum_{v\in W} c_v(t)\cdot(\Delta_w\mathfrak{S}_v)(t;t) = c_w(t).$$

□

In fact, there is a formula to express any $f(t;x) \in \mathbb{R}_W[t;x]$ as a $\mathbb{R}[t]$-linear combination of Schubert classes:

Proposition 4.6. *For $f(t;x) \in \mathbb{R}_W[t;x]$,*

$$f(t;x) = \sum_{w\in W} (\Delta_w(f)(t;t)\cdot\mathfrak{S}_w(t;x))$$

Proof. Suppose that $f(t;x) = \sum_{v\in W} c_v(t)\mathfrak{S}_v(t;x)$. Then $\Delta_w(f)(t;x) = \sum_{v\in W} c_v(t)\cdot\Delta_w(\mathfrak{S}_v)(t;x)$. Since $(\Delta_w\mathfrak{S}_v)(t;t) = \begin{cases}0 & (v \neq w)\\ 1 & (v = w)\end{cases}$, we have $\Delta_w(f)(t;t) = c_w(t)$. □

To show some properties of $\mathbb{R}_W[t;x]$, it's convenient to recall the definition of *GKM-ring*:

Definition 4.7 ([9])**.** *A subring of* $\bigoplus_{w\in W}\mathbb{R}[t]$ *called the GKM-ring for* W *is defined as:*

$$F_W := \left\{ \bigoplus_{w\in W} h_w(t) \in \bigoplus_{w\in W} \mathbb{R}[t] \mid h_w(t) - h_v(t) \text{ is divisible by } \beta(t) \in \Pi^+ \text{ when } w <_\beta v \right\}.$$

A $\mathbb{R}[t]$*-module map called the localization map* $\bigoplus_{w\in W} i_w^* : \mathbb{R}_W[t;x] \to F_W$ *is defined as* $i_w^*(f(t;x)) = f(t;w^{-1}(t))$*. Note that this map is well-defined because* $i_w^*(f(t;x)) - i_{s_\beta w}^*(f(t;x)) = f(t;w^{-1}(t)) - f(t;w^{-1}s_\beta(t))$ *is divisible by* $\beta(t)$*.*

Just as the cohomological localization map, it is injective.

Lemma 4.8. $\bigoplus_{w\in W} i_w^* : \mathbb{R}_W[t;x] \to F_W$ *is injective.*

Proof. Take $f(t;x) \in \mathbb{R}_W[t;x]$ such that $i_w^*(f) = 0$ $(\forall w \in W)$. By the definition of the divided difference operator, $i_w^*(\Delta_v(f)) = 0$ $(\forall v, w \in W)$, in particular, $i_e^*(\Delta_v(f)) = \Delta_v(f)(t;t) = 0$. Hence by Proposition 4.6, we have $f(t;x) = 0$. □

We show that the Schubert classes are characterized through the localization map.

Proposition 4.9.

$$i_v(\mathfrak{S}_w(t;x)) = \begin{cases} 0 & (l(v) \le l(w) \text{ and } v \ne w) \\ \prod_{\beta\in\Pi^+,\exists v<_\beta w} \beta(t) & (v = w). \end{cases}$$

On the other hand, if $h_w(t;x) \in \mathbb{R}_W^{2l(w)}[t;x]$ *satisfies* $h_w(t;x) = 0$ *when* $l(v) \le l(w)$ *and* $v \ne w$*, then* $h_w = c\mathfrak{S}_w$ *for some* $c \in \mathbb{R}$*.*

Proof. First, note that

$$\begin{aligned} i_w^*(\Delta_i f(t;x)) &= i_w^*\left(\frac{f(t;x) - f(t;s_i(x))}{-\alpha_i(x)} \right) \\ &= \frac{f(t;w^{-1}(t)) - f(t;s_i w^{-1}(t))}{-\alpha_i(w^{-1}(t))} \\ &= \frac{i_w^* f(t;x) - i_{ws_i}^* f(t;x)}{-\alpha_i(w^{-1}(t))}. \end{aligned}$$

We induct on the length of $w \in W$. Recall that $i_e^*\mathfrak{S}_w = 0$ for $w \ne e$. Take $u \in W$ such that $l(u) < l(w)$ and assume $i_u^*\mathfrak{S}_w = 0$. Then for any simple reflection s_i such that $l(us_i) = l(u) + 1$, we have

$$i_{us_i}^*\mathfrak{S}_w = \alpha_i(u^{-1}(t)) i_u^*(\Delta_i\mathfrak{S}_w) - i_u^*\mathfrak{S}_w = \alpha_i(u^{-1}(t)) i_u^*(\mathfrak{S}_{ws_i}) = \begin{cases} 0 & (u \ne ws_i) \\ \alpha_i(u^{-1}(t)) \prod_{\exists v<_\beta u} \beta(t) & (u = ws_i) \end{cases}.$$

So again by induction on the length of u and the Exchange condition, we have

$$i_v(\mathfrak{S}_w(t;x)) = \begin{cases} 0 & (l(v) \le l(w) \text{ and } v \ne w) \\ \prod_{k=1}^{l(w)} \alpha_{i_k}\left((s_{i_1} \cdots s_{i_{k-1}})^{-1}(t) \right) = \prod_{\beta\in\Pi^+,\exists v<_\beta w} \beta(t) & (v = w), \end{cases}$$

where $w = s_{i_1} \cdots s_{i_{l(w)}}$.

Let $h_w(t;x) \in \mathbb{R}_W^{2l(w)}[t;x]$ such that $h_w(t;x) = 0$ when $l(v) \le l(w)$ and $v \ne w$. Since $i_w^*(h(t;x)) - i_{s_\beta w}^*(h(t;x))$ is divisible by $\beta(t)$ and any two distinct positive roots are linearly independent,

$i_w^*(h_w(t;x))$ is divisible by $\prod_{\beta\in\Pi^+,\exists v<_\beta w}\beta(t)$. By degree reason, $i_w^*(h_w(t;x)) = c\prod_{\beta\in\Pi^+,\exists v<_\beta w}\beta(t)$. Put $h'_w(t;x) = h_w(t;x) - c\mathfrak{S}_w(t;x)$ then $i_v^*(h'_w(t;x)) = 0$ if $l(v) \le l(w)$. Let $u \in W$ be a minimal length element such that $i_u^*(h'_w(t;x)) \ne 0$. Then by the same argument above, $i_u^*(h'_w(t;x))$ should be divisible by $\prod_{\beta\in\Pi^+,\exists v<_\beta u}\beta(t)$. But $2l(u) > 2l(w)$ and by degree reason, this leads to contradiction. By the injectivity of the localization map, we have $h'_w(t;x) = 0$, i.e. $h_w(t;x) = c\mathfrak{S}_w(t;x)$. □

This and Proposition 4.2 assert that the Schubert class $\mathfrak{S}_w \in \mathbb{R}_W[t;x]$ we consider in the algebraic setting coincides with the Schubert class $Z_w \in H_T^*(G/B)$ in the topological setting when W is a Weyl group of a Lie group.

There are two interesting Corollaries to this Proposition.

Corollary 4.10. *The localization map gives an isomorphism between the GKM-ring F_W and the double coinvariant ring $\mathbb{R}_W[t;x]$.*

Proof. We only have to show surjectivity. Let $\bigoplus_{w\in W} h_w(t;x) \in F_W$. Take $v \in W$ such that $h_v(t;x) \ne 0$ and $h_u(t;x) = 0$ for $l(u) < l(v)$. Then the same argument as in the proof of the previous Proposition, $h_v(t;x)$ should be divisible by $\prod_{\beta\in\Pi^+,\exists u<_\beta v}\beta(t)$. Then put $\bigoplus_{w\in W} h'_w(t;x) = \bigoplus_{w\in W}\left(h_w(t;x) - \frac{h_v(t;x)}{\prod_{\beta\in\Pi^+,\exists u<_\beta v}\beta(t)}\cdot i_w^*(\mathfrak{S}_v(t;x))\right) \in F_W$ so that $h'_v(t;x) = 0$. Iterating this process shows that $\bigoplus_{w\in W} i_w^*$ is surjective. □

Corollary 4.11 (c.f. [3, 13])**.** *Let $v = s_{i_1}\cdots s_{i_{l(v)}}$ be a reduced word. The localization image of a Schubert class is determined to be*

$$i_v^*(\mathfrak{S}_w(t;x)) = \sum \beta_{j_1}\cdots\beta_{j_{l(w)}}$$

where $\beta_k = s_{i_1}\cdots s_{i-1}\alpha_{i_k}$ and the sum runs over $(1 \le j_1 < \cdots < j_{l(w)} \le l(v)$ such that $s_{i_{j_1}}\cdots s_{i_{j_{l(w)}}} = w$.

Proof. Using the Exchange condition, one can easily see that the right hand side resides in the GKM ring F_W. Because $\mathbb{R}_W[t;x] \cong F_W$, there is a lift $h(t;x) \in \mathbb{R}_W[t;x]$ which satisfies $i_v^*(h(t;x)) = \sum\beta_{j_1}\cdots\beta_{j_{l(w)}}$. This $h(t;x)$ trivially meets the condition in the previous Proposition. (In particular, the right hand side is independent of the choice for a reduced word.) □

5. Chevalley rule

Here we concern with the equivariant version of the structure constant $c_{uv}^w(t) \in \mathbb{R}[t]$, where

$$\mathfrak{S}_u\cdot\mathfrak{S}_v = \sum_{w\in W} c_{uv}^w(t)\mathfrak{S}_w.$$

Since $\mathfrak{S}_w(0;x) = \sigma_w(x)$, the equivariant version $c_{uv}^w(t)$ is a polynomial whose constant term is the ordinary structure constant c_{uv}^w.

Chevalley rule, which computes the product of any Schubert class and that of degree two, is well-known for the equivariant cohomology of flag varieties (see [13]). It can be slightly extended to this double coinvariant ring setting. First we identify the degree two Schubert classes.

Lemma 5.1.

$$\mathfrak{S}_{s_i}(t;x) = \omega_i(t) - \omega_i(x),$$

where the linear form $\omega_i \in \mathbb{R}^2[t]$ *is the* fundamental weight *defined by* $\langle \alpha_j, \omega_i \rangle = \begin{cases} 0 & (i \neq j) \\ |\alpha_j|^2/2 & (i = j) \end{cases}$.

Proof. Since $\sigma_{s_i}(x) = \omega_i(x)$, the assertion follows from the equation (4.2). □

Proposition 5.2 (Chevalley rule, c.f. [13])**.**

$$\mathfrak{S}_{s_i}\mathfrak{S}_w = \sum_{\beta \in \Pi^+, l(ws_\beta)=l(w)+1} \frac{2\langle \beta, \omega_i \rangle}{|\beta|^2} \mathfrak{S}_{ws_\beta} + \left(\omega_i(t) - \omega_i(w^{-1}(t))\right) \mathfrak{S}_w$$

To show the Proposition, we need the following direct but useful Lemma.

Lemma 5.3 ([2])**.** *The divided difference operators satisfy the following Leibniz rule:*

$$\Delta_i(f(t;x)g(t;x)) = \Delta_i(f(t;x))g(t;x) + f(t;s_i(x))\Delta_i(g(t;x)), \quad f(t;x), g(t;x) \in \mathbb{R}_W[t;x].$$

For a reduced word $v = s_{i_1} \cdots s_{i_{l(v)}}$ and a set $L \subset \{1, \ldots, l(v)\}$, we define a subword v^L of v by $s_{i_1}^{\epsilon_1} s_{i_2}^{\epsilon_2} \cdots s_{i_{l(v)}}^{\epsilon_{l(v)}}$, where $\epsilon_j = \begin{cases} 0 & (j \notin L) \\ 1 & (j \in L) \end{cases}$. Define Δ'_L as the composite $\phi_{i_1} \circ \phi_{i_2} \circ \cdots \circ \phi_{i_{l(v)}}$, where $\phi_{i_j} = \begin{cases} \Delta_{i_j} & (j \notin L) \\ s_{i_j} & (j \in L) \end{cases}$. Put $\Phi^w_v = \sum_L \Delta'_L$, where L runs over subsets of $\{1, \ldots, l(v)\}$ such that $v^L = w$. Then by iterating the Leibniz rule, we have

$$\Delta_v(\mathfrak{S}_u\mathfrak{S}_w)(t;t) = \Phi^w_v(\mathfrak{S}_u)(t;t) = \Phi^u_v(\mathfrak{S}_w)(t;t).$$

So by Proposition 4.6, we have

$$\mathfrak{S}_u\mathfrak{S}_w = \sum_{v \geq w} \Phi^w_v(\mathfrak{S}_u)(t;t) \cdot \mathfrak{S}_v(t;x).$$

Proof of Chevalley rule. By the argument above, we have

$$\mathfrak{S}_{s_i}\mathfrak{S}_w = \sum_{v \geq w} \Phi^w_v\left(\omega_i(t) - \omega_i(x)\right)(t;t) \cdot \mathfrak{S}_v(t;x).$$

By degree reason, $\Phi^w_v(\omega_i(t) - \omega_i(x))(t;t)$ vanish unless $v = w$ or $l(v) = l(w) + 1$. For $v = w$, $\Phi^w_w(\omega_i(t) - \omega_i(x))(t;t) = \left(\omega_i(t) - \omega_i(w^{-1}(x))\right)(t;t) = \left(\omega_i(t) - \omega_i(w^{-1}(t))\right)$. For $l(v) = l(w) + 1$, we can write $v = ws_\beta$ for some $\beta \in \Pi^+$. Then by Exchange condition, we have

$$\Phi^w_v(\omega_i(t) - \omega_i(x)) = \frac{\omega_i(t) - \omega_i(x) - (\omega_i(t) - \omega_i(s_\beta(x)))}{-\beta(x)} = \frac{2\langle \beta, \omega_i \rangle}{|\beta|^2}.$$

□

6. Symmetry between t and x

As one can see from (4.1), $H^*_T(G/B) \cong H^*(BT \times_{BG} BT)$ has a symmetry. This symmetry become clearer when we view it from the algebraic setting. The involution on $\mathbb{R}[t;x]$ defined by switching the variables x_i and t_i induces the involution τ on $\mathbb{R}_W[t;x]$ since I_W is stable.

What we show in this section is the following symmetry of the Schubert classes:

Proposition 6.1. $\tau(\mathfrak{S}_w(t;x)) = \mathfrak{S}_w(x;t) = (-1)^{l(w)}\mathfrak{S}_{w^{-1}}(t;x)$.

To show the Proposition, we use the *left divided difference operator* $\delta_w = (-1)^{l(w)}\tau \circ \Delta_w \circ \tau$. It is obvious that Δ_v and δ_w commute for any $w, v \in W$. The following Lemma explains why δ_w is called the left divided difference operator.

Lemma 6.2. $\delta_w \mathfrak{S}_v = \begin{cases} \mathfrak{S}_{wv} & (l(wv) = l(v) - l(w)) \\ 0 & (\textit{otherwise}) \end{cases}.$

Proof. By Proposition 4.6 and the commutativity,

$$\delta_i \mathfrak{S}_v = \sum_{u \in W} \delta_i \Delta_u(\mathfrak{S}_v)(t;t) \cdot \mathfrak{S}_u(t;x).$$

On the other hand, we have

$$(\delta_i \mathfrak{S}_u)(t;t) = \frac{\mathfrak{S}_u(t;t) - \mathfrak{S}_u(s_i t, t)}{\alpha_i(t)} = \frac{\mathfrak{S}_u(t;t) - s_i \mathfrak{S}_u(t, s_i t)}{\alpha_i(t)} = \begin{cases} 1 & (u = s_i) \\ 0 & (\text{otherwise}) \end{cases}.$$

Hence

$$\delta_i \Delta_u(\mathfrak{S}_v)(t;t) = \begin{cases} 1 & (u = s_i v, l(u) = l(v) - 1) \\ 0 & (\text{otherwise}) \end{cases},$$

and

$$\delta_i \mathfrak{S}_v = \begin{cases} \mathfrak{S}_{s_i v} & (l(s_i v) = l(v) - 1) \\ 0 & (\text{otherwise}) \end{cases}.$$

By induction on the length of w, we have the Proposition. □

Proof of Proposition 6.1. By Proposition 4.6 and the previous Lemma,

$$\begin{aligned} \tau \mathfrak{S}_w &= \sum_{v \in W} \Delta_v(\tau \mathfrak{S}_w)(t;t) \cdot \mathfrak{S}_v(t;x) \\ &= \sum_{v \in W} (-1)^{l(v)} (\tau \delta_v \mathfrak{S}_w)(t;t) \cdot \mathfrak{S}_v(t;x) \\ &= \sum_{v \in W} (-1)^{l(v)} (\mathfrak{S}_{vw})(t;t) \cdot \mathfrak{S}_v(t;x) \\ &= (-1)^{l(w^{-1})} \mathfrak{S}_{w^{-1}}(t;x) \\ &= (-1)^{l(w)} \mathfrak{S}_{w^{-1}}(t;x). \end{aligned}$$

□

7. Ordinary vs Equivariant Schubert classes

The equivariant cohomology $H_T^*(G/B)$ recovers the ordinary one $H^*(G/B)$ by the augmentation map

$$r_1 : H_T^*(G/B) \to \frac{H_T^*(G/B)}{H^+(BT)} \cong H^*(G/B),$$

which maps the equivariant Schubert classes to the ordinary ones. Similarly in our algebraic setting, it is easily seen from the definition that

$$r_1 : \mathbb{R}_W[t;x] \ni f(t;x) \mapsto f(0;x) \in \mathbb{R}_W[x]$$

maps the equivariant Schubert class $\mathfrak{S}_w$ to the ordinary one σ_w.

We have another map with a similar property. In the topological setting, we can consider the following composition:

$$H_T^*(G/B) \xrightarrow{r_2} H_T^*(G/B)^W \cong H_T^*(G/G) \cong H^*(BT) = H^*(BB) \xrightarrow{c^*} H^*(G/B),$$

where r_2 is Reynold's operator $Z \mapsto \frac{1}{|W|}\sum_{w\in W} w(Z)$ and c^* is the induced map of the fiber inclusion $G/B \xrightarrow{c} BB \to BG$.

Similarly in our algebraic setting,

$$r_2 : \mathbb{R}_W[t;x] \ni f(t;x) \mapsto \frac{1}{|W|}\sum_{w\in W} f(t;w^{-1}(x)) = \frac{1}{|W|}\sum_{w\in W} f(t;w^{-1}(t)) \in \mathbb{R}_W[t].$$

Here $\sum_{w\in W} f(t;w^{-1}(x)) = \sum_{w\in W} f(t;w^{-1}(t))$ in $\mathbb{R}_W[t;x]$ because $\sum_{w\in W} f(t;w^{-1}(x))$ is invariant under the action of W on x-variables.

Proposition 7.1. $r_2(\mathfrak{S}_{w^{-1}}(t;x)) = \sigma_w(-t)$.

Proof. Applying $\frac{1}{|W|}\Delta_{w^{-1}w_0}$ to the both hand sides of

$$\sum_{v\in W} \mathfrak{S}_{w_0}(v^{-1}(t);x) = \sum_{v\in W} \mathfrak{S}_{w_0}(v^{-1}(t);t) = \sum_{v\in W} i_v^*(\tau\mathfrak{S}_{w_0}) = (-1)^{l(w_0)}\sum_{v\in W} i_v^*(\mathfrak{S}_{w_0}) = (-1)^{l(w_0)}\prod_{\beta\in\Pi^+}\beta = |W|\sigma_{w_0}(t)$$

yields

$$\frac{1}{|W|}\sum_{v\in W} \mathfrak{S}_w(v^{-1}(t);x) = \sigma_w(t).$$

Again, applying τ to the both hand sides of the above equation yields

$$\frac{1}{|W|}\sum_{v\in W} \mathfrak{S}_{w^{-1}}(t;v^{-1}(x)) = (-1)^{l(w)}\sigma_w(t) = \sigma_w(-t).$$

□

8. Example

Presentation of Schubert classes $\mathfrak{S}_w(t;x) \in \mathbb{R}_W[t;x]$ has indeterminancy up to the ideal I_W. It is preferable to choose a simple and explicit presentation than the one given in Definition 4.3. For example, Lascoux and Schützenberger [16] defined the beautiful *double Schubert polynomial* for $W = A_{n-1}$ as

$$\mathfrak{S}_{w_0}(t;x) = \prod_{i+j<n}(x_i - t_j).$$

We can also easily verify that the polynomial

$$\mathfrak{S}_{w_0}(t;x) = c_n \prod_{i\geq j}(x_i - t_j)\prod_{i>j}(x_i + t_j)$$

is the top Schubert class for W of type B_n and C_n by Proposition 4.9, where

$$c_n = \begin{cases} 1/(-2)^n & (W = B_n) \\ (-1)^n & (W = C_n) \end{cases}.$$

Note that this representative is different as polynomials in $\mathbb{R}[t] \otimes \mathbb{R}[x]$ from the one given by Fulton and Pragacz [8], and Kresch and Tamvakis [14]; their constructions aim not only to represent Schubert classes but also to satisfy a lot of combinatorially desirable properties.

In this section, we try to find a simple presentation of the Schubert class $\mathfrak{S}_w$ for the Coxeter group of non-crystallographic type $I_2(m)$ in view of Proposition 4.9. The facts about this group are summarized as follows:

- W is the dihedral group of order $2m$.
- W is generated by s_1, s_2 with $(s_1s_2)^m = (s_2s_1)^m = 1$.
- $s_1s_2 = \beta_2$ and $s_2s_1 = \beta_{-2}$, where β_k is the rotation by $k\theta$ ($\theta = \pi/m$).
- the simple roots are $\alpha_1 = t_1, \alpha_2 = \beta_{(m-1)}(t_1)$.
- the fundamental weights are $\omega_1 = \dfrac{t_1}{2} + \dfrac{t_2}{2\tan\theta}, \omega_2 = \dfrac{t_2}{\sin\theta}$.
- the positive roots are $\beta_k(t_1)$ $(0 \leq k \leq m-1)$.
- the longest element is $w_0 = \begin{cases} (s_1s_2)^{m/2} & (m : \text{even}) \\ s_2(s_1s_2)^{(m-1)/2} & (m : \text{odd}) \end{cases}$.
- the double coinvariant ring is

$$\mathbb{R}_W[t;x] = \frac{\mathbb{R}[t_1,t_2,x_1,x_2]}{\left(t_1^2+t_2^2-x_1^2-x_2^2, \mathrm{Re}(t_1+\sqrt{-1}t_2)^m - \mathrm{Re}(x_1+\sqrt{-1}x_2)^m\right)}$$

We define

$$h(t;x) = (x_1 - t_1) \prod_{\substack{k=0,\dots,m-1 \\ k\neq m/2}} (x_2 - \beta_{2k}(t_2)) \quad (m : \text{even})$$

and

$$h(t;x) = (x_1 - \beta_{m+1}(t_1)) \prod_{\substack{k=0,\dots,m-1 \\ k\neq (m+1)/2}} (x_2 - \beta_{2k}(t_2)) \quad (m : \text{odd}).$$

From the following facts:

- the W-orbit of x_2 is $\{\beta_{2k}(x_2) \mid k = 0,1,\dots,m-1\}$

- $w_0(x_2) = s_1 w_0(x_2) = \begin{cases} \beta_m(x_2) = -x_2 & (m : \text{even}) \\ \beta_{m+1}(x_2) & (m : \text{odd}) \end{cases}$
- $s_1 w_0(x_1) = \begin{cases} -\beta_m(x_1) = x_1 & (m : \text{even}) \\ \beta_{m+1}(x_1) & (m : \text{odd}) \end{cases}$,

we can easily verify that $i_w^* h(t;x)$ doesn't vanish iff $w = w_0$. Hence by Proposition 4.9, $h(t;x)$ is the top Schubert class up to constant.

Next, we give the multiplication table for the classes using the result obtained in §5. Put $w'_k \in W$ ($w''_k \in W$) be the element of length k whose reduced word ends with s_1 (respectively, s_2), so that $W = \{e = w'_0 = w''_0\} \sqcup \{w'_k, w''_k \mid 1 \le k < m\} \sqcup \{w_0 = w'_m = w''_m\}$. Then Chevalley rule computes:

$$\begin{aligned}
\mathfrak{S}_1 \mathfrak{S}_{w'_k} &= \frac{\sin((k+1)\theta)}{\sin\theta} \mathfrak{S}_{w'_{k+1}} + \left(\omega_1(t) - \omega_1(w_k'^{-1}(t))\right) \mathfrak{S}_{w'_k} \\
\mathfrak{S}_2 \mathfrak{S}_{w'_k} &= \mathfrak{S}_{w''_{k+1}} + \frac{\sin(k\theta)}{\sin\theta} \mathfrak{S}_{w'_{k+1}} + \left(\omega_2(t) - \omega_2(w_k'^{-1}(t))\right) \mathfrak{S}_{w'_k} \\
\mathfrak{S}_1 \mathfrak{S}_{w''_k} &= \mathfrak{S}_{w'_{k+1}} + \frac{\sin(k\theta)}{\sin\theta} \mathfrak{S}_{w''_{k+1}} + \left(\omega_1(t) - \omega_1(w_k''^{-1}(t))\right) \mathfrak{S}_{w''_k} \\
\mathfrak{S}_2 \mathfrak{S}_{w''_k} &= \frac{\sin((k+1)\theta)}{\sin\theta} \mathfrak{S}_{w''_{k+1}} + \left(\omega_2(t) - \omega_2(w_k''^{-1}(t))\right) \mathfrak{S}_{w''_k}
\end{aligned}$$

Remark 8.1. *Note that the Weyl group of type G_2 is $I_2(6)$ upto a length normalization in the positive roots.*

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