Self-Organized Feature Extraction in a Three-Dimensional Discrete Reaction-Diffusion System

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We investigated self-organized patterns formed by a FitzHugh–Nagumo model in three-dimensional space. We found functional orders in the model. The edge plane structure was autonomously extracted from an initial cubic pattern. As in a two-dimensional model, the edge points and edge lines developed spontaneously in this model. These edge structures were obtained when parameters were in a specific region. We can interpret that the reaction-diffusion system has a function of self-organized feature extraction when the model is in a discrete condition of space.
Key words: FitzHugh–Nagumo Model, Edge Detection, Turing Pattern

1. Introduction

Living things form various patterns spontaneously, including body forms, skin-surface patterns, and heartbeat rhythms. It is known that the reaction-diffusion models also form patterns that have a similar structure to such self-organized patterns (e.g., Meinhardt and Klinger, 1987). In particular, Kondo and Asai (1995) successfully simulated the growth dynamics of skin-surface patterns on several types of fish using a reaction-diffusion model. The appearance of self-organized patterns is not limited to organisms. The Belousov-Zhabotinsky (BZ) reaction is a well-known example; the chemical reaction generates a spatio-temporal pattern spontaneously. It is interesting that a phenomenon on photosensitive BZ reaction generates the contour and/or reversal pattern of an input pattern induced by a light source (Kuhnert *et al.*, 1989). Reaction-diffusion models can also be applied to simulate the pattern formations appearing in these chemical reactions. Many numerical studies on the behavior of reaction-diffusion models were restricted to one or two spatial dimensions. However, several characteristic three-dimensional structures have been reported recently (Leppänen *et al.*, 2002; Shoji *et al.*, 2007).

The photosensitive BZ reaction's self-organization of the contour and/or reversal structure of the input pattern is a type of image processing. However, these specific structures do not remain stationary. For application to the field of image processing, it is important that the pattern obtained is stationary. If the pattern does not become stationary after sometime, we must be able to decide manually the time when the process should stop. Nomura *et al.* (1999) reported that a FitzHugh–Nagumo model produced stationary patterns like those obtained by typical image processing. The functions of edge point and edge line extractions and image segmentation were found in numerical simulations. These functions of the FitzHugh–Nagumo model were found in coarse calculations of approximation for spatial differences (Ebihara *et al.*, 2003). Our main interest in this study is to clarify the behaviors of the reaction-diffusion model in three dimensions that adopted rough approximation in space. Throughout this study, we use the term "discreteness" to indicate roughness for approximation of differential equations, and consider "discrete reaction-diffusion model" and "discrete system" as a system that has little spatial connection between neighboring cells: little diffusion or large spatial intervals. In the discrete reaction-diffusion models, there are few studies dealing with three-dimensional pattern formations. Many researchers regarded such discrete regions useless as an approximation of differential equations.

In this study, we investigated the discrete reaction-diffusion model in three dimensions. We found that the model spontaneously organized the edge plane structure of an initial cubic pattern. We also confirmed that the model extracted the edge points and edge lines of the pattern in a self-organized fashion. The edge plane structure was observed only in the three-dimensional models.

2. Related Research

Leppänen *et al.* (2002) compared pattern formations in three dimensions and two dimensions. They investigated a general Turing system and the Gray-Scott model in their numerical simulations. Ohta and co-workers studied a three-dimensional Turing pattern for a type of FitzHugh–Nagumo model, the Brusselator, and the Gray-Scott model in numerical simulations (Shoji *et al.*, 2007). They searched various parameters and reported several categories of three-dimensional structure. However, they did not report any feature extraction functions, which the discrete reaction-diffusion models have. Nomura *et al.* (1999) reported that the FitzHugh–Nagumo model (Eqs. (1) and (2)) behaved as a system realizing self-organized image processing, including edge point/line detection and image segmentation as stationary patterns. Although the FitzHugh–Nagumo equation was proposed originally as a model of active pulse trans-

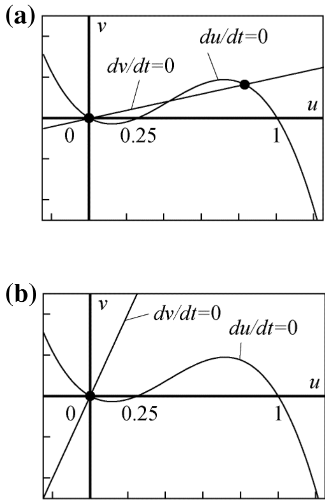

Fig. 1. Nullclines of a FitzHugh–Nagumo system that is bistable ($b = 10.0$) (a) and monostable ($b = 1.0$) (b). The filled circles indicate stable points. $a = 0.25$ in both cases.

mission on nerve axons (Nagumo *et al.*, 1962), they applied it as a model of feature extraction from images. Image-processing-like functions in the reaction-diffusion model were reported only in their study.

The FitzHugh–Nagumo model is represented by the following equations,

$$\frac{\partial u}{\partial t} = D_u \nabla^2 u + \frac{1}{\epsilon}\{u(u-a)(1-u) - v\}, \tag{1}$$

$$\frac{\partial v}{\partial t} = D_v \nabla^2 v + u - bv, \tag{2}$$

where activator u and inhibitor v are variables. D_u and D_v are diffusion coefficients of u and v, respectively. a, b, and ϵ are positive constants. Image-processing-like behaviors appeared when the system had a discrete nature (Ebihara *et al.*, 2003), i.e., the spatial interval Δx was moderately large or the diffusion coefficients D_u and D_v were moderately small. In addition, the functions of the model for edge detection or image segmentation have robustness against noise (Ebihara *et al.*, 2003). Thus, it is expected that the model can be applied to image or signal processing as a noise-robust model.

Figure 1 shows nullclines of Eqs. (1) and (2) when D_u and D_v equal zero. When the model has two stable points (i.e., the model is bistable; see Fig. 1(a)), it yields a segmented image as a binarization of the input pattern because the value of each element of the model settles down at either stable point in the bistable system. When the model has only one stable point (i.e., the model is monostable; Fig. 1(b)), it yields the edge points or edge lines of the input pattern. It appears that the diffusion coefficients D_u and D_v affect the number of stable points of the system, and the system turns bistable only in the edge regions (see detailed discussion in Subsec. 4.2).

3. Numerical Simulations and Results

We focused on the FitzHugh–Nagumo model for two reasons. (1) This study follows the previous research of Nomura *et al.* (1999) and Ebihara *et al.* (2003). It is easy to compare the results of this study with that of their research. (2) The model is a well-known system with a sufficiently studied mechanism (Rocoreanu *et al.*, 2000).

For computer simulation, we adopted the explicit Euler method of finite difference schemes (Smith, 1996) on the FitzHugh–Nagumo equation (Eqs. (1) and (2)). The cell size was Δx, the number of cells was $N = 50$, and the temporal interval was represented as Δt. There was no flux at any of the boundaries of the system; the so-called Neumann condition was adopted. The initial condition shaped a solid cube (see Fig. 2). We performed a numerical simulation varying the diffusion coefficients D_u and D_v, while keeping the ratio $D_v/D_u = 4.0$ because our purpose was to investigate the relationship between the system's discreteness and its behavior. We visualized spatial distributions of the magnitude of the activator u (from 0.0 to 1.0) using MayaVi[*1] ver. 1.5: a data visualizing software.

When D_u and D_v were sufficiently large ($D_u = 4.0 \times 10^{-2}$ and $D_v = 1.6 \times 10^{-1}$), three-dimensional wave propagation appeared as expected (see Fig. 3). On the other hand, Fig. 4 represents a result when D_u was at its smallest value ($D_u = 4.0 \times 10^{-5}$) in the current numerical simulations. The edge points of the given shape as the initial condition of the simulations appeared as a convergent pattern in this case. When D_u was 5.0×10^{-5}, the edge lines appeared (Fig. 5). These results were similar to the results in two dimensions (Nomura *et al.*, 1999; Ebihara *et al.*, 2003). Moreover, as shown in Fig. 6, we obtained the edge plane structure of the initial shape when D_u was 1.5×10^{-4}. The static edge plane structure can be regarded as a characteristic pattern self-organized in the three-dimensional model. In addition, we performed a numerical simulation for a sphere-shaped initial condition and obtained the edge plane on the same parameter values for the cube shape.

4. Discussion

4.1 Patterns obtained by the FitzHugh–Nagumo model

We found a function of self-organized feature extraction that appeared in a three-dimensional FitzHugh–Nagumo model. The function includes extraction of the edge planes of the initial shape in three-dimensional space. Other functions of feature extractions (edge point/line detection) were also confirmed in three dimensions as well in two dimensions when D_u and D_v were moderately small. When D_u and D_v were too small, the value of all the cells settled at zero. In the current numerical simulations, we observed the three-dimensional wave propagation, edge point, line, and plane structures. We also observed a so-called Turing pattern at $N = 100$ (see Fig. 7); however, we could not confirm the pattern clearly at $N = 50$. More efforts are required in parameter searching of numerical simulations of the FitzHugh–Nagumo model to observe various other Turing patterns and to compare these patterns with previous spatial patterns in reaction-diffusion systems.

We explained that the calculation steps of each simulation were sufficient for obtaining stationary patterns through discussion of a characteristic timescale τ of the

[*1] http://mayavi.sourceforge.net/

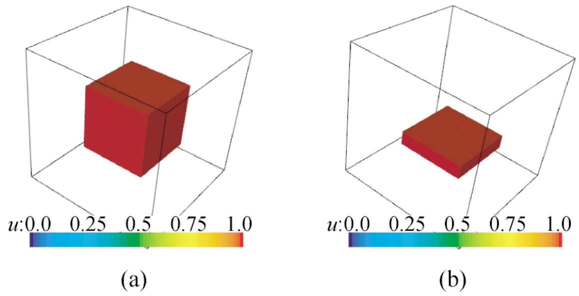

Fig. 2. A given shape as an initial condition of the system. (a) shows a distribution of u shaped solid cube with $u = 1.0$. (b) represents a sliced image of (a).

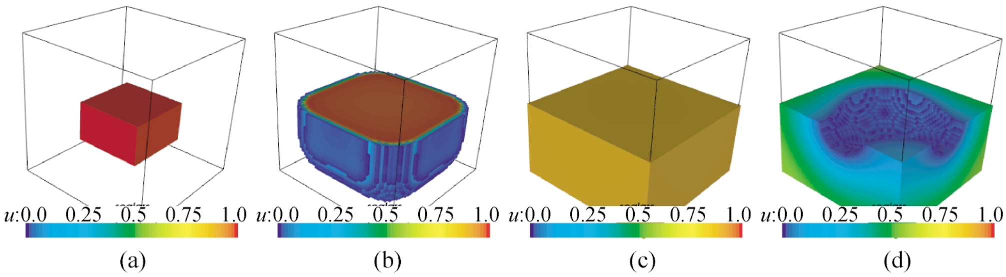

Fig. 3. Wave propagation in a three-dimensional FitzHugh–Nagumo model. The lower half of distributions of u at the 0th step (a), 100th step (b), 500th step (c), and 940th step (d) are shown. The parameters are $D_u = 4.0 \times 10^{-2}$, $D_v = 1.6 \times 10^{-1}$, $a = 0.1$, $b = 1.0$, $\epsilon = 1.0 \times 10^{-4}$, $\Delta x = 0.01$, and $\Delta t = 1.0 \times 10^{-4}$.

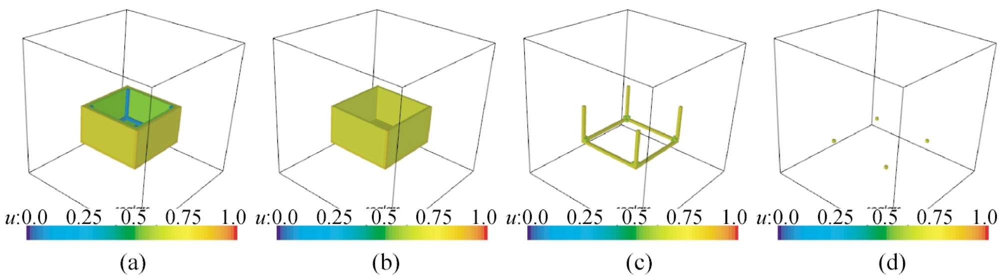

Fig. 4. A time series of the edge point structure observed in a discrete system with $D_u = 4.0 \times 10^{-5}$ and $D_v = 1.6 \times 10^{-4}$. The lower half of distributions of u at the 1600th step (a), 1700th step (b), 1900th step (c), and 5000th step (d) are visualized. The other parameters are $a = 0.1$, $b = 1.0$, $\epsilon = 1.0 \times 10^{-4}$, $\Delta x = 0.01$, and $\Delta t = 1.0 \times 10^{-4}$.

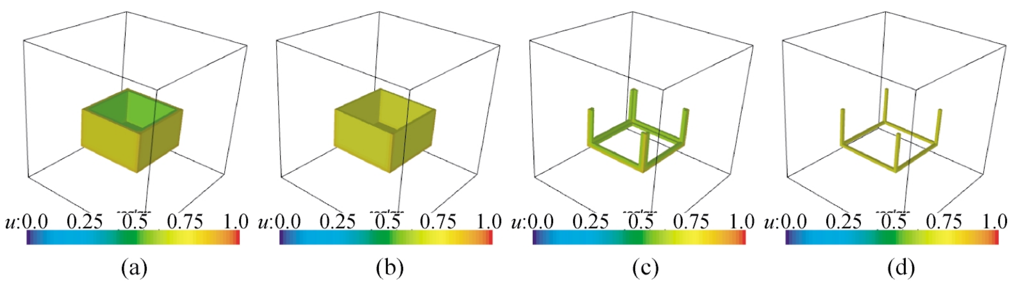

Fig. 5. A time series of the edge line structure observed in a discrete system with $D_u = 5.0 \times 10^{-5}$ and $D_v = 2.0 \times 10^{-4}$. The lower half of distributions of u at the 1600th step (a), 1700th step (b), 1900th step (c), and 5000th step (d) are visualized. The other parameters are $a = 0.1$, $b = 1.0$, $\epsilon = 1.0 \times 10^{-4}$, $\Delta x = 0.01$, and $\Delta t = 1.0 \times 10^{-4}$.

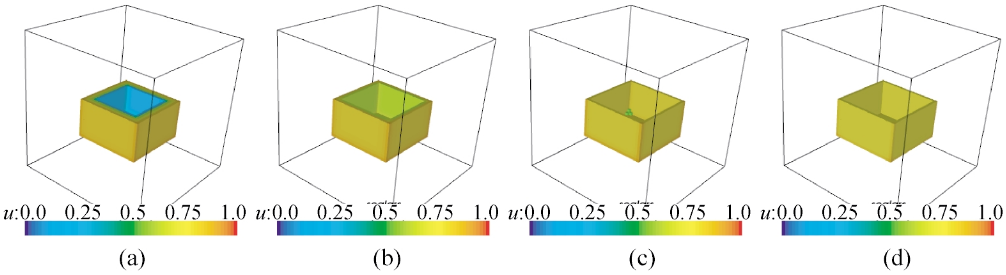

Fig. 6. A time series of the edge plane structure found at $D_u = 1.5 \times 10^{-4}$ and $D_v = 6.0 \times 10^{-4}$. The lower half of distributions of u at the 1600th step (a), 1700th step (b), 1900th step (c), and 5000th step (d) are visualized. The other parameters were $a = 0.1$, $b = 1.0$, $\epsilon = 1.0 \times 10^{-4}$, $\Delta x = 0.01$, and $\Delta t = 1.0 \times 10^{-4}$.

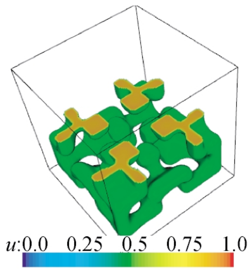

Fig. 7. A three-dimensional Turing pattern found at $a = 0.2$, $b = 5.0$, $\epsilon = 1.0 \times 10^{-3}$, $\Delta x = 0.01$, $\Delta t = 1.0 \times 10^{-5}$, $D_u = 1.0 \times 10^{-2}$, $D_v = 4.0 \times 10^{-2}$, and $N = 100$. The lower half of the isosurface $u = 0.5$ at the 300000th step is visualized.

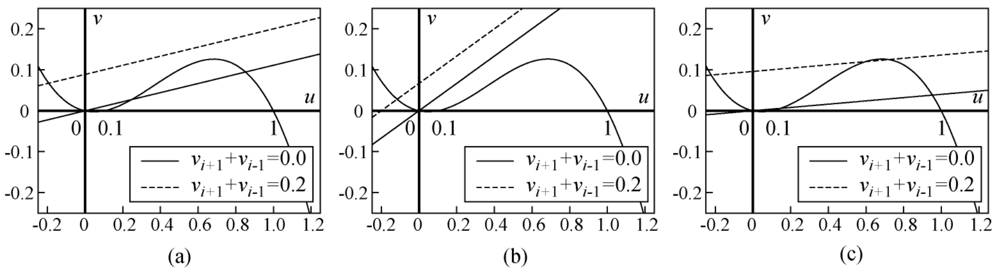

Fig. 8. Nullclines of Eqs. (3) and (4) at $D_v = 1.0$ (a), $D_v = 4.0$ (b), and $D_v = 12.0$ (c). $\Delta x = 0.01$ and $a = 0.1$ in each case. The solid lines indicate that $v_{i+1} + v_{i-1}$ is small (0.0), and the dashed lines show that $v_{i+1} + v_{i-1}$ is sufficiently large (0.2). Sufficiently large discreteness results in a non-uniform number of stable points in space according to distributions of the inhibitor value v, which almost depends on the activator value u (a). The number of stable points is identical in space in (b) and (c).

system (see details in Kitamori and Kitamura (1996)). If a dynamic system shows an exponential decay, the quantitative state of the system becomes $1/e$ after τ passes. In the FitzHugh–Nagumo model (Eqs. (1) and (2)), the estimated value of characteristic τ is 0.1 for parameters that yield both edge detection (Figs. 4 to 6) and a Turing pattern (Fig. 7). We also estimated τ for a Turing pattern generated by the Oregonetor model (refer figure 12 in Nomura *et al.*, 1997) and obtained $\tau = 1.0$. In each case (edge detection, the Turing pattern of the FitzHugh–Nagumo model, and that of the Oregonetor models), states of the patterns were still non-stationary just as τ passed, but settled into steady patterns after around tens of times τ passed in each case.

4.2 A brief theory of extracting the edge structures in the FitzHugh–Nagumo model

Here, we consider the FitzHugh–Nagumo system in one spatial dimension, for better understanding of the discussion. It seems that the generality of the theory of the one-dimensional system is retained when it is adopted in three dimensions. In addition, we assume that D_u is zero. We have confirmed that edge structures also appear in this case. First, we adopt a central difference in space and obtain the following Eqs. (3) and (4),

$$\frac{du_i}{dt} = \frac{1}{\epsilon}\{u_i(u_i - a)(1 - u_i) - v_i\}, \tag{3}$$

$$\frac{dv_i}{dt} = r'(v_{i+1} - 2v_i + v_{i-1}) + u_i - bv_i, \tag{4}$$

where r' represents $D_v/(\Delta x)^2$. The subscript $i = 1, 2, \ldots, N$ is a spatial index. Equations (5) and (6) represent nullclines of Eqs. (3) and (4), respectively (see also Fig. 8),

$$v_i = u_i(u_i - a)(1 - u_i), \tag{5}$$

$$v_i = \frac{1}{b + 2r'}u_i + \frac{r'(v_{i+1} + v_{i-1})}{b + 2r'}, \tag{6}$$

where b, r', v_{i+1}, and v_{i-1} are positive.

When D_v is moderately small or Δx is moderately large (i.e., r' is moderately small), the number of stable points varies spatially depending on the values of neighboring inhibitors v_{i+1} and v_{i-1}. The system has two stable points where the term $v_{i+1} + v_{i-1}$ is small, whereas the system has only one stable point around the origin where the value of $v_{i+1} + v_{i-1}$ is sufficiently large, because the line drawn by Eq. (6) moves too far into the upper region to maintain two stable points (see Fig. 8(a)). On one hand, when r' is too small, all spatial components of the system have one stable point, as shown in Fig. 8(b). On the other hand, when r' is too large, all components of the system have two stable points, as shown in Fig. 8(c). The sum of the neighboring cells' inhibitor values is smaller in the edge cells than in other cells. The reason is that outer-side edge cells have almost zero inhibitor value v, since the value almost depends on the activator value u. Thus, the model extracts the edge structures under moderate spatial discreteness. In three dimensions, the sum of neighboring inhibitors is larger than that in one dimension. Indeed, it becomes more complex when D_u is not zero. In such cases, the cubic curve drawn by Eq. (5) also changes its profile.

5. Conclusion

In this study, we have reported on functional pattern formations in three spatial dimensions for the FitzHugh–Nagumo model. We confirmed that the proposed model self-organized the edge plane structure and the edge point/line structure when the system was discrete; the diffusion coefficients were moderately small. The former appeared as a characteristic structure in the three-dimensional discrete reaction-diffusion system. The latter is a common structure in one and/or two dimensions. As the diffusion coefficients increased, the extracted pattern differed in the edge points, lines, and planes, in that order.

For a sphere-shaped initial condition, we also confirmed that we obtained the edge plane structure. According to a previous study (Nomura *et al.*, 2007), the model in two dimensions extracted the edge lines for arbitrary figures, e.g., a photograph of a building. We consider that edge detection for an arbitrary three-dimensional object is possible by analogy with the two-dimensional result. Nevertheless, we must confirm whether the model can obtain the edge planes for any arbitrary three-dimensional complex-shaped object. If the proposed model extracts it, the field of three-dimensional visualization, e.g., visualization of computed tomography, might be an application of this study. It is also important to confirm whether the behavior is influenced by the discretization scheme using other schemes, e.g., the finite element method.

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