Metric Adjusted Skew Information and Uncertainty Relation

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Abstract. We show that an uncertainty relation for Wigner-Yanase-Dyson skew information proved by Yanagi(2010)[10] can hold for an arbitrary quantum Fisher information under some conditions. This is a refinement of the result of Gibilisco and Isola(2011)[4].

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1 Introduction

Wigner-Yanase skew information

$$\begin{aligned} I_\rho(H) &= \frac{1}{2}Tr\left[\left(i\left[\rho^{1/2}, H\right]\right)^2\right] \\ &= Tr[\rho H^2] - Tr[\rho^{1/2} H \rho^{1/2} H] \end{aligned}$$

was defined in [9]. This quantity can be considered as a kind of the degree for non-commutativity between a quantum state ρ and an observable H. Here we denote the commutator by $[X, Y] = XY - YX$. This quantity was generalized by Dyson

$$\begin{aligned} I_{\rho,\alpha}(H) &= \frac{1}{2}Tr[(i[\rho^\alpha, H])(i[\rho^{1-\alpha}, H])] \\ &= Tr[\rho H^2] - Tr[\rho^\alpha H \rho^{1-\alpha} H], \alpha \in [0, 1] \end{aligned}$$

which is known as the Wigner-Yanase-Dyson skew information. Recently it is shown that these skew informations are connected to special choices of quantum Fisher

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information in [3]. The family of all quantum Fisher informations is parametrized by a certain class of operator monotone functions $\mathcal{F}_{op}$ which were justified in [7]. The Wigner-Yanase skew information and Wigner-Yanase-Dyson skew information are given by the following operator monotone functions

$$f_{WY}(x) = \left(\frac{\sqrt{x}+1}{2}\right)^2,$$

$$f_{WYD}(x) = \alpha(1-\alpha)\frac{(x-1)^2}{(x^\alpha - 1)(x^{1-\alpha}-1)}, \quad \alpha \in (0,1),$$

respectively. In particular the operator monotonicity of the function f_{WYD} was proved in [8]. On the other hand the uncertainty relation related to Wigner-Yanase skew information was given by Luo [6] and the uncertainty relation related to Wigner-Yanase-Dyson skew information was given by Yanagi [10], respectively. In this paper we generalize these uncertainty relations to the uncertainty relations related to quantum Fisher informations.

2 Operator Monotone Functions

Let $M_n = M_n(\mathbb{C})$(resp. $M_{n,sa} = M_{n,sa}(\mathbb{C})$) be the set of all $n \times n$ complex matrices (resp. all $n \times n$ self-adjoint matrices), endowed with the Hilbert-Schmidt scalar product $\langle A, B\rangle = Tr(A^*B)$. Let $\mathcal{D}_n$ be the set of strictly positive elements of M_n and $\mathcal{D}_n^1 \subset \mathcal{D}_n$ be the set of strictly positive density matrices, that is $\mathcal{D}_n^1 = \{\rho \in M_n | Tr\rho = 1, \rho > 0\}$. If it is not otherwise specified, from now on we shall treat the case of faithful states, that is $\rho > 0$.

A function $f : (0, +\infty) \to \mathbb{R}$ is said operator monotone if, for any $n \in \mathbb{N}$, and $A, B \in M_n$ such that $0 \le A \le B$, the inequalities $0 \le f(A) \le f(B)$ hold. An operator monotone function is said symmetric if $f(x) = xf(x^{-1})$ and normalized if $f(1) = 1$.

Definition 2.1 *$\mathcal{F}_{op}$ is the class of functions $f : (0, +\infty) \to (0, +\infty)$ such that*

(1) $f(1) = 1$,

(2) $tf(t^{-1}) = f(t)$,

(3) *f is operator monotone.*

Example 2.1 *Examples of elements of $\mathcal{F}_{op}$ are given by the following list*

$$f_{RLD}(x) = \frac{2x}{x+1}, \quad f_{WY}(x) = \left(\frac{\sqrt{x}+1}{2}\right)^2, \quad f_{BKM}(x) = \frac{x-1}{\log x},$$

$$f_{SLD}(x) = \frac{x+1}{2}, \quad f_{WYD}(x) = \alpha(1-\alpha)\frac{(x-1)^2}{(x^\alpha - 1)(x^{1-\alpha}-1)}, \ \alpha \in (0,1).$$

Remark 2.1 *Any $f \in \mathcal{F}_{op}$ satisfies*

$$\frac{2x}{x+1} \leq f(x) \leq \frac{x+1}{2}, \quad x > 0.$$

For $f \in \mathcal{F}_{op}$ define $f(0) = \lim_{x \to 0} f(x)$. We introduce the sets of regular and non-regular functions

$$\mathcal{F}_{op}^{r} = \{f \in \mathcal{F}_{op} | f(0) \neq 0\}, \quad \mathcal{F}_{op}^{n} = \{f \in \mathcal{F}_{op} | f(0) = 0\}$$

and notice that trivially $\mathcal{F}_{op} = \mathcal{F}_{op}^{r} \cup \mathcal{F}_{op}^{n}$.

Definition 2.2 *For $f \in \mathcal{F}_{op}^{r}$ we set*

$$\tilde{f}(x) = \frac{1}{2}\left[(x+1) - (x-1)^2 \frac{f(0)}{f(x)}\right], \quad x > 0.$$

Theorem 2.1 ([1], [3], [5]) *The correspondence $f \to \tilde{f}$ is a bijection between $\mathcal{F}_{op}^{r}$ and $\mathcal{F}_{op}^{n}$.*

3 Means, Fisher Information and Metric Adjusted Skew Information

In Kubo-Ando theory of matrix means one associates a mean to each operator monotone function $f \in \mathcal{F}_{op}$ by the formula

$$m_f(A, B) = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2},$$

where $A, B \in \mathcal{D}_n$. Using the notion of matrix means one may define the class of monotone metrics (also said quantum Fisher informtions) by the following formula

$$\langle A, B \rangle_{\rho, f} = Tr(A \cdot m_f(L_\rho, R_\rho)^{-1}(B)),$$

where $L_\rho(A) = \rho A, R_\rho(A) = A\rho$. In this case one has to think of A, B as tangent vectors to the manifold $\mathcal{D}_n^1$ at the point ρ (see [7], [3]).

Definition 3.1 *For $A \in M_{n,sa}$, we define as follows*

$$I_\rho^f(A) = \frac{f(0)}{2} \langle i[\rho, A], i[\rho, A] \rangle_{\rho, f},$$

$$C_\rho^f(A) = Tr(m_f(L_\rho, R_\rho)(A) \cdot A),$$

$$U_\rho^f(A) = \sqrt{V_\rho(A)^2 - (V_\rho(A) - I_\rho^f(A))^2}.$$

The quantity $I_\rho^f(A)$ is known as metric adjusted skew information.

Proposition 3.1 *Let $A_0 = A - Tr(\rho A)I$. The following hold:*

(1) $I^f_\rho(A) = I^f_\rho(A_0) = Tr(\rho A_0^2) - Tr(m_{\tilde{f}}(L_\rho, R_\rho)(A_0) \cdot A_0) = V_\rho(A) - C^{\tilde{f}}_\rho(A_0)$,

(2) $J^f_\rho(A) = Tr(\rho A_0^2) + Tr(m_{\tilde{f}}(L_\rho, R_\rho)(A_0) \cdot A_0) = V_\rho(A) + C^{\tilde{f}}_\rho(A_0)$,

(3) $0 \le I^f_\rho(A) \le U^f_\rho(A) \le V_\rho(A)$,

(4) $U^f_\rho(A) = \sqrt{I^f_\rho(A) \cdot J^f_\rho(A)}$.

Remark 3.1 $I^f_\rho(A)$ *is identified in [2] with* $Cov_\rho(A, A) - qCov^F_\rho(A, A)$.

4 The Main Result

Theorem 4.1 *For $f \in \mathcal{F}^r_{op}$, if*

$$\frac{x+1}{2} + \tilde{f}(x) \ge 2f(x), \tag{4.1}$$

then it holds

$$U^f_\rho(A) \cdot U^f_\rho(B) \ge f(0)|Tr(\rho[A, B])|^2, \tag{4.2}$$

where $A, B \in M_{n,sa}$.

In order to prove Theorem 4.1, we use several lemmas.

Lemma 4.1 *If (4.1) holds, then the following inequality is satisfied*

$$\left(\frac{x+y}{2}\right)^2 - m_{\tilde{f}}(x, y)^2 \ge f(0)(x - y)^2.$$

Proof. By (4.1) we have

$$\frac{x+y}{2} + m_{\tilde{f}}(x, y) \ge 2m_f(x, y). \tag{4.3}$$

Since

$$\begin{aligned} m_{\tilde{f}}(x, y) &= y\tilde{f}\left(\frac{x}{y}\right) \\ &= \frac{y}{2}\left\{\frac{x}{y} + 1 - \left(\frac{x}{y} - 1\right)^2 \frac{f(0)}{f(x/y)}\right\} \\ &= \frac{x+y}{2} - \frac{f(0)(x-y)^2}{2m_f(x, y)}, \end{aligned}$$

we have

$$
\begin{aligned}
& \left(\frac{x+y}{2}\right)^2 - m_{\tilde{f}}(x,y)^2 \\
= & \left\{\frac{x+y}{2} - m_{\tilde{f}}(x,y)\right\}\left\{\frac{x+y}{2} + m_{\tilde{f}}(x,y)\right\} \\
= & \frac{f(0)(x-y)^2}{2m_f(x,y)}\left\{\frac{x+y}{2} + m_{\tilde{f}}(x,y)\right\} \\
\geq & f(0)(x-y)^2. \quad (by \quad (4.3))
\end{aligned}
$$

□

Lemma 4.2 *Let* $\{|\phi_1\rangle, |\phi_2\rangle, \cdots, |\phi_n\rangle\}$ *be a basis of eigenvectors of* ρ*, corresponding to the eigenvalues* $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$*. We put* $a_{jk} = \langle\phi_j|A_0|\phi_k\rangle, b_{jk} = \langle\phi_j|B_0|\phi_k\rangle$*. By Corollary 6.1 in [1],*

$$
I_\rho^f(A) = \frac{1}{2}\sum_{j,k}(\lambda_j+\lambda_k)a_{jk}a_{kj} - \sum_{j,k} m_{\tilde{f}}(\lambda_j,\lambda_k)a_{jk}a_{kj},
$$

$$
J_\rho^f(A) = \frac{1}{2}\sum_{j,k}(\lambda_j+\lambda_k)a_{jk}a_{kj} + \sum_{j,k} m_{\tilde{f}}(\lambda_j,\lambda_k)a_{jk}a_{kj},
$$

$$
(U_\rho^f(A))^2 = \frac{1}{4}\left(\sum_{j,k}(\lambda_j+\lambda_k)|a_{jk}|^2\right)^2 - \left(\sum_{j,k} m_{\tilde{f}}(\lambda_j,\lambda_k)|a_{jk}|^2\right)^2.
$$

Proof of Theorem 4.1. Since

$$
Tr(\rho[A,B]) = Tr(\rho[A_0,B_0]) = \sum_{j,k}(\lambda_j-\lambda_k)a_{jk}b_{kj},
$$

we have

$$
\begin{aligned}
& f(0)|Tr(\rho[A,B])|^2 \\
\leq & \left(\sum_{j,k} f(0)^{1/2}|\lambda_j-\lambda_k||a_{jk}||b_{kj}|\right)^2 \\
\leq & \left(\sum_{j,k}\left\{\left(\frac{\lambda_j+\lambda_k}{2}\right)^2 - m_{\tilde{f}}(\lambda_j,\lambda_k)^2\right\}^{1/2}|a_{jk}||b_{kj}|\right)^2 \\
\leq & \left(\sum_{j,k}\left\{\frac{\lambda_j+\lambda_k}{2} - m_{\tilde{f}}(\lambda_j,\lambda_k)\right\}|a_{jk}|^2\right)
\end{aligned}
$$

$$\times\left(\sum_{j,k}\left\{\frac{\lambda_j+\lambda_k}{2}+m_{\tilde{f}}(\lambda_j,\lambda_k)\right\}|b_{kj}|^2\right)$$
$$= I_\rho^f(A)J_\rho^f(B).$$

We also have

$$I_\rho^f(B)J_\rho^f(A) \geq f(0)|Tr(\rho[A,B])|^2.$$

Hence we have the final result (4.2). □

By putting

$$f_{WYD}(x)=\alpha(1-\alpha)\frac{(x-1)^2}{(x^\alpha-1)(x^{1-\alpha}-1)}, \quad \alpha\in(0,1),$$

we obtain the following uncertainty relation:

Corollary 4.1 ([10]) *For $A,B\in M_{n,sa}$,*

$$U_\rho^{f_{WYD}}(A)U_\rho^{f_{WYD}}(B)\geq \alpha(1-\alpha)|Tr(\rho[A,B])|^2.$$

Proof. Since

$$f_{WYD}(x)=\alpha(1-\alpha)\frac{(x-1)^2}{(x^\alpha-1)(x^{1-\alpha}-1)},$$

it is clear that

$$\tilde{f}_{WYD}(x)=\frac{1}{2}\{x+1-(x^\alpha-1)(x^{1-\alpha}-1)\}.$$

By Lemma 3.3 in [10] we have for $0\leq\alpha\leq 1$ and $x>0$,

$$(1-2\alpha)^2(x-1)^2-(x^\alpha-x^{1-\alpha})^2\geq 0.$$

Then we can rewrite as follows

$$(x^{2\alpha}-1)(x^{2(1-\alpha)}-1)\geq 4\alpha(1-\alpha)(x-1)^2.$$

Thus

$$\begin{aligned}
&\frac{x+1}{2}+\tilde{f}_{WYD}(x)\\
=\ & x+1-\frac{1}{2}(x^\alpha-1)(x^{1-\alpha}-1)\\
=\ & \frac{1}{2}(x^\alpha+1)(x^{1-\alpha}+1)\\
\geq\ & 2\alpha(1-\alpha)\frac{(x-1)^2}{(x^\alpha-1)(x^{1-\alpha}-1)}\\
=\ & 2f_{WYD}(x).
\end{aligned}$$

It follows from Theorem 4.1 that we can give the aimed result. □

Remark 4.1 *In [4], the following result was given. Even if (4.1) does not necessarily hold, then*

$$U_\rho^f(A)U_\rho^f(B) \geq f(0)^2|Tr[(\rho[A,B])|^2, \tag{4.4}$$

where $A, B \in M_{n,sa}$. *Since* $f(0) < 1$, *it is easy to show (4.4) is weaker than (4.2).*

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