Uncertainty Relation on Wigner-Yanase-Dyson Skew Information

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Abstract. We give a trace inequality related to the uncertainty relation of Wigner-Yanase-Dyson skew information. This inequality corresponds to a generalization of the uncertainty relation derived by S.Luo [7] for the quantum uncertainty quantity excluding the classical mixture.

Key Words: Uncertainty relation, Wigner-Yanase-Dyson skew information

1 Introduction

Wigner-Yanase skew information

$$\begin{aligned} I_\rho(H) &= \frac{1}{2}Tr\left[\left(i\left[\rho^{1/2}, H\right]\right)^2\right] \\ &= Tr[\rho H^2] - Tr[\rho^{1/2}H\rho^{1/2}H] \end{aligned}$$

was defined in [9]. This quantity can be considered as a kind of the degree for non-commutativity between a quantum state ρ and an observable H. Here we denote the commutator by $[X, Y] = XY - YX$. This quantity was generalized by Dyson

$$\begin{aligned} I_{\rho,\alpha}(H) &= \frac{1}{2}Tr[(i[\rho^\alpha, H])(i[\rho^{1-\alpha}, H])] \\ &= Tr[\rho H^2] - Tr[\rho^\alpha H\rho^{1-\alpha}H], \alpha \in [0, 1] \end{aligned}$$

which is known as the Wigner-Yanase-Dyson skew information. It is famous that the convexity of $I_{\rho,\alpha}(H)$ with respect to ρ was successfully proven by E.H.Lieb in [6]. From the physical point of view, an observable H is generally considered to be an unbounded opetrator, however in the present paper, unless otherwise stated, we

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consider $H \in B(\mathcal{H})$ represents the set of all bounded linear operators on the Hilbert space $\mathcal{H}$, as a mathematical interest. We also denote the set of all self-adjoint operators (observables) by $\mathcal{L}_h(\mathcal{H})$ and the set of all density operators (quantum states) by $\mathcal{S}(\mathcal{H})$ on the Hilbert space $\mathcal{H}$. The relation between the Wigner-Yanase skew information and the uncertainty relation was studied in [8]. Moreover the relation between the Wigner-Yanase-Dyson skew information and the uncertainty relation was studied in [4, 10]. In our paper [10], we defined a generalized skew information and then derived a kind of an uncertainty relation. In the section 2, we discuss various properties of the Wigner-Yanase-Dyson skew information. Finally in section 3, we give our main result and its proof.

2 Trace inequalities of Wigner-Yanase-Dyson skew information

We review the relation between the Wigner-Yanase skew information and the uncertainty relation. In quantum mechanical system, the expectation value of an observable H in a quantum state ρ is expressed by $Tr[\rho H]$. It is natural that the variance for a quantum state ρ and an observable H is defined by $V_\rho(H) = Tr[\rho(H - Tr[\rho H]I)^2] = Tr[\rho H^2] - Tr[\rho H]^2$. It is famous that we have

$$V_\rho(A)V_\rho(B) \geq \frac{1}{4}|Tr[\rho[A, B]]|^2 \tag{2.1}$$

for a quantum state ρ and two observables A and B. The further strong results was given by Robertson and Schrödinger

$$V_\rho(A)V_\rho(B) - |Cov_\rho(A, B)|^2 \geq \frac{1}{4}|Tr[\rho[A, B]]|^2,$$

where the covariance is defined by $Cov_\rho(A, B) = Tr[\rho(A - Tr[\rho A]I)(B - Tr[\rho B]I)]$. However, the uncertainty relation for the Wigner-Yanase skew information failed. (See [8, 4, 10])

$$I_\rho(A)I_\rho(B) \geq \frac{1}{4}|Tr[\rho[A, B]]|^2.$$

Recently, S.Luo introduced the quantity $U_\rho(H)$ representing a quantum uncertainty excluding the classical mixture:

$$U_\rho(H) = \sqrt{V_\rho(H)^2 - (V_\rho(H) - I_\rho(H))^2}, \tag{2.2}$$

then he derived the uncertainty relation on $U_\rho(H)$ in [7]:

$$U_\rho(A)U_\rho(B) \geq \frac{1}{4}|Tr[\rho[A, B]]|^2. \tag{2.3}$$

Note that we have the following relation

$$0 \le I_\rho(H) \le U_\rho(H) \le V_\rho(H). \tag{2.4}$$

The inequality (2.3) is a refinement of the inequality (2.1) in the sense of (2.4). In this section, we study one-parameter extended inequality for the inequality (2.3).

Definition 2.1 *For $0 \le \alpha \le 1$, a quantum state ρ and an observable H, we define the Wigner-Yanase-Dyson skew information*

$$\begin{aligned} I_{\rho,\alpha}(H) &= \frac{1}{2}Tr[(i[\rho^\alpha, H_0])(i[\rho^{1-\alpha}, H_0])] \\ &= Tr[\rho H_0^2] - Tr[\rho^\alpha H_0 \rho^{1-\alpha} H_0] \end{aligned} \tag{2.5}$$

and we also define

$$\begin{aligned} J_{\rho,\alpha}(H) &= \frac{1}{2}Tr[\{\rho^\alpha, H_0\}\{\rho^{1-\alpha}, H_0\}] \\ &= Tr[\rho H_0^2] + Tr[\rho^\alpha H_0 \rho^{1-\alpha} H_0], \end{aligned} \tag{2.6}$$

where $H_0 = H - Tr[\rho H]I$ and we denote the anti-commutator by $\{X, Y\} = XY + YX$.

Note that we have

$$\frac{1}{2}Tr[(i[\rho^\alpha, H_0])(i[\rho^{1-\alpha}, H_0])] = \frac{1}{2}Tr[(i[\rho^\alpha, H])(i[\rho^{1-\alpha}, H])]$$

but we have

$$\frac{1}{2}Tr[\{\rho^\alpha, H_0\}\{\rho^{1-\alpha}, H_0\}] \ne \frac{1}{2}Tr[\{\rho^\alpha, H\}\{\rho^{1-\alpha}, H\}].$$

Then we have the following inequalities:

$$I_{\rho,\alpha}(H) \le I_\rho(H) \le J_\rho(H) \le J_{\rho,\alpha}(H), \tag{2.7}$$

since we have $Tr[\rho^{1/2} H \rho^{1/2} H] \le Tr[\rho^\alpha H \rho^{1-\alpha} H]$. (See [1, 2] for example.) If we define

$$U_{\rho,\alpha}(H) = \sqrt{V_\rho(H)^2 - (V_\rho(H) - I_{\rho,\alpha}(H))^2}, \tag{2.8}$$

as a direct generalization of Eq.(2.2), then we have

$$0 \le I_{\rho,\alpha}(H) \le U_{\rho,\alpha}(H) \le U_\rho(H) \tag{2.9}$$

due to the first inequality of (2.7). We also have

$$U_{\rho,\alpha}(H) = \sqrt{I_{\rho,\alpha}(H) J_{\rho,\alpha}(H)}.$$

From the inequalities (2.4),(2.8),(2.9), our situation is that we have

$$0 \leq I_{\rho,\alpha}(H) \leq I_\rho(H) \leq U_\rho(H)$$

and

$$0 \leq I_{\rho,\alpha}(H) \leq U_{\rho,\alpha}(H) \leq U_\rho(H).$$

Our concern is to show an uncertainty relation with respect to $U_{\rho,\alpha}(H)$ as a direct generalization of the inequality (2.3) such that

$$U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) \geq \frac{1}{4}|Tr[\rho[A,B]]|^2 \tag{2.10}$$

On the other hand, we introduced a generalized Wigner-Yanase skew information which is a generalization of the inequality (2.10), but different from the Wigner-Yanase-Dyson skew information defined in (2.5) and gave the following theorem in [11].

Theorem 2.1 *For $0 \leq \alpha \leq 1$, a quantum state ρ and an observable H, we define a generalized Wigner-Yanase skew information by*

$$K_{\rho,\alpha}(H) = \frac{1}{2}Tr\left[\left(i\left[\frac{\rho^\alpha + \rho^{1-\alpha}}{2}, H_0\right]\right)^2\right]$$

and we also define

$$L_{\rho,\alpha}(H) = \frac{1}{2}Tr\left[\left(i\left\{\frac{\rho^\alpha + \rho^{1-\alpha}}{2}, H_0\right\}\right)^2\right],$$

and

$$W_{\rho,\alpha}(H) = \sqrt{K_{\rho,\alpha}(H)L_{\rho,\alpha}(H)}.$$

Then for a quantum state ρ and observables A, B and $\alpha \in [0,1]$, we have

$$W_{\rho,\alpha}(A)W_{\rho,\alpha}(B) \geq \frac{1}{4}\left|Tr\left[\left(\frac{\rho^\alpha + \rho^{1-\alpha}}{2}\right)^2 [A,B]\right]\right|^2.$$

3 Main Theorem

We give the main theorem as follows;

Theorem 3.1 *For a quantum state ρ and observables A, B and $0 \leq \alpha \leq 1$, we have*

$$U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) \geq \alpha(1-\alpha)|Tr[\rho[A,B]]|^2. \tag{3.1}$$

We use the several lemmas to prove the theorem 3.1. By spectral decomposition, there exists an orthonormal basis $\{\phi_1, \phi_2, \ldots\}$ consisting of eigenvectors of ρ. Let $\lambda_1, \lambda_2, \ldots$ be the corresponding eigenvalues, where $\sum_{i=1}^{\infty} \lambda_i = 1$ and $\lambda_i \geq 0$. Thus, ρ has a spectral representation

$$\rho = \sum_{i=1}^{\infty} \lambda_i |\phi_i\rangle\langle\phi_i|. \tag{3.2}$$

Lemma 3.1

$$I_{\rho,\alpha}(H) = \sum_{i<j} (\lambda_i + \lambda_j - \lambda_i^{\alpha}\lambda_j^{1-\alpha} - \lambda_i^{1-\alpha}\lambda_j^{\alpha}) |\langle\phi_i|H_0|\phi_j\rangle|^2.$$

Proof of Lemma 3.1. By (3.2),

$$\rho H_0^2 = \sum_{i=1}^{\infty} \lambda_i |\phi_i\rangle\langle\phi_i| H_0^2.$$

Then

$$Tr[\rho H_0^2] = \sum_{i=1}^{\infty} \lambda_i \langle\phi_i|H_0^2|\phi_i\rangle = \sum_{i=1}^{\infty} \lambda_i \|H_0|\phi_i\rangle\|^2. \tag{3.3}$$

Since

$$\rho^{\alpha} H_0 = \sum_{i=1}^{\infty} \lambda_i^{\alpha} |\phi_i\rangle\langle\phi_i| H_0$$

and

$$\rho^{1-\alpha} H_0 = \sum_{i=1}^{\infty} \lambda_i^{1-\alpha} |\phi_i\rangle\langle\phi_i| H_0,$$

we have

$$\rho^{\alpha} H_0 \rho^{1-\alpha} H_0 = \sum_{i,j=1}^{\infty} \lambda_i^{\alpha}\lambda_j^{1-\alpha} |\phi_i\rangle\langle\phi_i|H_0|\phi_j\rangle\langle\phi_j| H_0.$$

Thus

$$\begin{aligned} Tr[\rho^{\alpha} H_0 \rho^{1-\alpha} H_0] &= \sum_{i,j=1}^{\infty} \lambda_i^{\alpha}\lambda_j^{1-\alpha} \langle\phi_i|H_0|\phi_j\rangle\langle\phi_j|H_0|\phi_i\rangle \\ &= \sum_{i,j=1}^{\infty} \lambda_i^{\alpha}\lambda_j^{1-\alpha} |\langle\phi_i|H_0|\phi_j\rangle|^2. \end{aligned} \tag{3.4}$$

From (2.5), (3.3), (3.4),

$$\begin{aligned} I_{\rho,\alpha}(H) &= \sum_{i=1}^{\infty} \lambda_i \| H_0 |\phi_i\rangle \|^2 - \sum_{i,j=1}^{\infty} \lambda_i^{\alpha} \lambda_j^{1-\alpha} |\langle \phi_i | H_0 | \phi_j \rangle|^2 \\ &= \sum_{i,j=1}^{\infty} (\lambda_i - \lambda_i^{\alpha}\lambda_j^{1-\alpha}) |\langle \phi_i | H_0 | \phi_j \rangle|^2 \\ &= \sum_{i<j} (\lambda_i + \lambda_j - \lambda_i^{\alpha}\lambda_j^{1-\alpha} - \lambda_i^{1-\alpha}\lambda_j^{\alpha}) |\langle \phi_i | H_0 | \phi_j \rangle|^2. \end{aligned}$$

□

Lemma 3.2

$$J_{\rho,\alpha}(H) \geq \sum_{i<j} (\lambda_i + \lambda_j + \lambda_i^{\alpha}\lambda_j^{1-\alpha} + \lambda_i^{1-\alpha}\lambda_j^{\alpha}) |\langle \phi_i | H_0 | \phi_j \rangle|^2.$$

Proof of Lemma 3.2. By (2.6), (3.3), (3.4), we have

$$\begin{aligned} J_{\rho,\alpha}(H) &= \sum_{i=1}^{\infty} \lambda_i \| H_0 |\phi_i\rangle \|^2 + \sum_{i,j=1}^{\infty} \lambda_i^{\alpha} \lambda_j^{1-\alpha} |\langle \phi_i | H_0 | \phi_j \rangle|^2 \\ &= \sum_{i,j=1}^{\infty} (\lambda_i + \lambda_i^{\alpha}\lambda_j^{1-\alpha}) |\langle \phi_i | H_0 | \phi_j \rangle|^2 \\ &= 2\sum_{i=1}^{\infty} \lambda_i |\langle \phi_i | H_0 | \phi_i \rangle|^2 + \sum_{i \neq j} (\lambda_i + \lambda_i^{\alpha}\lambda_j^{1-\alpha}) |\langle \phi_i | H_0 | \phi_j \rangle|^2 \\ &= 2\sum_{i=1}^{\infty} \lambda_i |\langle \phi_i | H_0 | \phi_i \rangle|^2 + \sum_{i<j} (\lambda_i + \lambda_j + \lambda_i^{\alpha}\lambda_j^{1-\alpha} + \lambda_i^{1-\alpha}\lambda_j^{\alpha}) |\langle \phi_i | H_0 | \phi_j \rangle|^2 \\ &\geq \sum_{i<j} (\lambda_i + \lambda_j + \lambda_i^{\alpha}\lambda_j^{1-\alpha} + \lambda_i^{1-\alpha}\lambda_j^{\alpha}) |\langle \phi_i | H_0 | \phi_j \rangle|^2. \end{aligned}$$

□

Lemma 3.3 *For any $t > 0$ and $0 \leq \alpha \leq 1$, the following inequality holds;*

$$(1-2\alpha)^2 (t-1)^2 - (t^{\alpha} - t^{1-\alpha})^2 \geq 0. \tag{3.5}$$

Proof of Lemma 3.3. If $\alpha = 0$ or $\frac{1}{2}$ or 1, then it is clear that (3.5) is satisfied. Now we put

$$F(t) = (1-2\alpha)^2(t-1)^2 - (t^\alpha - t^{1-\alpha})^2.$$

We have

$$F^{'}(t) = 2(1-2\alpha)^2 t - 2\alpha t^{2\alpha-1} - 2(1-\alpha)t^{1-2\alpha} + 8\alpha(1-\alpha).$$

And we also have

$$F^{''}(t) = 2(1-2\alpha)^2 - 2\alpha(2\alpha-1)t^{2\alpha-2} - 2(1-\alpha)(1-2\alpha)t^{-2\alpha}$$

and

$$\begin{aligned} & F^{'''}(t) \\ = \; & 4\alpha(1-2\alpha)(1-\alpha)t^{-2\alpha-1} - 4\alpha(1-2\alpha)(1-\alpha)t^{2\alpha-3} \\ = \; & 4\alpha(1-2\alpha)(1-\alpha)\left(\frac{1}{t^{1+2\alpha}} - \frac{1}{t^{3-2\alpha}}\right). \end{aligned}$$

If $\frac{1}{2} < \alpha < 1$, then $1+2\alpha > 3-2\alpha$. Then it is easy to show that $F^{'''}(t) < 0$ for $t < 1$ and $F^{'''}(t) > 0$ for $t > 1$. On the other hand if $0 < \alpha < \frac{1}{2}$, then $1+2\alpha < 3-2\alpha$. Then it is easy to show that $F^{'''}(t) < 0$ for $t < 1$ and $F^{'''}(t) > 0$ for $t > 1$. Since $F^{''}(1) = 0$, we can get $F^{''}(t) > 0$. Since $F^{'}(1) = 0$, we also have $F^{'}(t) < 0$ for $t < 1$ and $F^{'}(t) > 0$ for $t > 1$. Since $F(1) = 0$, we finally get $F(t) \geq 0$ for all $t > 0$. Therefore we have (3.5). $\square$

Proof of Theorem 3.1. We put $t = \dfrac{\lambda_i}{\lambda_j}$ in (3.5). Then we have

$$(1-2\alpha)^2\left(\frac{\lambda_i}{\lambda_j} - 1\right)^2 - \left(\left(\frac{\lambda_i}{\lambda_j}\right)^\alpha - \left(\frac{\lambda_i}{\lambda_j}\right)^{1-\alpha}\right)^2 \geq 0.$$

And we get

$$(1-2\alpha)^2(\lambda_i - \lambda_j)^2 - (\lambda_i^\alpha\lambda_j^{1-\alpha} - \lambda_i^{1-\alpha}\lambda_j^\alpha)^2 \geq 0$$

and

$$(\lambda_i - \lambda_j)^2 - (\lambda_i^\alpha\lambda_j^{1-\alpha} - \lambda_i^{1-\alpha}\lambda_j^\alpha)^2 \geq 4\alpha(1-\alpha)(\lambda_i - \lambda_j)^2$$

and

$$(\lambda_i + \lambda_j)^2 - (\lambda_i^\alpha\lambda_j^{1-\alpha} + \lambda_i^{1-\alpha}\lambda_j^\alpha)^2 \geq 4\alpha(1-\alpha)(\lambda_i - \lambda_j)^2. \tag{3.6}$$

Since

$$Tr[\rho[A,B]] \;=\; Tr[\rho[A_0,B_0]]$$

$$
\begin{aligned}
&= 2iImTr[\rho A_0 B_0] \\
&= 2iIm\sum_{i<j}(\lambda_i - \lambda_j)\langle\phi_i|A_0|\phi_j\rangle\langle\phi_j|B_0|\phi_i\rangle \\
&= 2i\sum_{i<j}(\lambda_i - \lambda_j)Im\langle\phi_i|A_0|\phi_j\rangle\langle\phi_j|B_0|\phi_i\rangle,
\end{aligned}
$$

$$
\begin{aligned}
|Tr[\rho[A,B]]| &= 2|\sum_{i<j}(\lambda_i - \lambda_j)Im\langle\phi_i|A_0|\phi_j\rangle\langle\phi_j|B_0|\phi_i\rangle| \\
&\leq 2\sum_{i<j}|\lambda_i - \lambda_j||Im\langle\phi_i|A_0|\phi_j\rangle\langle\phi_j|B_0|\phi_i\rangle|.
\end{aligned}
$$

Then we have

$$
|Tr[\rho[A,B]]|^2 \leq 4\{\sum_{i<j}|\lambda_i - \lambda_j||Im\langle\phi_i|A_0|\phi_j\rangle\langle\phi_j|_0|\phi_i\rangle|\}^2.
$$

By (3.6) and Schwarz inequality,

$$
\begin{aligned}
& \alpha(1-\alpha)|Tr[\rho[A,B]]|^2 \\
\leq\ & 4\alpha(1-\alpha)\{\sum_{i<j}|\lambda_i - \lambda_j||Im\langle\phi_i|A_0|\phi_j\rangle\langle\phi_j|B_0|\phi_i\rangle|\}^2 \\
=\ & \{\sum_{i<j}2\sqrt{\alpha(1-\alpha)}|\lambda_i - \lambda_j||Im\langle\phi_i|A_0|\phi_j\rangle\langle\phi_j|B_0|\phi_i\rangle|\}^2 \\
\leq\ & \{\sum_{i<j}2\sqrt{\alpha(1-\alpha)}|\lambda_i - \lambda_j||\langle\phi_i|A_0|\phi_j\rangle||\langle\phi_j|B_0|\phi_i\rangle|\}^2 \\
\leq\ & \{\sum_{i<j}\{(\lambda_i + \lambda_j)^2 - (\lambda_i^{\alpha}\lambda_j^{1-\alpha} + \lambda_i^{1-\alpha}\lambda_j^{\alpha})^2\}^{1/2}|\langle\phi_i|A_0|\phi_j\rangle||\langle\phi_j|B_0|\phi_i\rangle|\}^2 \\
\leq\ & \sum_{i<j}(\lambda_i + \lambda_j - \lambda_i^{\alpha}\lambda_j^{1-\alpha} - \lambda_i^{1-\alpha}\lambda_j^{\alpha})|\langle\phi_i|A_0|\phi_j\rangle|^2 \\
& \times\sum_{i<j}(\lambda_i + \lambda_j + \lambda_i^{\alpha}\lambda_j^{1-\alpha} + \lambda_i^{1-\alpha}\lambda_j^{\alpha})|\langle\phi_i|B_0|\phi_j\rangle|^2.
\end{aligned}
$$

Then we have

$$
I_{\rho,\alpha}(A)J_{\rho,\alpha}(B) \geq \alpha(1-\alpha)|Tr[\rho[A,B]]|^2.
$$

We also have

$$
I_{\rho,\alpha}(B)J_{\rho,\alpha}(A) \geq \alpha(1-\alpha)|Tr[\rho[A,B]]|^2.
$$

Hence we have the final result (3.1). □

Remark 3.1 *We remark that (2.3) is derived by putting $\alpha = 1/2$ in (3.1). Then Theorem 3.1 is a generalization of the result of Luo [7].*

Remark 3.2 *We remark that Conjecture 2.3 in [11] does not hold in genaral. The Conjecture is (2.10). A counterexample is given as follows. Let*

$$\rho = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}, \ A = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \alpha = \frac{1}{3}.$$

We have

$$I_{\rho,\alpha}(A)J_{\rho,\alpha}(B) = I_{\rho,\alpha}(B)I_{\rho,\alpha}(A) = 0.22457296\cdots$$

and $|Tr[\rho[A,B]]|^2 = 1$. *These imply*

$$U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) = 0.22457296\cdots < \frac{1}{4}|Tr[\rho[A,B]]|^2 = 0.25.$$

On the other hand we have

$$U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) > \alpha(1-\alpha)|Tr[\rho[A,B]]|^2 = 0.2222222\cdots.$$

We also give a counterexample for Conjecture 2.10 in [11]. The inequality

$$U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) \geq \frac{1}{4}|Tr[(\frac{\rho^{\alpha}+\rho^{1-\alpha}}{2})^2[A,B]]|^2$$

is not correct in general, because $LHS = 0.22457296\cdots, RHS = 0.23828105995\cdots$.

Remark 3.3 *In the recent literature another generalization for inequality (2.3) has been proved in [5] as follows; for any* ρ, A, B *and* $0 \leq \alpha \leq 1$

$$U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) \geq \frac{1}{4}|Tr[(\rho - \rho^{|2\alpha-1|})[A,B]]|^2.$$

However we gave the counter example for this inequality. Let

$$\rho = \begin{pmatrix} \frac{1}{64} & 0 & 0 \\ 0 & \frac{1}{16} & 0 \\ 0 & 0 & \frac{59}{64} \end{pmatrix}, A = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \alpha = \frac{3}{4}.$$

Then we have

$$U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) = 0.00170898\cdots,$$

$$\frac{1}{4}|Tr[(\rho - \rho^{|2\alpha-1|})[A,B]]|^2 = 0.00610351\cdots.$$

References

[1] J.C.Bourin, *Some inequalities for norms on matrices and operators*, Linear Algebra and its Applications, vol.292(1999), pp.139-154.

[2] J.I.Fujii, *A trace inequality arising from quantum information theory*, Linear Algebra and its Applications, vol.400(2005), pp.141-146.

[3] W.Heisenberg, *Über den anschaulichen Inhat der quantummechanischen Kinematik und Mechanik*, Zeitschrift für Physik, vol.43(1927), pp.172-198.

[4] H.Kosaki, *Matrix trace inequality related to uncertainty principle*, Internatonal Journal of Mathematics, vol.16(2005), pp.629-646.

[5] L.Li, X.Li, F.Wang, H.Huang X.Li and L.C.Kwek, *Uncertainty relation of mixed states by means of Wigner-Yanase-Dyson information*, Physical Review A, vol.79(2009), pp.052106-1-4.

[6] E.H.Lieb, *Convex trace functions and the Wigner-Yanase-Dyson conjecture*, Adv. Math., vol.11(1973), pp.267-288.

[7] S.Luo, *Heisenberg uncertainty relation for mixed states*, Phys. Rev. A, vol.72(2005), p.042110.

[8] S.Luo and Q.Zhang, *On skew information*, IEEE Trans. Information Theory, vol.50(2004), pp.1778-1782, and *Correction to "On skew information"*, IEEE Trans. Information Theory, vol.51(2005), p.4432.

[9] E.P.Wigner and M.M.Yanase, *Information content of distribution*, Proc. Nat. Acad. Sci. U,S,A., vol.49(1963), pp.910-918.

[10] K.Yanagi, S.Furuichi and K.Kuriyama, *A generalized skew information and uncertainty relation*, IEEE Trans. Information Theory, vol.51(2005), pp.4401-4404.

[11] S.Furuichi, K.Yanagi and K.Kuriyama, *Trace inequalities on a generalized Wigner-Yanase skew information*, J. Math. Anal. Appl., vol.356(2009), pp.179-185.